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Secondary School

MATHEMATICS

Grade Eleven

Edition Two

P. R. BEESACK
W. B. MacLEAN
W. L. MUMFORD
D. W. ALEXANDER

SECONDARY SCHOOL **Mathematics** GRADE ELEVEN

Edition Two

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AUTHORS

P. R. BEESACK
Professor of Mathematics
Carleton University, Ottawa

W. B. MacLEAN
Professor of Methods in Mathematics
Ontario College of Education
University of Toronto

D. L. MUMFORD
Associate Professor of Methods in Mathematics
Ontario College of Education
University of Toronto

D. W. ALEXANDER
Instructor of Mathematics
University of Toronto Schools, Toronto

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PREFACE

Secondary School Mathematics Grade Eleven, Edition Two, is the fifth text in the Copp Clark Modern Mathematics Series from Grade 7 through Grade 12.

The main theme of the text is the development of the concept of function. This is accomplished by the study of linear, exponential, logarithmic, quadratic, and trigonometric functions.

The emphasis in this text is on discovery and on understanding of general principles and processes of mathematics. Many discovery exercises and pre-assignment practice examples are included which students can work independently and then compare with solutions provided at the back of the book.

The study of the circle and its properties by both synthetic and analytic methods provides an instructive contrast.

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P.R.B.
W.B.M.
D.L.M.
C.W.A.

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MATHEMATICAL SYMBOLS

$\because$	since; because
$\therefore$	therefore
$\ldots$	and so on
.	decimal point
$.\dot{3}$ *or* $.\overline{3}$	repeating decimal
$+$	plus; add; positive
$-$	minus; subtract; negative
$\pm$	plus or minus; positive or negative
$\times$ *or* $\cdot$	multiply
$\div$	divide
$=$	is equal to
$\neq$	is not equal to
$\doteq$	is approximately equal to
$>$	is greater than
$\ngtr$	is not greater than
$<$	is less than
$\nless$	is not less than
$\geqq$	is greater than or equal to
$\ngeqq$	is not greater than or equal to
$\leqq$	is less than or equal to
$\nleqq$	is not less than or equal to
$a : b$	the ratio of a to b
$\lvert\ \rvert$	the absolute value of
$\sqrt{\ }$	the principal square root of
$\sqrt[n]{\ }$	the principal nth root of
$(\ \ )$	round brackets
$[\ \ ]$	square brackets
$\overline{a+b}$	bar bracket
%	per cent
N	the set of natural numbers
N_0	the set of natural numbers and zero
I	the set of integers
^{+}I	the set of positive integers
^{-}I	the set of negative integers

Q	the set of rational numbers
R	the set of real numbers
^{+}R	the set of positive real numbers
^{-}R	the set of negative real numbers
C	the set of complex numbers
i	$i^2 = -1$
(ClA)	Closure Law for Addition
(ClM)	Closure Law for Multiplication
(CA)	Commutative Law for Addition
(CM)	Commutative Law for Multiplication
(AA)	Associative Law for Addition
(AM)	Associative Law for Multiplication
(D)	Distributive Law
$\{\quad\}$	brace brackets; the set
$\{\mid\}$	the set builder
$\in$	belongs to; is a member of; is an element of
$\notin$	is not a member of
$\emptyset$	null set
$\subset$	is a proper subset of
$\subseteq$	is a subset of
$\cup$	cup; the union of
$\cap$	cap; the intersection of
$\sim A$	not A; the complement of A
$A \times B$	the Cartesian product of A and B
$\forall$	for all; for every; for any
$\exists$	there exists; for some; there is at least one
$\wedge$	and (conjunction)
$\vee$	or (disjunction)
$\rightarrow$	if . . . then; implies; only if; is a sufficient condition for (conditional)
$\leftrightarrow$	is equivalent to; if and only if; is a necessary and sufficient condition for (biconditional)
iff	if and only if
$\sim A$	the negation of A (negation)
$f(x)$	the value of the function f at x
log	common logarithm (base 10)
ln	natural (or Napierian) logarithm (base e)

e	2.71828 . . .
π	$\frac{22}{7}$ (3 digits); 3.1416 (5 digits)
$^\circ$	degree
$'$	minute; foot
$''$	second; inch
f_n	the nth term of the sequence f
$\{f_k\}_{k=1}^{\infty}$	the infinite sequence $f_1, f_2\, f_3, \ldots$
$\sum_{k=1}^{k=50} f_k$	the finite series $f_1 + f_2 + f_3 + \ldots + f_{50}$
$\cong$	is congruent to; is identical to
$\sim$	is similar to (as in $\triangle ABC \sim \triangle DEF$)
$\|\|$	is parallel to
$\nparallel$	is not parallel to
$\perp$	is perpendicular to
$\angle$	angle
$\triangle$	triangle
$\|\|^{gm}$	parallelogram

Chapter I

REAL NUMBERS
EQUATIONS AND INEQUATIONS

***1·1* Real numbers.** The set of real numbers may be developed through extensions from the natural numbers. The development produces the following sets of numbers.

Natural numbers	$N = \{1, 2, 3, \ldots\}$
Whole numbers	$N_0 = \{0, 1, 2, 3, \ldots\}$
Integers	$I = \{\pm a \mid a \in N\} \cup \{0\}$
Rational numbers	$Q = \left\{\frac{a}{b} \mid a, b \in I, b \neq 0\right\}$
Real numbers	R is the union set of

(i) the set of all rational numbers
and (ii) the set of all irrational numbers.

The real numbers may be characterized as the set of numbers represented by

the set of all decimals,

which is composed of:

(i) the set of all periodic decimals (rational numbers),
and (ii) the set of all non-periodic decimals (irrational numbers).

The set of real numbers contains subsets, the elements of which behave under addition and multiplication as do the natural numbers, whole numbers, integers, and rational numbers. In this sense, the real numbers contain these other sets of numbers as subsets.

Under addition and multiplication the real number system has the following properties.

If $a, b, c \in R$:

PROPERTY	ADDITION	MULTIPLICATION
Closure	$(a + b)$ *represents a unique real number.* (*ClA*)	(ab) *represents a unique real number.* (*ClM*)
Commutative	$a + b = b + a$ (*CA*)	$ab = ba$ (*CM*)
Associative	$(a+b) + c = a+(b+c)$ (*AA*)	$(ab)c = a(bc)$ (*AM*)
Distributive	$a(b + c) = ab + ac$ (*D*)	
Identity Element	0 *is the unique element such that* $0 + a = a + 0 = a.$	1 *is the unique element such that* $1 \cdot a = a \cdot 1 = a.$
Inverse Element	*For every* a *there is a unique element* $-a$ *such that* $a+(-a) = (-a)+a = 0.$	*For every* a $(a \neq 0)$ *there is a unique element* $\frac{1}{a}$ *such that* $a \cdot \frac{1}{a} = \frac{1}{a} \cdot a = 1.$

Order: If $a, b \in R$, then

$a > b$ if $a - b$ is a positive real number

or $a > b$ if $a = b + c$ where $c \in {}^{+}R$ (a positive real number).

$b < a \leftrightarrow a > b.$

Trichotomy property: One and only one of the following is true:

$$a > b \quad \text{or} \quad a = b \quad \text{or} \quad a < b.$$

Transitive property: If $a > b$ and $b > c$, then $a > c$.

Completeness: The real number system is complete: that is, each point on a real number line corresponds to a real number, and to each real number there corresponds a point on the line.

Products involving zero:

(i) if $a = 0$ or $b = 0$, then $a \cdot b = 0$;

(ii) if $a \cdot b = 0$, then $a = 0$ or $b = 0$.

Thus, $$a \cdot b = 0 \leftrightarrow a = 0 \text{ or } b = 0.$$

***1·2* Number system; field.** The first seven properties of the list in Section 1·1 are also shared by the systems of natural numbers, whole numbers, integers, and rational numbers.

A number system is:

(i) a set of *entities* (usually called numbers),

(ii) on which are defined two *binary operations* called *addition* (+) and *multiplication* (×), such that

(iii) the set is *closed* under each operation, and

(iv) each operation is *commutative* and *associative*, and

(v) multiplication is *distributive* over addition.

Some of these number systems have other properties also. The rational and real number systems have, along with the seven properties listed above, the following four properties:

(i) an identity element (zero) for addition;

(ii) an identity element (1) for multiplication;

(iii) for each a there is a unique inverse element $(-a)$ for addition;

(iv) for each a $(a \neq 0)$ there is a unique inverse element $\left(\frac{1}{a}\right)$ for multiplication.

A number system which has all eleven properties is called a *field*.

***1·3* The equality relation.** The equality relation, "*is equal to*", over the set of real numbers is an *equivalence relation* since it possesses the following properties:

If $a, b, c \in R$, then

(i) $a = a$; (Reflexive property)

(ii) if $a = b$, then $b = a$; (Symmetric property)

(iii) if $a = b$ and $b = c$, then $a = c$. (Transitive property)

Many of the relations in the work which follows involve equality. These properties are made use of without further reference.

***1·4* The solution of linear equations in one variable; graph of the solution set.** The sentence

$$p \rightarrow q$$

in which p and q represent statements is called a *conditional sentence*.

For example: If $3x + 2 = x - 4$, then $x = -3$. (1)

The sentence

$$q \rightarrow p$$

is the *converse conditional* of $p \rightarrow q$.

For example: If $x = -3$, then $3x + 2 = x - 4$ (2)

is the converse of sentence (1).

If it is possible to prove both implications, then the *biconditional statement*

$$p \leftrightarrow q$$

which states that "p is equivalent to q" is proved. (The statement $p \leftrightarrow q$ is also read, "p if and only if q" and written p iff q.)

For example: $3x + 2 = x - 4 \leftrightarrow x = -3$ (3)

since it is possible to show that both of the implications (1) and (2) are true.

A logical consequent of statement (3) is that $x = -3$, that is, -3 is the *root* of the equation $3x + 2 = x - 4$, or $\{-3\}$ is the *solution set* of the equation.

To *solve an equation* in one variable it is sufficient to obtain simpler equivalent equations until one is obtained from which the solution set or root may be read.

The closure property of the set of real numbers under addition, subtraction, multiplication, and division justifies the following statement:

If $a, b, c \in R$, then

(i) $a = b$
(ii) $a + c = b + c$
(iii) $a - c = b - c$
(iv) $a \times c = b \times c, c \neq 0$
(v) $a \div c = b \div c, c \neq 0$

are equivalent equations.

Thus if $x \in R$, the solution set of the equation

$$3x + 2 = x - 4$$

is obtained by writing the following sequence of equivalent equations:

$$\begin{aligned} & 3x + 2 = x - 4 \\ \leftrightarrow\ & 2x + 2 = -4 \\ \leftrightarrow\ & 2x = -6 \\ \leftrightarrow\ & x = -3. \end{aligned}$$

$\therefore$ the solution set is $\{-3\}$.

The symbol $\leftrightarrow$ indicates:

(i) that the equation following is, in each case, equivalent to the previous one;

and (ii) that the operations performed in obtaining the equation are reversible.

A verification (or check) that no computational error has been made and that -3 satisfies the equation may be carried out as follows:

Verification. L.S. $= 3(-3) + 2$ R.S. $= -3 - 4$
$= -9 + 2$ $= -7.$
$= -7.$

The graph of the solution set is shown in *Fig. 1-1.*

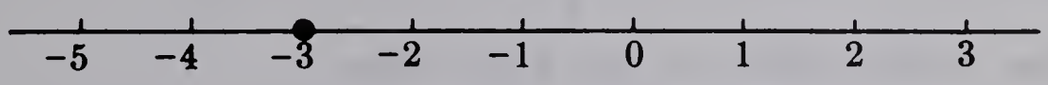


Fig. 1-1.

Example 1. Solve $\dfrac{x+3}{3} - \dfrac{2x-4}{4} = 1, x \in R.$

Solution.

$$\frac{x+3}{3} - \frac{2x-4}{4} = 1$$

$\leftrightarrow$ $4(x+3) - 3(2x-4) = 12$ (Multiplication by 12)

$\leftrightarrow$ $-2x + 24 = 12$

$\leftrightarrow$ $-2x = -12$

$\leftrightarrow$ $x = 6.$

$\therefore$ 6 is the root of the equation.

Example 2. Solve $\dfrac{x+1}{x-1} = \dfrac{5-2x}{7-2x}, x \in R.$

Solution. Since division by zero is not defined,

$\therefore x - 1 \neq 0$ (or $x \neq 1$) and $7 - 2x \neq 0$ (or $x \neq 3\frac{1}{2}$).

For $x \neq 1$ and $x \neq 3\frac{1}{2}$,

$$\frac{x+1}{x-1} = \frac{5-2x}{7-2x}$$

$\leftrightarrow$ $(x+1)(7-2x) = (5-2x)(x-1)$ (Multiplication by $(x-1)(7-2x)$)

$\leftrightarrow$ $5x + 7 - 2x^2 = 7x - 2x^2 - 5$

$\leftrightarrow$ $-2x = -12$

$\leftrightarrow$ $x = 6.$

$\therefore$ the root of the equation is 6.

Example 3. Solve $9(c-2)+c=6(c-2)+2(2c-3), c \in R$.

Solution.

$$\begin{aligned} & 9(c-2)+c=6(c-2)+2(2c-3) \\ \leftrightarrow \quad & 10c-18=10c-18 \\ \leftrightarrow \quad & 0 \cdot c=0 . \end{aligned}$$

Since this equation is true for all real numbers, every real number is a root. The solution set is R.

An equation which is true for all values of the replacement set is called an *identical equation* (or simply an *identity*) in that set.

Example 4. Solve $13-\dfrac{2}{a+2}=\dfrac{4a+6}{a+2}, a \in R$.

Solution. Since division by zero is not defined,

$\therefore a+2 \neq 0$ (or $a \neq -2$).

For $a \neq -2$,

$$\begin{aligned} & 13-\frac{2}{a+2}=\frac{4a+6}{a+2} \\ \leftrightarrow \quad & 13=\frac{4a+8}{a+2} && \text{(Addition of } \frac{2}{a+2}\text{)} \\ \leftrightarrow \quad & 13a+26=4a+8 && \text{(Multiplication by } a+2\text{)} \\ \leftrightarrow \quad & 9a=-18 \\ \leftrightarrow \quad & a=-2 . \end{aligned}$$

Since $a \neq -2$, there is no real root of the equation.

Write solutions for the following; compare your solutions with those on page 403.

1. Find $\{x \mid \frac{1}{3}x-\frac{4}{5}x=\frac{7}{8}, x \in R\}$, and draw the graph of the solution set. The defining sentence contains an equation with fractional coefficients.
2. Solve the equation $3ax+4bx=c$ for x, $x \in R$, if $a, b, c \in R$. Verify. This equation is one with literal coefficients.

Unless otherwise specified, it will be understood that the variables are real numbers in the remainder of this text.

Exercise 1-1

(B)

Solve each of the following; draw the graph of the solution set:

1. $3(x-2)+2(x-1)=6$
2. $\frac{1}{2}x+\frac{3}{4}x=6$
3. $(x-2)(x+2)-(x+3)(x-2)=7$
4. $3(x-2)(x+4)-(3x+4)(x+2)=3$

Find and verify:

5. $\left\{x \mid \frac{1}{3}(x-2)+\frac{2}{5}(x+3)=\frac{x-5}{2}, x \in R\right\}$

6. $\{x \mid (3x+4)(2x-5)-3(2x+3)(x-2)+(x-2)(x+2)-(x+3)(x+4)=4, x \in R\}$

7. $\left\{x \left| \frac{x-3}{5}+\frac{x-3}{2}-\frac{x+4}{3}=\frac{3}{2}, x \in R\right.\right\}$

Solve:

8. $\frac{x+3}{x-2}=\frac{5+3x}{3x-2}$

9. $3(a-2)+2a=7(a+4)-2(a+17)$

10. $7+\frac{6}{a-3}=\frac{5a-9}{a-3}$

Solve for x; the coefficients represent real numbers with restrictions as indicated:

11. $ax=3, a \neq 0$

12. $ax=0, a \neq 0$

13. $ax=b, a \neq 0$

14. $ax-3=0, a=0$

15. $ax=b, b \neq 0, a=0$

16. $ax-b=0, a=0, b=0$

Solve for x; the coefficients represent real numbers:

17. $ax+bx+cx=6$

18. $3mx+2mx=p$

19. $a^2x-a=b^2x-b$

20. $c^2x+2cdx+d^2x=c+d$

***1·5* Summary; the solution of the general equation of the first degree in one variable.** All examples of equations of the first degree in one variable can be expressed in the form:

$$ax+b=0, \qquad a, b \in R.$$

To study the real solutions (if any) of equations of this type we must consider three cases:

Case 1. $a \neq 0$

For $a \neq 0$,

$$ax+b=0$$
$$\leftrightarrow \quad ax=-b$$
$$\leftrightarrow \quad x=-\frac{b}{a}.$$

$\therefore$ the solution set is $\left\{-\frac{b}{a}\right\}$.

Thus, it is seen that all such equations have one and only one solution. The graph of the solution set is a point on a real number line corresponding to the real number represented by $-\frac{b}{a}$ (*Fig. 1-2*).

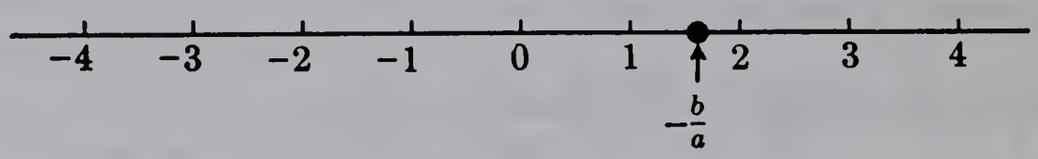


Fig. 1-2

Case 2. $a = 0, b \neq 0$

For $a = 0, b \neq 0$,

$$0 \cdot x + b = 0$$
$$\leftrightarrow \quad 0 \cdot x = -b .$$

Since $b \neq 0$ and $0 \cdot x = 0$ for all $x \in R$, the equation has *no root*; the solution set is $\emptyset$.

Case 3. $a = 0, b = 0$

For $a = 0, b = 0$,

$$0 \cdot x + 0 = 0$$
$$\leftrightarrow \quad 0 \cdot x = 0 .$$

Since $0 \cdot x = 0$ is true for all $x \in R$, the solution set is R, an infinite set.

***1·6* The Factor Theorem.** The indicated product $(x - 1)(x - 3)$, when expanded, is $x^2 - 4x + 3$. Thus, for all real replacements of x, the two expressions are equivalent, that is:

$$x^2 - 4x + 3 = (x - 1)(x - 3) \text{ for all } x \in R .$$

This means that $x^2 - 4x + 3 = (x - 1)(x - 3)$ is an equation with the solution set $\{x \mid x \in R\}$. Equations whose solution sets are the set of numbers under consideration are called *identical equations* or identities in that set.

Since $x^2 - 4x + 3 = (x - 1)(x - 3)$ is an identity in the set of real numbers:

if $x = 1$,	if $x = 3$,
$x^2 - 4x + 3 = (1 - 1)(1 - 3)$	$x^2 - 4x + 3 = (3 - 1)(3 - 3)$
$= 0(-2)$	$= 2(0)$
$= 0$	$= 0.$

No real replacements for x other than 1 or 3 make the expression equal to zero. Only the replacements 1 or 3 make either factor $(x - 1)$ or $(x - 3)$ equal to zero.

This illustrates that:

(i) $x = 1$ makes $x^2 - 4x + 3$ equal to zero, and that $(x - 1)$ is a factor of $x^2 - 4x + 3$;

(ii) $x = 3$ makes $x^2 - 4x + 3$ equal to zero, and that $(x - 3)$ is a factor of $x^2 - 4x + 3$.

This suggests the following theorem:

In general, if a polynomial in x is zero when $x = a$, then $x - a$ is a factor of the polynomial.

This theorem, called the *Factor Theorem*, is not proved here, but it is accepted and used as a means of factoring some polynomial expressions.

Example 1. Find the factors of $x^3 + 2x^2 - 5x - 6$.

By the Factor Theorem, if a replacement a can be found for x which makes the expression equal to zero, then $x - a$ is a factor. Also since an indicated product like $(x - 1)(x - 2)(x - 3)$ produces a constant term in its expansion which is the product of $(-1)(-2)(-3) = -6$, it is seen that the only possible factors $(x - a)$ having a an integer are those for which a is an integral factor of the constant term. In this case the set of integral factors of -6 is

$$\{1, -1, 2, -2, 3, -3, 6, -6\}.$$

Solution. If $x = 1$, the expression $= 1 + 2 - 5 - 6 \neq 0$.

$\therefore\ x - 1$ is not a factor of the expression.

If $x = -1$, the expression $= (-1)^3 + 2(-1)^2 - 5(-1) - 6$

$= -1 + 2 + 5 - 6$

$= 0$.

$\therefore\ (x + 1)$ is a factor of the expression.

Similarly, other factors may be found, or, having found one factor, the second factor may be found by division.

$$\begin{array}{r|l} & x^2 + x - 6 \\ \cline{2-2} x + 1 & x^3 + 2x^2 - 5x - 6 \\ & \underline{x^3 + x^2} \\ & \quad x^2 - 5x \\ & \quad \underline{x^2 + x} \\ & \quad\quad -6x - 6 \\ & \quad\quad \underline{-6x - 6} \end{array}$$

$\therefore\ x^3 + 2x^2 - 5x - 6$

$= (x + 1)(x^2 + x - 6)$

$= (x + 1)(x - 2)(x + 3)$.

Example 2. Show by the Factor Theorem that $x - 1$ is a factor of $2x^3 - 4x^2 + 6x - 4$.

Solution. If $x = 1$, the expression $= 2 - 4 + 6 - 4 = 0$.

$\therefore\ x - 1$ is a factor of the expression.

Write solutions for the following problems and compare them with those on page 403.

1. Factor $2x^3 - 11x^2 + 5x + 4$.
2. Factor: (i) $x^3 - y^3$ (the difference of two cubes);
   (ii) $x^3 + y^3$ (the sum of two cubes).
3. Show by the Factor Theorem that $(y + 3)$ is a factor of $y^3 + 2y^2 - 7y - 12$.

Exercise 1-2

(B)

Factor by means of the Factor Theorem:

1. $2x^3 - 5x^2 + x + 2$
2. $a^3 + a - 2$
3. $x^3 - 1$
4. $x^3 + 1$
5. $x^3 - 3x + 2$
6. $p^3 - 6p^2 - 9p + 14$
7. $4x^3 + 8x^2 - 11x + 3$
8. $6x^3 + 13x^2 + x - 2$
9. Use the Factor Theorem to show that $a - b$, $a - 2b$, and $a + 3b$ are factors of $a^3 - 7ab^2 + 6b^3$.
10. If $a^4 + a^3 + xa^2 + ya - 3$ is divisible by $a - 1$ and $a - 3$, find x and y and the remaining factor.

1·7 Solutions of equations of higher degree than the first in one variable.

Example 1. Solve $x^2 - 3x + 2 = 0$, $x \in R$ and draw the graph of the solution set.

The solution of this equation depends on the following property:

$$\text{If } a, b \in R,\ a \cdot b = 0 \ \leftrightarrow\ a = 0 \text{ or } b = 0.$$

Solution.

$$\begin{aligned} & x^2 - 3x + 2 = 0 \\ \leftrightarrow\ & (x - 2)(x - 1) = 0 \\ \leftrightarrow\ & x - 2 = 0 \quad \text{or} \quad x - 1 = 0 \\ \leftrightarrow\ & x = 2 \quad \text{or} \quad x = 1. \end{aligned}$$

$\therefore$ the roots are 1 and 2,
and the solution set is $\{1, 2\}$.

Verification.

If $x = 2$, then
L.S. $= 4 - 6 + 2$
$= 0.$
R.S. $= 0.$

If $x = 1$, then
L.S. $= 1 - 3 + 2$
$= 0.$
R.S. $= 0.$

The graph of the solution set is shown in *Fig. 1-3*.

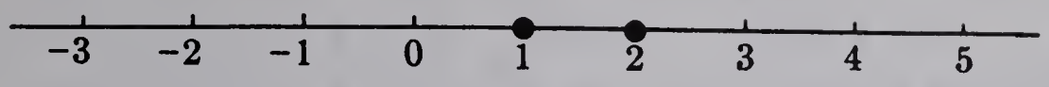


Fig. 1-3

Example 2. Solve $x^3 - 6x^2 + 11x - 6 = 0$ and draw the graph of the solution set.

Solution. If $x = 1$, the expression $= 1 - 6 + 11 - 6 = 0$,
$\therefore$ $x - 1$ is a factor by the Factor Theorem.

By division the second factor is $x^2 - 5x + 6$.

$$\therefore\ x^3 - 6x^2 + 11x - 6 = (x - 1)(x^2 - 5x + 6)$$
$$= (x - 1)(x - 2)(x - 3).$$

Thus,
$$x^3 - 6x^2 + 11x - 6 = 0$$
$$\leftrightarrow\quad (x - 1)(x - 2)(x - 3) = 0$$
$$\leftrightarrow\quad x - 1 = 0 \text{ or } x - 2 = 0 \text{ or } x - 3 = 0$$
$$\leftrightarrow\quad x = 1 \text{ or } x = 2 \text{ or } x = 3.$$

$\therefore$ the roots of the equation are 1, 2, and 3,
and the solution set is $\{1, 2, 3\}$.

This may be checked, or verified, by substitution.
The graph of the solution set is shown in *Fig. 1-4*.

Fig. 1-4

Exercise 1-3

(B)

Solve:

1. $4(x + 2) = 3 - 3(2x - 5)$

2. $7(4x - 5) = 8(3x - 5) + 9$

3. $(x+2)(x+3) = (x+3)(x+5)$

4. $(3x + 2)(x + 3) = 3x(x + 5)$

5. $\dfrac{7x + 2}{5} = \dfrac{4x - 1}{2}$

6. $\dfrac{3a - 1}{3} + \dfrac{5}{12} = \dfrac{a}{4} + \dfrac{2a + 1}{5}$

7. $x^2 - x - 2 = 0$

8. $x^2 - 5x = -6$

9. $(2x - 5)(x + 1) = x^2 - 5$

10. $5(x^2 + 1)^2 = 4[(x^2 + 1)^2 + 1]$

11. $x^3 - 7x + 6 = 0$ **12.** $x^3 + 2x^2 - 3x = 0$

13. $2x^3 - 3x^2 - 11x + 6 = 0$ **14.** $6x^3 - 17x^2 = 3 - 14x$

15. $\dfrac{a+2}{a-1} - \dfrac{7}{3} = \dfrac{4-a}{2a}$ **16.** $\dfrac{3a-2}{2a-3} = \dfrac{1}{2} + \dfrac{a+17}{a+10}$

17. $x^2 + (2x - 3)(x + 5) + 10 = (3x - 1)(x + 3) - 3$

18. $x(x + 1)^2 + 13 = (2x + 1)^2 + 2(x + 3)$

19. $\dfrac{6x - 12}{-x + 5} + \dfrac{2x - 11}{2x - 5} = 7$

20. $(a - x)(b + x) - (a + x)(b - x) = 2ab$

***1·8* The inequality relation.** The set of real numbers is ordered in terms of the relation "*is greater than*" or "*is less than*". Any positive real number is considered to be greater than zero and any negative real number to be less than zero. Any real number, a, is defined to be greater than another real number, b, if and only if

$$a - b > 0 \quad \text{or} \quad a = b + c, \quad c \in {}^+R.$$

Thus $3.5 > 2.2$ because $3.5 - 2.2 > 0$, or $3.5 = 2.2 + 1.3$.

This relation is illustrated on a real number line (*Fig. 1-5*), in which the coordinate of a point to the right of a given point is greater than the coordinate of the given point.

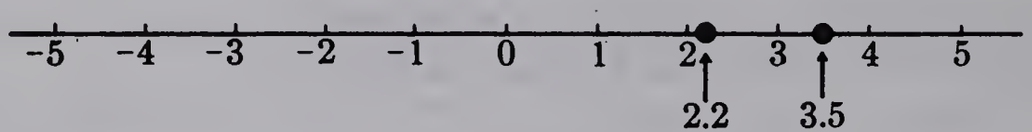


Fig. 1-5

On a number line the point with coordinate 3.5 is to the right of the point with coordinate 2.2.

Although only integral points are marked on the number line in *Fig. 1-5*, it is assumed that there is a unique real coordinate for every point on the line and that to each real number there corresponds a unique point on the line.

On the basis of this assumption and the definition of order it is assumed that:

if $a, b \in R$, then precisely one of the following alternatives holds:

$$a < b \quad \text{or} \quad a = b \quad \text{or} \quad a > b.$$

This assumption is called the *trichotomy assumption.*

It follows that if $a, b, c \in R$, the relation "is greater than"

(i) is not reflexive, that is, a is not greater than a;

(ii) is not symmetric, that is, a cannot both be greater than and less than b;

(iii) is transitive, that is, if $a > b$ and $b > c$, then $a > c$.

The closure property of the set of real numbers under addition, subtraction, multiplication, and division, and the definition of inequality justify the following statements.

If $a, b, c \in R$, then

(i) $a > b$ or $b < a$

(ii) $a + c > b + c$

(iii) $a - c > b - c$

(iv) $ac > bc$, if $c > 0$
$ac < bc$, if $c < 0$

(v) $\frac{a}{c} > \frac{b}{c}$, if $c > 0$
$\frac{a}{c} < \frac{b}{c}$, if $c < 0$

are equivalent inequations.

1·9 The solution of inequations in one variable; graph of the solution set.

Example. Find the solution set defined by

$$2x + 8 \leqq 8x - 4, \; x \in R$$

and draw the graph of the solution set.

Solution.

$$2x + 8 \leqq 8x - 4$$
$$\leftrightarrow \quad -6x \leqq -12$$
$$\leftrightarrow \quad x \geqq 2 .$$

$\therefore$ the solution set is $\{x \mid x \geqq 2, x \in R\}$.

The graph of the solution set is shown in *Fig. 1-6*.

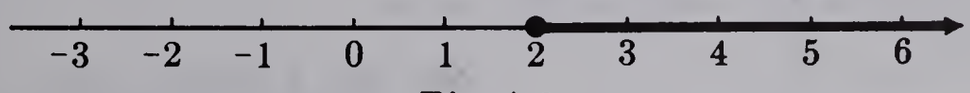


Fig. 1-6

Exercise 1-4

(B)

Find the solution set defined by each of the following and draw its graph:

1. $3x - 2 \geqq 7(x - 4) + 2$

2. $5(x - 3) \leqq 3(x - 4)$

3. $x(2x + 1) - 2(x + 2)(x - 4) \leqq 2$

Find the solution set defined by each of the following and draw its graph:

4. $(3x + 1)(2x - 7) \geqq 6(x - 3)^2 + 7$

5. $(3x - 7)(2x - 6) \geqq (3x - 8)(2x - 5)$ 6. $\frac{1}{x} + \frac{1}{2x} \geqq \frac{1}{3x} + \frac{7}{3}, x \neq 0$

7. $2(x + 2)^2 - (x - 1)(x - 2) > (x - 3)(x + 1) - 4$

Find and draw the graph of each of the following:

8. $\{x \mid 2(x - 1)^2 - 3(x - 2)(x + 3) \geqq 32 - (x - 3)(x - 4), x \in R\}$
9. $\{x \mid 2(x + 3)^2 - (x - 5)(x + 2) < 8 + (x + 8)(x - 1), x \in R\}$
10. $\{x \mid x^2 + (2x - 3)(x + 5) + 10 \leqq (3x - 1)(x + 3) - 3, x \in R\}$

Exercise 1-5

(Review of operations with polynomials)

(B)

Simplify:

1. $2a(a + 3)$ 2. $2xy(3x - 5y)$ 3. $5x(2x - y)$
4. $(a - b)(2a + b)$ 5. $(p + q)(p + q)$ 6. $(x + y)^2$
7. $(2x + 3y)(5x - 2y)$ 8. $(x + y)(x^2 - xy + y^2)$
9. $(x + y)(x^2 + 2xy + y^2)$ 10. $(p + q)^3$
11. $(x - y)(x^2 + xy + y^2)$ 12. $(x - y)^3$
13. $(2a - 3q)^2$ 14. $(x + y)(x - y)$
15. $(2a - 1)^3$ 16. $(2a - 3)(4a^2 + 6a + 9)$
17. $[(a + b) + c][(a + b) - c]$ 18. $(a + b + c)^2$
19. $(2p + 7q + r)^2$ 20. $(x + y + z)^2$
21. $(a - 2b - c)^2$ 22. $(2a - 3b + c)^2$
23. $(x + 3y)^2 - x(x + 6y)$
24. $(x - 2y)(x + 5y) - 3(y - x)(y + 2x)$
25. $(a + 1)(a - 1)$ 26. $(3b + 2)(3b - 2)$
27. $(4x - 5y)(4x + 5y)$ 28. $(7x - 2y)(7x + 2y)$
29. $[a + (b - 2)][a - (b - 2)]$ 30. $[(m - 3) + n][(m - 3) - n]$
31. $(x + 2)^2 - (x + 3)(x - 3)$ 32. $\frac{1}{x} + \frac{1}{y}$
33. $\frac{a}{b} + \frac{c}{d}$ 34. $\frac{2}{x} + \frac{x + y}{x^2 - y^2}$

Simplify:

35. $\dfrac{3p+q}{x-y} - \dfrac{3p+q}{y-x}$

36. $\dfrac{2a-3b}{a-b} - \dfrac{2a+3b}{a+b}$

37. $\dfrac{a-b}{a+b} \times \dfrac{5(a+b)}{2(a-b)}$

38. $\dfrac{(a+b+c)^2}{(x+y)} \times \dfrac{x^2+2xy+y^2}{(x-y)(a+b+c)}$

Exercise 1-6

(Review of factoring of polynomials)

(B)

Factor as expressions with a common factor:

1. ax^2+bx

2. $a(x-y)+b(x-y)$

3. $3(p-2q)-p(p-2q)$

4. $am+an+bm+bn$

5. $3px-15py-2qx+10qy$

6. $ax+2bx-3ay-6by$

Factor as trinomial squares:

7. $a^2+2ab+b^2$

8. $4x^2+4x+1$

9. $9x^2+12xy+4y^2$

10. $(x-y)^2+2(x-y)+1$

11. x^4+4x^2+4

12. a^3-2a^2+a

Factor as the difference of squares:

13. a^2-b^2

14. $4x^2-9y^2$

15. $a^2+2ab+b^2-c^2$

16. $a^2-b^2-2bc-c^2$

17. $9a^2-b^2-4c^2-4bc$

18. $4x^2-p^2+6pq-9q^2$

Factor the following trinomials:

19. x^2+3x+2

20. $6x^2+5x+1$

21. x^2-2x-8

22. $12a^2-23ab+10b^2$

23. $(x-y)^2-(x-y)-2$

24. $a^2+2ab+b^2+2a+2b-8$

Factor the following incomplete squares:

25. x^4+x^2+1

26. a^4+2a^2+9

27. a^4-19a^2+9

28. $4x^4+1$

29. $9a^4+8a^2b^2+16b^4$

30. $x^4y^4-15x^2y^2+9$

Factor by means of the Factor Theorem:

31. $x^3 - x^2 - x - 2$

32. $3x^3 - 22x^2 + 43x - 12$

33. $10a^3 + 3a^2b - 6ab^2 + b^3$

34. $x^3 - 6x^2 - 9x + 14$

35. $2x^3 + 5x^2y + 4xy^2 + y^3$

36. $2x^3 + 15x^2 - 16x - 1$

37. If $x - 1$ is a factor of $x^3 + 3x^2 + x - k$, find k.

Factor as the sum or difference of cubes:

38. $8a^3 - b^3$

39. $27a^3 + 125b^3$

40. $4a^3 + 108$

41. $x^6 + y^6$

42. $x^3 - (x - y)^3$

43. $a^3 - 8(a - 2)^3$

Factor:

44. $4a^2 - 20ab + 25b^2$

45. $(a - 2y)^2 - 4(a - 2y)$

46. $x^4 + y^4 - 18x^2y^2$

47. $a^3 - 11a^2 + 7a + 3$

48. $x^4y - x^2y^3 - x^3y^2 + xy^4$

49. $108x^3 - 500$

50. $x^6 - y^6$

51. $x^2 + 2 + \frac{1}{x^2} - y^2 + 2 - \frac{1}{y^2}$

52. $4x^4 - 29x^2 + 1$

53. $a^3 - 6a^2 + 4a + 5$

54. $a^2x^3 - x^3 + a^2 - 1$

55. $(p^2 - pb)^2 - 8b^2(p^2 - pb) + 12b^4$

56. $15p^2 + 11pq - 60q^2$

57. $x^2 - x^3 + 8x^4 - 8$

58. $a^4 - 7a^2b^2 - 18b^4$

59. $15xy + 12yz - 25x^2 + 16z^2$

60. $(x^2 - 5x)^2 + 10(x^2 - 5x) + 24$

61. $a^2 + b^2 + c^2 + 2ab + 2bc + 2ac$

Simplify:

62. $\frac{1 - x^2}{1 + x} + \frac{x - 2}{x^2 - 3x + 2}$

63. $\frac{4}{x^2 - x} + \frac{3}{x - 1} - \frac{1}{x}$

64. $\frac{y + 1}{y - 5} + \frac{y}{10 - 2y}$

65. $\frac{a}{a^2 - 9} + \frac{2a - 5}{a^2 - 4a + 3} - \frac{3a}{a^2 + 2a - 3}$

66. $\dfrac{\frac{x^2}{y^2} - 2 + \frac{y^2}{x^2}}{\frac{x^2}{y^2} - \frac{y^2}{x^2}} \times \frac{x^2 + y^2}{x - y}$

67. $\frac{x}{x - y} - \frac{2y - x}{y + x} + \frac{x^2 + y^2}{y^2 - x^2}$

68. $(1 - \frac{1}{a + 1})(1 + \frac{1}{a - 1})$

Chapter II

EXPONENTS

SQUARE ROOTS

RADICALS

***2·1* Definition of a^n, $n \in {}^+I$, $a \in R$.** In earlier studies, the definition

$$2^5 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 ,$$

or in general terms

$$a^n = a \cdot a \cdot a \ldots a, \ (na\text{'s}) ,$$

where $a \in R$ and n represents a positive integer greater than 1 provided a very useful shorthand for expressing the products of equal factors where the factors were real numbers.

A number such as thirty-two when represented by the numeral

$$2^5$$

is said to be expressed in *exponential form;*

2 is called the *base;*
5 is called the *exponent* of the base;
and the number thirty-two is said to be the *fifth power* of the base two.

The following definition

$$\begin{cases} a^1 = a,\ a \in R, \\ a^{n+1} = a \cdot a^n,\ n \in {}^+I, \end{cases}$$

makes precise the meaning of

$$a^n,\ a \in R,\ n \in {}^+I,$$

and implies

(i) $a^1 = a$,

(ii) $a^{1+1} = a^2 = a \cdot a$,
$a^{2+1} = a^3 = a \cdot a^2 = a \cdot a \cdot a$,
$a^{3+1} = a^4 = a \cdot a^3 = a \cdot a \cdot a \cdot a$,
$\vdots$
$a^{n-1+1} = a^n = a \cdot a^{n-1} = a \cdot a \cdot a \ldots a$, ($na$'s).

In the sections which immediately follow, the laws for operating with integral exponents are derived from particular examples. Unless otherwise indicated, the base is considered to be a real number.

2·2 The law of a product involving integral exponents.

The product $a^2 \cdot a^3 = (a \cdot a)(a \cdot a \cdot a)$ (Definition)
$= a^5$ (Definition)
$= a^{2+3}$.

This and similar examples suggest the general law:

$$\boldsymbol{a^n \times a^m = a^{n+m},\ n,\ m \in {}^+I}.$$

This law may be developed in a general manner from the definition of Section 2.1 as follows:

$a^n = a \cdot a \cdot a \ldots a$, ($na$'s) (Definition)

$a^m = a \cdot a \cdot a \ldots a$, ($ma$'s) (Definition)

$\therefore\ a^n \times a^m = [a \cdot a \cdot a \ldots a, (na\text{'s})]\,[a \cdot a \cdot a \ldots a, (ma\text{'s})]$

$= a \cdot a \cdot a \ldots a, [(n + m)\ a\text{'s}]$

$= a^{n+m}$. (Definition)

$\therefore\ a^n \times a^m = a^{n+m},\ n, m \in {}^+I$.

This law, called the *law of a product*, may be extended to any number of factors. Thus:

$$a^n \cdot a^m \cdot a^p \ldots = a^{n+m+p+\cdots} .$$

If we wish to extend this law to include zero as an exponent, then it must be true that:

$$(-3)^3(-3)^0 = (-3)^{3+0} \qquad \text{or in general} \qquad a^n \times a^0 = a^{n+0}$$
$$= (-3)^3 \qquad\qquad\qquad\qquad = a^n$$

and a^0 must be 1, the identity element for multiplication. Therefore, we extend the law to a zero exponent by defining

$$a^0 = 1 .$$

If we replace a by zero in the statement $a^n \times a^0 = a^n$, then

$$0^n \times 0^0 = 0^n .$$

Since zero to any positive integral exponent is zero, this statement is equivalent to

$$0 \times 0^0 = 0 .$$

Since the product of any real number and zero is zero, this statement is true if 0^0 is any real number. Consequently, we will not attempt to define 0^0 but will restrict the definition of a^0 so that $a \neq 0$. Thus,

$$a^0 = 1, \; a \neq 0 .$$

If we wish to extend this law to include negative integral exponents, then it must be true that:

$$2^2 \times 2^{-2} = 2^{2+(-2)} \qquad \text{or in general} \qquad a^n \times a^{-n} = a^{n+(-n)}$$
$$= 2^0 \qquad\qquad\qquad\qquad = a^0, \quad a \neq 0$$
$$= 1 \qquad\qquad\qquad\qquad = 1 .$$

Since the product of *reciprocals* is 1, the law of a product involving exponents may be extended to include negative integral exponents if and only if we define a^n and a^{-n} to be reciprocals; that is:

$$a^{-n} = \frac{1}{a^n} \qquad \textit{for all } n \in I, \; a \neq 0 .$$

Simplify each of the following and compare your solutions with those on page 404.

1. $2^2 \cdot 2^4$
2. $(-3)^4 \cdot (-3)^0$
3. $5^3 \times 5^{-2}$
4. $(-16)^3(-16)^{-3}$
5. $x^3 \cdot x^{-4}$
6. $x^{-2} \cdot x^{-3}$.

2·3 The law of a quotient involving integral exponents. Since division is the inverse operation of multiplication:

(i) the quotient

$(-3)^5 \div (-3)^2 = x$ if and only if $(-3)^5 = (-3)^2 \cdot x,\ x \in R$.

Since $(-3)^2(-3)^3 = (-3)^5$ (Law of a product)

$\therefore\ (-3)^5 \div (-3)^2 = (-3)^3$

$= (-3)^{5-2}$;

(ii) the quotient

$2^3 \div 2^{-4} = x$ if and only if $2^3 = 2^{-4} \cdot x,\ x \in R$.

Since $2^{-4} \times 2^7 = 2^3$ (Law of a product)

$\therefore\ 2^3 \div 2^{-4} = 2^7$

$= 2^{3-(-4)}$;

(iii) the quotient

$5^{-3} \div 5^{-2} = x$ if and only if $5^{-3} = 5^{-2} \cdot x,\ x \in R$.

Since $5^{-2} \times 5^{-1} = 5^{-3}$ (Law of a product)

$\therefore\ 5^{-3} \div 5^{-2} = 5^{-1}$

$= 5^{-3-(-2)}$.

(iv) In general, if $n, m \in I$, the quotient

$a^n \div a^m = x$ if and only if $a^n = a^m \cdot x,\ x \in R,\ a \neq 0$.

Since $a^m \cdot a^{n-m} = a^{m+n-m}$ (Law of a product)

$= a^n$

$\therefore\ a^n \div a^m = a^{n-m},\quad a \neq 0$.

Thus, the law of a quotient involving integral exponents is

$$a^n \div a^m = a^{n-m},\ a \neq 0\ .$$

The base a cannot be zero because division by zero is not defined.

Simplify each of the following and compare your solutions with those on page 404. *Assume* $x, y, z \neq 0$.

1. $3^5 \div 3^2$ **2.** $(-2)^3 \div (-2)^0$ **3.** $(-5)^2 \div (-5)^{-3}$

4. $x^0 \div x^{-2}$ **5.** $y^{-2} \div y^{-4}$ **6.** $z^0 \div z^0$

2·4 Summary.

If $n, m \in I,\ a \neq 0$, then

(i) *Law of a product:* $a^n \cdot a^m = a^{n+m}$;

(ii) *Zero exponent:* $a^0 = 1$;

(iii) *Negative exponent:* $a^{-n} = \dfrac{1}{a^n}$;

(iv) *Law of division:* $a^n \div a^m = a^{n-m}$.

Obtain simpler equivalent expressions for each of the following and compare your solutions with those on page 404.

1. $(a^2b^3c^4)(a^0b^{-2}c^2)$ **2.** $3m^{-2}p^0q^2 \times 4m^4p^{-3}q^{-4}$

3. $x^4y^2z^0 \div x^{-2}y^4z^3$ **4.** $12r^4s^{-5} \div 3r^{-2}s^5$ . .

Exercise 2-1

(The variables are real numbers for which the expressions are defined.)

(A)

Simplify each of the following:

1. $a^3 \times a^5$ **2.** $a^8 \div a^2$ **3.** $a^2 \div a^6$ **4.** $\dfrac{a^2b^5c^2}{a^6bc^2}$

5. $(3m^2b)(2mb^4)$ **6.** $3x^2 \times 4x^3$ **7.** $\dfrac{b^{5n}}{b^{2n}}$ **8.** x^0y^0

9. $b^n \cdot b^3$ **10.** $x^2 \cdot y^3 \cdot x \cdot y$ **11.** $(-3)^5$ **12.** $(-2)^4(-2)^{-3}$

Express with positive exponents:

13. a^{-3} **14.** a^2b^{-4} **15.** $\dfrac{a^2b^{-1}}{c}$ **16.** $\dfrac{m^{-1}n^{-2}}{c^{-3}}$

17. $\dfrac{7}{a^{-4}}$ **18.** $x^{-2}y^{-3}$ **19.** $5b^{-10}$ **20.** $\dfrac{m^{-3}}{p^{-2}}$

21. $\dfrac{2x^{-4}}{3y^{-2}}$ **22.** $a^{-1} + b^{-1}$ **23.** $\dfrac{1}{a^{-1}} - \dfrac{1}{b^{-2}}$ **24.** $\dfrac{1}{a^{-1} + b^{-1}}$

Simplify:

25. 7^0 **26.** $(-5)^0$ **27.** $\dfrac{1}{(-3)^0}$ **28.** $\dfrac{2^0}{3^0}$

29. 5^{-3} **30.** $\dfrac{2^{-1}}{3^{-1}}$ **31.** $6^0 \cdot 2$ **32.** $5 \div 8^0$

33. $5 \cdot z^0$ **34.** $\dfrac{4}{3^{-1}}$ **35.** $6 \cdot 8^0$ **36.** $\dfrac{4}{2x^0}$

37. $7^0 \cdot 3^0$ **38.** $(-4)^{-2}$ **39.** $\dfrac{1 + a^0}{2}$ **40.** $32 \cdot 4^{-2}$

41. $32 \cdot 2^{-4}$ **42.** $3 \cdot 10^{-4}$ **43.** $x^{-3} \div x^{-3}$ **44.** $\dfrac{1}{2^0 + b^0}$

45. $10^{-3} \cdot 10^4$ **46.** $10{,}000 \times 10^{-4}$ **47.** $(25)^{-2}$ **48.** $(-4)^{-2}$

49. $a^3b^4 \times a^{-2}b^{-5}$ **50.** $\dfrac{x^3y^4}{x^2y^5}$ **51.** $\dfrac{12m^{-2}p^3}{3m^3p^{-2}}$

52. $\dfrac{x^4y^0z^6}{x^2y^{-2}z^4}$ **53.** $\dfrac{36a^3b^4c^5}{72a^{-3}b^{-4}c^{-5}}$ **54.** $\dfrac{125m^{-3}p^0q^5}{75p^0q^5}$

55. $13a^2b^3c^{-3} \times 4a^{-2}b^{-4}c^3$ **56.** $\dfrac{(x-1)^2(y-3)^0}{(x-1)^3(y-3)^{-2}}$

57. $\dfrac{(a+b+c)^3(a-b+c)^4}{(a+b+c)^0(a-b+c)^4}$

2·5 The power law for integral exponents.

$$\begin{aligned} \text{The power } (a^2)^4 &= a^2 \cdot a^2 \cdot a^2 \cdot a^2 && \text{(Definition)} \\ &= a^{2+2+2+2} && \text{(Law of a product)} \\ &= a^8 \\ &= a^{2\times 4} \ . \end{aligned}$$

This and similar examples suggest the general law:

$$(a^n)^m = a^{nm}, \quad n, m \in I\,,\ a \neq 0,$$

called the *power law.*

2·6 The law of the power of a product for integral exponents.

$$\begin{aligned} \text{The power } (a \cdot b)^3 &= (a \cdot b)(a \cdot b)(a \cdot b) && \text{(Definition)} \\ &= (a \cdot a \cdot a)(b \cdot b \cdot b) && (CM)\ (AM) \\ &= a^3b^3 \ . && \text{(Definition)} \end{aligned}$$

This and similar examples suggest the general law:

$$(a \cdot b)^n = a^n \cdot b^n, n \in I\,, ab \neq 0,$$

called the *power of a product.*

2·7 The law of the power of a quotient for integral exponents.

$$\begin{aligned} \text{The power } \left(\frac{a}{b}\right)^4 &= \left(\frac{a}{b}\right)\left(\frac{a}{b}\right)\left(\frac{a}{b}\right)\left(\frac{a}{b}\right),\ b \neq 0 && \text{(Definition)} \\ &= \frac{a^4}{b^4} \ . && \text{(Definition)} \end{aligned}$$

This and similar examples suggest the general law:

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}\,,\ a \neq 0,\ b \neq 0,\ n \in I\,,$$

called the *power of a quotient.*

2·8 Summary of the laws of exponents for integral exponents.

If $n, m \in I$ and $a \neq 0, b \neq 0$, then

(i) *Law of a product:* $a^n \times a^m = a^{n+m}$;

(ii) *Zero exponent:* $a^0 = 1$;

(iii) *Negative exponent:* $a^{-n} = \frac{1}{a^n}$;

(iv) *Law of a quotient:* $a^n \div a^m = a^{n-m}$;

(v) *Power law:* $(a^n)^m = a^{nm}$;

(vi) *Power of a product:* $(a \cdot b)^n = a^n b^n$;

(vii) *Power of a quotient:* $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$.

Obtain simpler equivalent expressions for each of the following and compare your solutions with those on page 405.

1. $3(a^2)^{10}$ **2.** $(3m^2n^3p^4)^3$ **3.** $(4x^{-2}z^0y^2)^{-3}$

4. $\left(\frac{r^2}{t^3}\right)^3$ **5.** $\left(\frac{6s^3t^{-4}}{3mn^2}\right)^{-1}$ **6.** $\left(\frac{a^0}{b^0}\right)^3$.

Exercise 2-2

(The variables are real numbers for which the expressions are defined.)

(A)

State simpler equivalent expressions for each of the following:

1. $(a^2)^3$ **2.** $(a^{-2})^3$ **3.** $(a^{-2})^{-3}$

4. $(x^2y^3)^0$ **5.** $(x^2y^3)^2$ **6.** $(x^2y^3)^{-1}$

7. $(a^0b^0c^0)^0$ **8.** $(a^2)^2(b^3)^3$ **9.** $(-2)^3$

10. $(-12x^3)^2$ **11.** $(5ab^2c^3)^2$ **12.** $(-3xy^2)^4$

13. $\left(\frac{x}{y}\right)^{10}$ **14.** $\frac{x^{23}}{x^{12}}$ **15.** $\left(\frac{2a}{b}\right)^3$

16. $\left(\frac{2m}{5n}\right)^4$ **17.** $\frac{a^3}{b^3}$ **18.** $\left(\frac{p}{q}\right)^2\left(\frac{q}{p}\right)^3$

19. $(-x)^{3021}$ **20.** $(-y)^{3022}$ **21.** $-(-3x)^3$

22. $(3^2)^2$ **23.** $(2x^2)^2$ **24.** $(2x)^3(2x)^3$

25. $(-6a^2b^2)^2$ **26.** $-\frac{1}{2}(-\frac{1}{2})^3$ **27.** $(-2a^4)^3$

28. $\frac{b^{25}}{b^{35}}$ **29.** $\frac{3a^3}{9a^8}$ **30.** $(-3ab^2)^{-3}$

31. $2^5 \cdot 2^{-8}$ **32.** $3^5 \cdot 3^7 \cdot 3^{-10}$ **33.** $\frac{(3a-b)^4}{(3a-b)^2}$

34. $\dfrac{(2a)^{12}}{(2a)^{10}}$ **35.** $(x^{2y})^{y}$ **36.** $(a^{3p})^{2p}$

37. $\dfrac{3^6 \cdot 2^4}{3^3 \cdot 2^3}$ **38.** $\dfrac{-(mn)^4}{(-mn)^4}$ **39.** $\left(\dfrac{a^3}{2}\right)^3$

40. $\left(\dfrac{2p}{m}\right)^2$ **41.** $8a^3 \div 4a^2$ **42.** $(\frac{1}{4}pq^2)^2$

43. $(2a)^2 - 3a^2$ **44.** $ab \cdot 3a^2b \cdot 2a^3b$ **45.** $6a^5 + 2a^3$

46. $2a^2$ is a factor of $16a^3b$. What is the other factor?

47. $5m^2n^3$ is a factor of $35mn^4$. What is the other factor?

48. $6x^0y^5$ is a factor of $30x^4y^2$. What is the other factor?

49. $3x^{-2}y^{-3}$ is a factor of $9x^{-4}y^0$. What is the other factor?

Obtain as simple a form as possible for each of the following:

50. $ab + 2ab$ **51.** $3abc \cdot 2b^2 \cdot 3a^0c^2d$

52. $a^4 + a^3$ **53.** $6a^5b^5 \div 3a^2b^3$

54. $2r^3 \cdot 2p^5$ **55.** $-20pq^4 \div -5p^2q^3$

56. $ab + a$ **57.** $4a^2 \div 4a$

58. $4a^2 + 4a$ **59.** $xy \cdot x$

60. $xy \div x$ **61.** $(5a)(6a)$

62. $\dfrac{45a^4b^3}{-5a^3b^4}$ **63.** $2b^2 + 2b^2 + 4b$

64. $2b^2 \cdot 2b^2 \cdot 4b$ **65.** $\dfrac{6mnp^2}{9m^4np}$

66. $(4c^2d^3)^2$ **67.** $\left(\dfrac{a^2}{3}\right)^3$

68. $4a^2 \cdot \dfrac{9}{4a}$ **69.** $(4x^2yz^3)(-3x^{-2}y^{-1}z^{-3})$

2·9 The principal square root of a non-negative real number. The *square* of any real number is a non-negative real number. For example:

(i) $(-5)^2 = 25$, (ii) $0^2 = 0$, (iii) $(+5)^2 = 25$.

Any positive real number has two *square roots*. For example:

(i) $25 = (-5)^2$ and (ii) $25 = (+5)^2$,

and hence both $+5$ and -5 are square roots of 25. Zero has only one square root, zero.

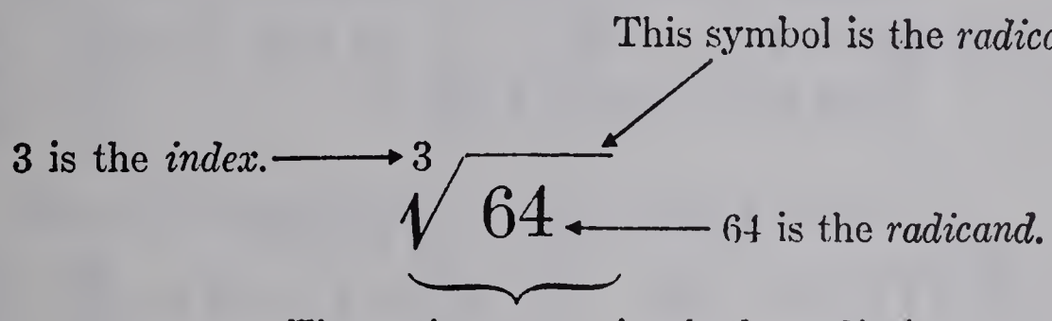
In general,

if $a, b \in R$, $a \geqq 0$,

b is called a square root of a if and only if $b^2 = a$.

The example illustrates that b may be positive, negative, or zero, but a must be positive or zero.

The *positive square root* of any positive real number a is called the *principal square root* of a and is designated $\sqrt{a}$. The *radical sign* (or square root sign), $\sqrt{\ }$, always implies the principal square root. The indicated square root of a positive real number, $\sqrt{a}$, $a \in {}^{+}R$, is called *a radical* or *a surd of the second order*. The number a is called the *radicand*. The cube root of 5, $\sqrt[3]{5}$, and the fourth root of 5, $\sqrt[4]{5}$, are examples of radicals or surds of the third and fourth order, respectively. The numeral written above the radical sign to indicate the order of the root is called the *index* of the radical. The negative square root of a positive real number a is indicated by $-\sqrt{a}$. Our discussions will be concerned only with radicals of the second order.



In the following sections, methods of operating with radicals involving square roots and the simplification of expressions involving such radicals are discussed.

2·10 The product of two square roots. It should be recalled that the (principal) square root of any non-negative real number is a real number; hence, in combining radicals of the second order, the laws of addition and multiplication for real numbers and the laws of exponents apply.

(i) By definition, $(\sqrt{a})^2 = a$ or $\sqrt{a} \times \sqrt{a} = \sqrt{a^2} = a$ if $a \geqq 0$.

(ii) $(\sqrt{2} \times \sqrt{3})^2 = (\sqrt{2})^2(\sqrt{3})^2$ (Power of a product)
$= 2 \times 3$ (Definition)
$= 6$.

Thus $\sqrt{2} \times \sqrt{3} = \sqrt{2 \times 3} = \sqrt{6}$. (Definition)

Similarly, if $a, b \geqq 0$, then

$(\sqrt{a} \times \sqrt{b})^2 = (\sqrt{a})^2(\sqrt{b})^2$ (Power of a product)
$= a \cdot b$. (Definition)
$\therefore \sqrt{a} \times \sqrt{b} = \sqrt{ab}$.

Example 1. Simplify the following:

(i) $\sqrt{3} \times \sqrt{7}$ (ii) $\sqrt{5} \times \sqrt{11}$ (iii) $\sqrt{12}$ (iv) $\sqrt{75}$.

Solution. (i) $\sqrt{3} \times \sqrt{7} = \sqrt{3 \times 7}$
$= \sqrt{21}$.

(ii) $\sqrt{5} \times \sqrt{11} = \sqrt{5 \times 11}$
$= \sqrt{55}$.

(iii) $\sqrt{12} = \sqrt{4 \times 3}$
$= \sqrt{4} \times \sqrt{3}$
$= 2\sqrt{3}$.

(iv) $\sqrt{75} = \sqrt{25 \times 3}$
$= \sqrt{25} \times \sqrt{3}$
$= 5\sqrt{3}$.

In parts (iii) and (iv), each of the radicands has a factor which is a perfect square. Since in general, $\sqrt{ab} = \sqrt{a} \times \sqrt{b}$, such a factor may be separated as the examples indicate. Numerals such as $2\sqrt{3}$, $5\sqrt{3}$ are called *mixed radicals* or *mixed surds*. Mixed radicals may be written as *entire radicals* or *entire surds* by reversing this process. For example:

$2\sqrt{3} = \sqrt{4}\sqrt{3} = \sqrt{12}$ and $5\sqrt{3} = \sqrt{25}\sqrt{3} = \sqrt{75}$.

Example 2. Simplify: (i) $5\sqrt{2} \times 3\sqrt{3}$ (ii) $7\sqrt{5} \times 6\sqrt{7}$
(iii) $m\sqrt{n} \times p\sqrt{q}$, $n, q \geqq 0$.

Solution.

(i) $5\sqrt{2} \times 3\sqrt{3}$
$= 5 \times 3 \times \sqrt{2} \times \sqrt{3}$ *(CM)*
$= 15\sqrt{6}$. *(AM)*

(ii) $7\sqrt{5} \times 6\sqrt{7}$
$= 7 \times 6 \times \sqrt{5} \times \sqrt{7}$ *(CM)*
$= 42\sqrt{35}$. *(AM)*

(iii) $m\sqrt{n} \times p\sqrt{q} = m \times p \times \sqrt{n} \times \sqrt{q}$ *(CM)*
$= mp\sqrt{nq}$.

Example 3. Simplify: (i) $\sqrt{x^2}$ (ii) $\sqrt{x^2 + 2x + 1}$.

Solution. In each example the principal square root is required, and this is a positive number or zero. Thus:

(i) $\sqrt{x^2} = |x|$

(ii) $\sqrt{x^2 + 2x + 1} = \sqrt{(x + 1)^2}$
$= |x + 1|$.

The absolute value signs ensure that, whatever number x represents, the root is positive or zero.

In the worked examples of the text, the authorities which are most significant in the development are given in brackets. When writing a solution, you should think of the authorities for the steps, but it is not necessary to include them in your solution.

Exercise 2-3

(The variables are real numbers for which the expressions are defined.)

(A)

1. In $b^2 = a$ how many numbers does b represent?
2. If b is the principal square root of a, what quality does it possess?

3. What is the name of the symbol $\sqrt{\ }$ and what does it imply?
4. How is the negative square root of a positive real number indicated?
5. What is the radicand in the symbol $-\sqrt{3a}$?
6. What is the order of the following radicals or surds?

(i) $\sqrt{3}$ (ii) $\sqrt[5]{6}$ (iii) $\sqrt[10]{7}$ (iv) $\sqrt[n]{64}$

State the principal square root of each of the following:

7. 36	8. 64	9. 81 10. 0
11. $16x^2$	12. $4(x-1)^2$	13. $a^2-2ab+b^2$
14. x^2+4x+4	15. $(2a-3b)^4$	16. $a^2b^4c^6$

State each of the following as entire surds or radicals:

17. $2\sqrt{3}$	18. $3\sqrt{5}$	19. $4\sqrt{a}$
20. $a\sqrt{b}$	21. $b^2\sqrt{c}$	22. $b^4\sqrt{x^2y}$
23. $6\sqrt{6}$	24. $2^2\sqrt{3}$	25. $3a^4b^2\sqrt{x}$

State each of the following as mixed radicals or surds:

26. $\sqrt{8}$	27. $\sqrt{12}$	28. $\sqrt{18}$
29. $\sqrt{32}$	30. $\sqrt{24}$	31. $\sqrt{48}$
32. $\sqrt{54}$	33. $\sqrt{68}$	34. $\sqrt{44}$
35. $\sqrt{72}$	36. $\sqrt{125}$	37. $\sqrt{80}$
38. $\sqrt{52}$	39. $\sqrt{75}$	40. $\sqrt{75x}$
41. $\sqrt{18b^2}$	42. $\sqrt{98}$	43. $\sqrt{8a^2}$
44. $\sqrt{90y^3}$	45. $\sqrt{\frac{12y}{5}}$	46. $\sqrt{\frac{8}{49}}$
47. $\sqrt{\frac{3}{4}}$	48. $\sqrt{\frac{4}{81}}$	49. $\sqrt{\frac{200}{16}}$
50. $\sqrt{2a^2b^4}$	51. $\sqrt{60x^4y}$	52. $\sqrt{x^3}$
53. $\sqrt{2a^2+4a+2}$	54. $\sqrt{5(b-2)^2}$	55. $\sqrt{(a-b)(a+b)^2}$

Expand the indicated products:

56. $\sqrt{3}\times\sqrt{3}$	57. $2\sqrt{5}\cdot\sqrt{3}$	58. $\sqrt{6}\cdot\sqrt{2}$
59. $\sqrt{6}\times\sqrt{\frac{2}{3}}$	60. $\sqrt{2xy}\cdot\sqrt{3x^2y^2}$	61. $\sqrt{b}\cdot\sqrt{2b^2}$
62. $\sqrt{5}\times\sqrt{\frac{3}{5}}$	63. $\sqrt{5}\cdot\sqrt{2}$	64. $\sqrt{2a}\cdot\sqrt{3b}$
65. $\sqrt{2}\times\sqrt{18}$	66. $3\sqrt{5}\times4\sqrt{2}$	67. $4\sqrt{7}\cdot3\sqrt{8}$
68. $\sqrt{\frac{5}{7}}\cdot\sqrt{\frac{5}{7}}$	69. $\sqrt{2x}\cdot\sqrt{5x}$	70. $x\sqrt{a}\cdot\sqrt{b}$

(B)

Write the principal square root of each of the following:

71. x^2+6x+9	72. $(x-3)^2$	73. $4x^2+4x+1$

74. $9x^2 + 12x + 4$ **75.** $4x^2 - 20x + 25$ **76.** $\frac{x^2}{4} - \frac{x}{4} + \frac{1}{16}$

77. $25 + 30x + 9x^2$ **78.** $49x^2 - 14x + 1$ **79.** $64x^2 + 48x + 9$

Write each of the following as an entire surd or radical (assume that the coefficients of the radicals are positive):

80. $a\sqrt{b}$ **81.** $3\sqrt{c}$ **82.** $3x\sqrt{3y}$

83. $(a - b)\sqrt{3}$ **84.** $(a + b)\sqrt{2}$ **85.** $(x + 3)\sqrt{y}$

86. $(a - b)\sqrt{(a + b)}$ **87.** $(2x + 4)\sqrt{5}$ **88.** $(3x + y)\sqrt{z}$

Write each of the following as a mixed radical or surd:

89. $\sqrt{99}$ **90.** $\sqrt{3a^2}$ **91.** $\sqrt{2(x - 1)^2}$

92. $\sqrt{3(x^2+16x+64)}$ **93.** $\sqrt{5x^2 + 10x + 5}$ **94.** $\sqrt{b^2a-8ba+16a}$

95. $\sqrt{\frac{(a - b)^2}{(a + b)^2}}$ **96.** $\sqrt{12(a + ab)^2}$ **97.** $\sqrt{3x^2 + 6x + 3}$

Multiply as indicated and write the result with simplest radicand:

98. $\sqrt{2xy} \cdot \sqrt{3xy}$ **99.** $\sqrt{x^3} \cdot \sqrt{x^2}$ **100.** $2\sqrt{3} \cdot 3\sqrt{4}$

101. $\sqrt{18} \times \sqrt{32}$ **102.** $\sqrt{b} \cdot \sqrt{3b}$ **103.** $\sqrt{45} \cdot \sqrt{80}$

104. $\sqrt{8} \times \sqrt{6} \times \sqrt{3}$ **105.** $\sqrt{2} \cdot \sqrt{18}$ **106.** $\sqrt{10} \cdot \sqrt{4}$

107. $3\sqrt{6} \cdot 5\sqrt{5} \cdot 2\sqrt{15}$ **108.** $6\sqrt{14} \cdot 2\sqrt{7}$ **109.** $20\sqrt{20} \cdot 5\sqrt{5}$

110. $\sqrt{\frac{14}{27}} \cdot \sqrt{\frac{4}{21}}$ **111.** $\sqrt{\frac{96}{25}} \cdot \sqrt{\frac{125}{54}}$ **112.** $\sqrt{\frac{10}{11}} \cdot \sqrt{\frac{33}{15}}$

2·11 **Addition of second order radicals.** The mixed radicals $3\sqrt{2}$ and $5\sqrt{2}$ are said to be similar, since they have the same radical coefficient. The indicated sum of two or more such radicals may be simplified by applying the distributive property of multiplication over addition. For example:

(i) $3\sqrt{2} + 5\sqrt{2} = (3 + 5)\sqrt{2}$
$= 8\sqrt{2}$

(ii) $7\sqrt{5} + 3\sqrt{5} - 2\sqrt{5}$
$= (7 + 3 - 2)\sqrt{5}$
$= 8\sqrt{5}$.

Simplify each of the following and compare your solutions with those on page 405.

1. $3\sqrt{2} + 4\sqrt{8} - 5\sqrt{18}$ **2.** $3\sqrt{8} + 5\sqrt{32} - 2\sqrt{72}$

3. $\sqrt{75} - \sqrt{27} + \sqrt{200}$

Exercise 2-4

(The variables are real numbers for which the expressions are defined.)

(A)

State the sum in simplified form:

1. $2\sqrt{3} + 3\sqrt{3}$ **2.** $4\sqrt{5} + 6\sqrt{5}$ **3.** $3\sqrt{2} - 2\sqrt{2}$

4. $3\sqrt{6}+5\sqrt{6}-2\sqrt{6}$ **5.** $2\sqrt{7}-3\sqrt{7}+5\sqrt{7}$ **6.** $2\sqrt{3}+4\sqrt{3}-10\sqrt{3}$

7. $3\sqrt{x} + 2\sqrt{x} - 4\sqrt{x}$ **8.** $11\sqrt{y} + 2\sqrt{y} - 3\sqrt{y}$

9. $12\sqrt{6} - 7\sqrt{6} + 2\sqrt{2} - 3\sqrt{2}$ **10.** $7\sqrt{a} - 4\sqrt{b} + 2\sqrt{a} - 3\sqrt{b}$

11. $3\sqrt{c} - 4\sqrt{d} + 3\sqrt{d} - 6\sqrt{c}$ **12.** $3\sqrt{b} - \sqrt{b} + \sqrt{c}$

13. $5\sqrt{x} - 3\sqrt{y} + 2\sqrt{x}$ **14.** $6\sqrt{a} + 7\sqrt{b} - 4\sqrt{a} - 5\sqrt{b}$

15. $5\sqrt{ab} - 2\sqrt{ab} + 3\sqrt{ab}$

(B)

Simplify the following expressions:

16. $4\sqrt{2} - 2\sqrt{2}$ **17.** $\sqrt{98} + \sqrt{32}$ **18.** $2\sqrt{50} + 4\sqrt{18}$

19. $7\sqrt{3} + 3\sqrt{3}$ **20.** $\sqrt{45} - \sqrt{245}$ **21.** $2\sqrt{27} - \sqrt{12}$

22. $\sqrt{28} + 2\sqrt{7}$ **23.** $x\sqrt{2} - x$ **24.** $2\sqrt{50} + \sqrt{98}$

25. $\sqrt{\frac{4}{5}} + 2\sqrt{\frac{4}{5}}$ **26.** $\frac{3}{4}\sqrt{45} - \frac{1}{3}\sqrt{20}$ **27.** $\sqrt{40} - \sqrt{90}$

28. $\sqrt{8} + \frac{1}{2}\sqrt{2}$ **29.** $\sqrt{b} + \sqrt{4b}$ **30.** $\sqrt{2a^2} + \sqrt{8a^2}$

31. $\sqrt{6} + 2\sqrt{24} - \sqrt{54}$ **32.** $6\sqrt{24} + 2\sqrt{18} - 3\sqrt{50}$

33. $\sqrt{72} - \sqrt{32} + \sqrt{8}$ **34.** $\sqrt{32} + \sqrt{128} - \sqrt{18} + \sqrt{50}$

35. $2\sqrt{128} + 3\sqrt{32} - \sqrt{18}$ **36.** $\sqrt{32} + 3\sqrt{8} - 2\sqrt{18}$

37. $\sqrt{6} + 3\sqrt{6} + 5\sqrt{6}$ **38.** $4\sqrt{20}+2\sqrt{45}-3\sqrt{80}+\sqrt{32}$

39. $3\sqrt{32} - 2\sqrt{8} + 5\sqrt{72}$ **40.** $\dfrac{\sqrt{4}}{2} + \dfrac{\sqrt{12}}{3} + \dfrac{2\sqrt{3}}{3}$

2•12 Square roots of fractions.

(i) $$\left(\frac{\sqrt{3}}{\sqrt{2}}\right)^2 = \frac{(\sqrt{3})^2}{(\sqrt{2})^2}$$ (Power of a quotient)

$$= \frac{3}{2} .$$ (Definition)

Thus, $$\frac{\sqrt{3}}{\sqrt{2}} = \sqrt{\frac{3}{2}} .$$ (Definition)

Similarly, if $a \geqq 0,\ b > 0$

$$\left(\frac{\sqrt{a}}{\sqrt{b}}\right)^2 = \frac{(\sqrt{a})^2}{(\sqrt{b})^2} \qquad \text{(Power of a quotient)}$$

$$= \frac{a}{b}. \qquad \text{(Definition)}$$

Thus, in general if $a \geqq 0,\ b > 0$, then $\dfrac{\sqrt{a}}{\sqrt{b}} = \sqrt{\dfrac{a}{b}}$.

Write solutions to the following problems and compare them with those on page 405.

1. Write each of the following in the form $\sqrt{\dfrac{a}{b}}$:

(i) $\dfrac{\sqrt{3}}{\sqrt{7}}$ (ii) $\dfrac{\sqrt{5}}{\sqrt{10}}$ (iii) $\dfrac{2\sqrt{7}}{\sqrt{3}}$ (iv) $\dfrac{3\sqrt{5}}{2\sqrt{11}}$

2. Perform each of the following divisions:

(i) $\sqrt{12} \div \sqrt{3}$ (ii) $4\sqrt{15} \div 2\sqrt{5}$

***2·13* Rationalizing a denominator.** The expression $\dfrac{\sqrt{3}}{\sqrt{2}}$ may be simplified to the form $\sqrt{\dfrac{3}{2}}$ or $\sqrt{1.5}$. It may also be simplified in another manner.

$$\frac{\sqrt{3}}{\sqrt{2}} = \frac{\sqrt{3} \cdot \sqrt{2}}{\sqrt{2} \cdot \sqrt{2}} = \frac{\sqrt{6}}{2}. \qquad \text{(Principle of equivalent fractions)}$$

The latter simplification has produced a form involving a rational denominator and a radical numerator. If the radicand in such a denominator is a rational number, this process will always produce a rational denominator and is sometimes referred to as *rationalizing the denominator*. This procedure facilitates further simplification by division as follows:

$$\frac{\sqrt{6}}{2} \doteqdot \frac{2.449}{2} \doteqdot 1.225.$$

Exercise 2-5

(The variables are real numbers for which the expressions are defined.)

(A)

Simplify by division:

1. $\dfrac{\sqrt{6}}{\sqrt{3}}$ **2.** $\dfrac{\sqrt{8}}{\sqrt{2}}$ **3.** $\dfrac{\sqrt{15}}{\sqrt{5}}$ **4.** $\dfrac{21\sqrt{14}}{2\sqrt{7}}$

Simplify by division:

5. $\dfrac{5\sqrt{18}}{3\sqrt{3}}$ **6.** $\dfrac{\sqrt{8x^3}}{\sqrt{2x^2}}$ **7.** $\dfrac{3\sqrt{a^2-b^2}}{2\sqrt{a+b}}$ **8.** $\dfrac{\sqrt{250x^4y^2z^3}}{\sqrt{25x^2y^2z^2}}$

State the simplest number necessary to multiply numerator and denominator in order to rationalize the denominator:

9. $\dfrac{1}{\sqrt{2}}$ **10.** $\dfrac{\sqrt{3}}{\sqrt{5}}$ **11.** $\dfrac{3\sqrt{2}}{2\sqrt{3}}$ **12.** $\dfrac{3}{\sqrt{7}}$

13. $\dfrac{4}{a\sqrt{2}}$ **14.** $\dfrac{3\sqrt{2}}{\sqrt{3}}$ **15.** $\dfrac{11}{\sqrt{12}}$ **16.** $\dfrac{7}{\sqrt{8}}$

17. $\dfrac{5}{\sqrt{50}}$ **18.** $\dfrac{7}{\sqrt{18}}$ **19.** $\dfrac{2}{\sqrt{8a}}$ **20.** $\dfrac{6b}{\sqrt{b}}$

State an equivalent expression in the form $\sqrt{\dfrac{a}{b}}$ *for each of the following:*

21. $\dfrac{\sqrt{2}}{\sqrt{3}}$ **22.** $\dfrac{2\sqrt{2}}{\sqrt{5}}$ **23.** $\dfrac{a\sqrt{b}}{\sqrt{c}}$ **24.** $\dfrac{3\sqrt{x}}{2\sqrt{y}}$

25. $\dfrac{a\sqrt{b}}{b\sqrt{a}}$ **26.** $\dfrac{\sqrt{11}}{\sqrt{3}}$ **27.** $\dfrac{(a-b)\sqrt{3}}{\sqrt{a-b}}$ **28.** $\dfrac{3x\sqrt{y}}{5\sqrt{x}}$

(B)

Rationalize the denominators of the following expressions:

29. $\dfrac{2}{\sqrt{2}}$ **30.** $\dfrac{\sqrt{5}}{\sqrt{3}}$ **31.** $\dfrac{\sqrt{2}}{\sqrt{3}}$ **32.** $\dfrac{b}{\sqrt{b}}$

33. $\dfrac{2\sqrt{3}}{5\sqrt{2}}$ **34.** $\dfrac{8x}{\sqrt{x}}$ **35.** $\dfrac{2}{\sqrt{5}}$ **36.** $\dfrac{\sqrt{11}}{\sqrt{7}}$

37. $\dfrac{4}{\sqrt{8}}$ **38.** $\dfrac{\sqrt{45}}{\sqrt{12}}$ **39.** $\dfrac{\sqrt{3}}{5\sqrt{20}}$ **40.** $\dfrac{\sqrt{5}}{\sqrt{14}}$

41. $\dfrac{15}{\sqrt{50}}$ **42.** $\dfrac{3}{\sqrt{18}}$ **43.** $\dfrac{6\sqrt{5}}{3\sqrt{2}}$ **44.** $\dfrac{48}{5\sqrt{32}}$

If $\sqrt{2} = 1.414$, $\sqrt{3} = 1.732$, $\sqrt{5} = 2.236$, *find correct to two decimal places the numerical value of:*

45. $\dfrac{1}{\sqrt{2}}$ **46.** $\dfrac{2}{\sqrt{5}}$ **47.** $\dfrac{2}{5\sqrt{2}}$ **48.** $\dfrac{15}{\sqrt{45}}$ **49.** $\dfrac{3}{\sqrt{8}}$

2·14 **The product of radical expressions.** Since second order radicals represent real numbers if the radicand is positive or zero, the laws of addition and multiplication apply.

Write solutions to the following problems and compare them with those on page 405. Consider the radicands to be positive or zero.

1. Expand each of the following using the distributive law:
   (i) $\sqrt{2}(\sqrt{3}+\sqrt{5})$ (ii) $3\sqrt{2}(\sqrt{7}+5\sqrt{11})$
2. Expand the indicated product by means of the distributive law and simplify: $(3\sqrt{2}+5\sqrt{3})^2$
3. Expand the following squares:
   (i) $(3\sqrt{2}+5\sqrt{3})^2$ (ii) $(\sqrt{5}-2\sqrt{7})^2$
4. Expand the indicated product: $(\sqrt{2}+\sqrt{3})(2\sqrt{2}-\sqrt{3})$
5. Expand using the distributive law: $(\sqrt{2}-\sqrt{3})(\sqrt{2}+\sqrt{3})$
6. Expand:
   (i) $(\sqrt{2}-\sqrt{3})(\sqrt{2}+\sqrt{3})$ (ii) $(3\sqrt{2}+4\sqrt{5})(3\sqrt{2}-4\sqrt{5})$,
   (iii) $(\sqrt{x}+\sqrt{y})(\sqrt{x}-\sqrt{y})$ (iv) $(a\sqrt{x}+b\sqrt{y})(a\sqrt{x}-b\sqrt{y})$,

Two radical expressions of the form $\sqrt{x}+\sqrt{y}$ and $\sqrt{x}-\sqrt{y}$ or $a\sqrt{x}+b\sqrt{y}$ and $a\sqrt{x}-b\sqrt{y}$ are called *conjugate radicals.* If a, b, x, y represent rational numbers, then the product of these conjugate radical expressions represent a rational number. Thus:

(i) $(\sqrt{x}+\sqrt{y})(\sqrt{x}-\sqrt{y}) = x-y$,

(ii) $(a\sqrt{x}+b\sqrt{y})(a\sqrt{x}-b\sqrt{y}) = a^2x-b^2y$,

and $x-y$ and a^2x-b^2y represent rational numbers.

Exercise 2-6

(The variables are real numbers for which the expressions are defined.)

(B)

Expand the indicated products:

1. $\sqrt{2}(\sqrt{2}+2)$
2. $\sqrt{3}(2\sqrt{3}-2)$
3. $\sqrt{5}(1+2\sqrt{5})$
4. $\sqrt{2}(5+\sqrt{2})$
5. $2\sqrt{3}(\sqrt{2}+\sqrt{5})$
6. $\sqrt{2}(2\sqrt{5}-3\sqrt{8})$
7. $(\sqrt{12}+\sqrt{5})\sqrt{5}$
8. $(\sqrt{2}+1)(\sqrt{2}-1)$
9. $(2+\sqrt{3})(2-\sqrt{3})$
10. $(\sqrt{2}+3)^2$
11. $(\sqrt{3}-\sqrt{2})^2$
12. $(\sqrt{2}+2)(\sqrt{2}-2)$
13. $(2+\sqrt{3})^2$
14. $(\sqrt{6}+\sqrt{2})^2$
15. $(\sqrt{6}+\sqrt{3})(\sqrt{6}-\sqrt{3})$
16. $\dfrac{\sqrt{7}+2}{3}\cdot\dfrac{\sqrt{7}-2}{3}$
17. $\left(\dfrac{\sqrt{11}+1}{2}\right)^2$
18. $\left(\dfrac{2-\sqrt{3}}{2}\right)^2$

Expand the indicated products:

19. $(2\sqrt{7} - 5\sqrt{2})(2\sqrt{7} + 5\sqrt{2})$
20. $(\sqrt{3} + 3\sqrt{2})(4\sqrt{3} - 2\sqrt{2})$
21. $(3\sqrt{2} - 4\sqrt{3})^2$
22. $(3\sqrt{x} - \sqrt{y})^2$
23. $(5\sqrt{3} - 3\sqrt{2})^2$
24. $2\sqrt{7}(1 - 2\sqrt{7})$
25. $(\sqrt{x-1})^2$
26. $(3\sqrt{5} - 1)(3\sqrt{5} + 1)$
27. $(\sqrt{x} - \sqrt{y})^2$
28. $(\sqrt{a} + \sqrt{b})^2$
29. $\sqrt{5+3\sqrt{2}}\ \sqrt{5-3\sqrt{2}}$
30. $\dfrac{-3+\sqrt{2}}{3} \cdot \dfrac{-3-\sqrt{2}}{3}$
31. $(\sqrt{2} + \sqrt{5})(2\sqrt{2} + 3\sqrt{5})$
32. $(3\sqrt{x} - \sqrt{y})(2\sqrt{x} + 3\sqrt{y})$
33. $(4\sqrt{5} - 2\sqrt{7})(3\sqrt{5} - \sqrt{7})$
34. $(\sqrt{a} + \sqrt{b})(\sqrt{c} + \sqrt{d})$
35. $(3\sqrt{5} - \sqrt{11})(2\sqrt{5} + 3\sqrt{11})$
36. $(3\sqrt{2} + \sqrt{10})(2\sqrt{2} + \sqrt{10})$

***2·15* Rationalizing a denominator which is a binomial containing radicals.** The fact that the product of two conjugate (binomial) radicals, $(\sqrt{a} + \sqrt{b})(\sqrt{a} - \sqrt{b})$, is a rational number, $a - b$, if $a, b \in Q$, may be used to rationalize the denominator of a fraction when it is a binomial containing radicals.

Example. Simplify by multiplying both terms of the fraction by the conjugate of the denominator: (i) $\dfrac{3}{3+\sqrt{2}}$ (ii) $\dfrac{5}{\sqrt{5}-\sqrt{3}}$.

Solution. (i) $\dfrac{3}{3+\sqrt{2}}$

$= \dfrac{3(3-\sqrt{2})}{(3+\sqrt{2})(3-\sqrt{2})}$

$= \dfrac{9-3\sqrt{2}}{9-2}$

$= \dfrac{9-3\sqrt{2}}{7}$.

(ii) $\dfrac{5}{\sqrt{5}-\sqrt{3}}$

$= \dfrac{5(\sqrt{5}+\sqrt{3})}{(\sqrt{5}-\sqrt{3})(\sqrt{5}+\sqrt{3})}$

$= \dfrac{5\sqrt{5}+5\sqrt{3}}{5-3}$

$= \dfrac{5\sqrt{5}+5\sqrt{3}}{2}$.

Write solutions for each of the following and compare them with those on page 406.

Simplify:

1. $\dfrac{\sqrt{2}}{3\sqrt{7}-\sqrt{5}}$
2. $\dfrac{3\sqrt{3}}{3\sqrt{5}-4\sqrt{2}}$
3. $\dfrac{3+\sqrt{11}}{2\sqrt{7}+3\sqrt{6}}$
4. $\dfrac{5\sqrt{2}-3\sqrt{3}}{2\sqrt{5}+3\sqrt{2}}$

Exercise 2-7

(The variables are real numbers for which the expressions are defined.)

(B)

Simplify by multiplying both terms of the fraction by the conjugate of the denominator:

1. $\dfrac{1}{\sqrt{2}+1}$
2. $\dfrac{3}{3-\sqrt{3}}$
3. $\dfrac{6}{2+\sqrt{5}}$
4. $\dfrac{5}{3-\sqrt{7}}$
5. $\dfrac{2}{x-\sqrt{y}}$
6. $\dfrac{\sqrt{5}}{2-\sqrt{6}}$
7. $\dfrac{\sqrt{7}}{\sqrt{5}-3}$
8. $\dfrac{2\sqrt{3}}{\sqrt{7}+2}$
9. $\dfrac{1}{2\sqrt{2}-3}$
10. $\dfrac{3}{\sqrt{2}+\sqrt{3}}$
11. $\dfrac{\sqrt{5}}{\sqrt{5}+\sqrt{2}}$
12. $\dfrac{\sqrt{3}}{2\sqrt{2}-\sqrt{7}}$
13. $\dfrac{\sqrt{2}+\sqrt{3}}{\sqrt{2}-\sqrt{3}}$
14. $\dfrac{4-\sqrt{5}}{4+\sqrt{5}}$
15. $\dfrac{\sqrt{5}+\sqrt{3}}{\sqrt{5}-\sqrt{3}}$
16. $\dfrac{\sqrt{3}-2\sqrt{2}}{\sqrt{3}+2\sqrt{2}}$
17. $\dfrac{\sqrt{6}-\sqrt{5}}{\sqrt{5}-\sqrt{6}}$
18. $\dfrac{2\sqrt{2}-3\sqrt{5}}{\sqrt{5}+\sqrt{7}}$
19. $\dfrac{4\sqrt{7}-5\sqrt{3}}{4\sqrt{7}+5\sqrt{3}}$
20. $\dfrac{2\sqrt{2}-3\sqrt{3}}{2\sqrt{2}+3\sqrt{3}}$
21. $\dfrac{4\sqrt{3}-3\sqrt{5}}{\sqrt{5}+\sqrt{3}}$

(C)

22. $\dfrac{x+y\sqrt{z}}{x-y\sqrt{z}}$
23. $\dfrac{\sqrt{a^2+3}-2}{\sqrt{a^2+3}+2}$
24. $\dfrac{\sqrt{a}-\sqrt{b}}{\sqrt{a}+\sqrt{b}}$
25. $\dfrac{\sqrt{x+1}+\sqrt{x-1}}{\sqrt{x+1}-\sqrt{x-1}}$
26. $\dfrac{ay-by}{\sqrt{a}+\sqrt{b}}$
27. $\dfrac{a-3\sqrt{b}}{\sqrt{b}+a}$
28. $\dfrac{1+\sqrt{2}+\sqrt{3}}{1+\sqrt{2}-\sqrt{3}}$

2·16 **Radical equations in one variable.** A radical equation is one in which the variable occurs in a radicand. For example:

(i) $\sqrt{x}=3$ (ii) $\sqrt{2x-5}+2=3$ (iii) $\sqrt{4y^2-7}+3=2y$

are each radical equations.

Our method of solving equations has been to write simpler equivalent equations until the solution set has been obtained. To do this, we used the uniqueness property of closure for real numbers under addition, multiplication, subtraction, and division.

Consider the equation:

(i) $\sqrt{4y^2 - 7} + 3 = 2y$.

The equation may be written:

(ii) $\sqrt{4y^2 - 7} = 2y - 3$. (Subtraction property)

To obtain the solution set, it is necessary to obtain an equation not involving a radical. This may be done by squaring both sides of the equation.

(iii) $\therefore\ 4y^2 - 7 = 4y^2 - 12y + 9$ (Squaring)

(iv) $\therefore\ 12y = 16$ (Subtraction, addition)

(v) $\therefore\ y = \frac{4}{3}$. (Division property)

Verification.

$$\text{L.S.} = \sqrt{4(\tfrac{4}{3})^2 - 7} + 3 \qquad \text{R.S.} = 2 \cdot \tfrac{4}{3}$$

$$= \sqrt{\frac{64 - 63}{9}} + 3 \qquad = \frac{8}{3} .$$

$$= \frac{1}{3} + 3$$

$$= \frac{10}{3} .$$

Upon verification we see that $\frac{4}{3}$ does not satisfy the equation and is, therefore, *not* a root of the equation. In fact, the given equation has no root.

The fact that $\frac{4}{3}$ is not a root of the equation means that

$$\sqrt{4y^2 - 7} + 3 = 2y \text{ and } y = \tfrac{4}{3}$$

are not equivalent equations. Which step in the procedure of solving is not reversible? To get from step (ii) to (iii) it was necessary to square both sides of the equation. Is squaring and taking square roots a reversible operation? If not, then equations (ii) and (iii) are not equivalent.

Consider the following example. The only root of the equation $x = 3$ is 3. If both sides are squared, then $x^2 = 9$. This equation has the two roots 3 and -3. That is, taking the square root of both sides produces the result, $x = \pm 3$, which is not the original equation.

Thus, the squaring of both sides of an equation may not produce an equivalent equation. The equation produced will contain among its roots the root or roots of the original equation, if it has any. It may, however, be satisfied by numbers which are not roots of the original equation. Thus, when the squaring process is introduced, it is necessary to verify or check the numbers obtained to determine the roots, if any, of the original equation.

In the previous example, $\frac{4}{3}$ is a root of the equation $4y^2 - 7 = 4y^2 - 12y + 9$, determined by squaring both members of the original equation, $\sqrt{4y^2 - 7} + 3 = 2y$. $\frac{4}{3}$ is said to be *extraneous* to the equation $\sqrt{4y^2 - 7} + 3 = 2y$ and has been introduced by the squaring process.

Example. Solve $\sqrt{a + 3} - 4 = 0$.

Solution.
$$\sqrt{a + 3} - 4 = 0$$
$$\therefore \sqrt{a + 3} = 4$$
$$\therefore a + 3 = 16 \qquad \text{(Squaring both sides)}$$
$$\therefore a = 13.$$

Verification.

L.S. $= \sqrt{13 + 3} - 4$ R.S. $= 0$.
$= 4 - 4$
$= 0$.

$\therefore$ 13 is the root of the equation,
and the solution set is $\{13\}$.

Exercise 2-8

(B)

Solve:

1. $\sqrt{2a} - 6 = 0$
2. $\sqrt{b} + 1 = 0$
3. $\sqrt{y + 3} - 5 = 0$
4. $3\sqrt{m + 1} - 8 = 0$
5. $\sqrt{a^2 + 11} - 1 = a$
6. $\sqrt{2p + 1} - 3 = 0$
7. $\sqrt{y^2 + 6} - y + 3 = 0$
8. $\sqrt{x - 2} = 3$
9. $\sqrt{x + 1} = -3$
10. $\sqrt{x^2 + 5} + x = 5$
11. $\frac{4}{\sqrt{y - 5}} - \sqrt{y - 5} = 0$
12. $\frac{6}{\sqrt{a + 5}} = \sqrt{a + 5}$

2·17 **Radical equations in one variable (continued).**

Example. Solve $\sqrt{x + 3} + \sqrt{x - 2} = 5$.

Solution. $\sqrt{x + 3} + \sqrt{x - 2} = 5$

We may square both sides as they stand, but trial will show that the arithmetic is simpler if one radical occurs in each side. Therefore, write:

$$\sqrt{x+3} = 5 - \sqrt{x-2} \qquad \text{(Subtraction property)}$$

$$\therefore\ x + 3 = 25 - 10\sqrt{x-2} + x - 2 \qquad \text{(Squaring)}$$

$$\therefore\ 10\sqrt{x-2} = 20$$

$$\therefore\ \sqrt{x-2} = 2$$

$$\therefore\ x - 2 = 4 \qquad \text{(Squaring)}$$

$$\therefore\ x = 6\ .$$

Verification.

$$\text{L.S.} = \sqrt{6+3} + \sqrt{6-2} \qquad \text{R.S.} = 5\ .$$

$$= 3 + 2 = 5\ .$$

$\therefore$ the solution set is $\{6\}$.

Note that both sides were squared twice in the process of solving the equation.

Solve the following equations and compare your solutions with those on page 407.

1. $\sqrt{b+5} + \sqrt{b} - \sqrt{4b+9} = 0$
2. $\sqrt{x^2+7x+17} - \sqrt{x^2+3x+5} = 2$

Exercise 2-9

(B)

Solve:

1. $\sqrt{3b+1} = 5$
2. $\sqrt{x-5} - 5 = \sqrt{x}$
3. $\sqrt{y-9} + \sqrt{y} = 1$
4. $\sqrt{4a+5} = \sqrt{7a+2}$
5. $\sqrt{9m^2+4} - 3m = 2$
6. $\sqrt{y-3} - 3 = -\sqrt{y}$
7. $\sqrt{5a+34} - \sqrt{5a+6} = 2$
8. $\sqrt{25y-29} - 3\sqrt{y} = \sqrt{4y-11}$
9. $\sqrt{a-3} + \sqrt{a+4} = 7$
10. $\sqrt{x^2+12} - x = 2$

(C)

11. $\sqrt{2-y} - \dfrac{3}{\sqrt{3-y}} = -\sqrt{3-y}$
12. $\sqrt{b+60} - 2\sqrt{b+5} = \sqrt{b}$
13. $\sqrt{m-5} + 2\sqrt{4m-15} = \sqrt{9m-5}$
14. $\dfrac{a}{\sqrt{a-y}} - \sqrt{a-y} = \sqrt{b-y},\ a, b > 0$

Chapter III

RELATIONS

3•1 **Binary relations.** The familiar words "relation" and "relationship" bring to mind a great many properties which relate one individual to another: for example,

Bill "is older than" Jim.
John "is taller than" George.
Jane "is the sister of" Mary.
Jack "likes" Jill.
George "is heavier than" John.
Helen "is the mother of" Mary.
Sally "is cleverer than" Ann.
Mr. Smith "is the employer of" Mr. Jones.

These relationships apply between two people. Other relations, such as, "is at war with", "trades with", "is opposed to", "is a supporter of", could apply between two countries. Still other relations, such as, "is greater than", "is equal to", "is less than", "is the square of", apply between two numbers.

The common characteristic of the above relations is that each relation applies between pairs of things of the same general type. For this reason, these relations are called *binary relations*.

If the relation "is older than" is examined as it applies to students in a particular classroom, it would not be difficult to list the names of all pairs of students, such that the student named second in each pair is older than the student named first. This would involve forming all possible pairs of the form (Jim, Bill), (Bill, Mary), (Bill, Jack), (Jim, Mary), (Jim, Jack),

(Mary, Jack), and so on, such that the second member of each pair is older than the first. Since the order in which the names are written in each of these pairs is important, each pair is called an *ordered pair*. The set of all ordered pairs formed as indicated above specifies completely the relation "is older than" as it applies to the students in the particular classroom under consideration.

The basic idea of a binary relation is an *association* from one element of a set to another. Such an association determines a set of ordered pairs. Thus, it is natural and convenient to study a binary relation by studying the set of ordered pairs of the relation.

It should be understood that there are relations which involve three members. These are called *ternary relations* and determine a set of *ordered triples*. Similarly, there are relations which involve more than three members. However, the only relations discussed in this book are binary relations.

Example 1. Given $A = \{2, -5, 7, -7, -2, 5\}$, list the ordered pairs of each of the following relations:

(i) R_1 "is greater than", on A;

(ii) R_2 "is the additive inverse of", on A;

(iii) R_3 "is the square of", on A.

In listing the ordered pairs of a relation, the second element in each pair is in the given relation to the first element.

Solution. (i) $R_1 = \{(2, 7), (2, 5), (-5, 2), (-5, 7), (-5, -2), (-5, 5), (-7, 2), (-7, -5), (-7, 7), (-7, -2), (-7, 5), (-2, 2), (-2, 7), (-2, 5), (5, 7)\}$.

(ii) $R_2 = \{(2, -2), (-2, 2), (-5, 5), (5, -5), (7, -7), (-7, 7)\}$.

(iii) $R_3 = \emptyset$.

Example 2. Given $A = \{1, 2, 3, 4, 5\}$, list the ordered pairs of the relation R "is equal to", on A.

Solution. $R = \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5)\}$.

Exercise 3-1

(B)

1. The following is a list of the heights of the boys on a basketball team: Andy 6′ 1″, George 6′ 2″, Michael 5′ 11″, Ted 6′ 4″, Martin 5′ 7″. List the ordered pairs such that the second member in each pair "is shorter than" the first.

2. The following is a list of the densities of certain materials in grams per cubic centimetre: Aluminium 2.72, Copper 8.90, Diamond 3.5, Gold 19.34, Iron 7.85 . List the ordered pairs such that the density of the second member in each pair "is greater than" the density of the first.

3. Given the set $A = \{-3, -1, 0, 1, 3, 5\}$, list the set of ordered pairs of each of the following relations on A :

(i) R_1 "is three times" (ii) R_2 "is equal to"

(iii) R_3 "is the square of" (iv) R_4 "is two more than"

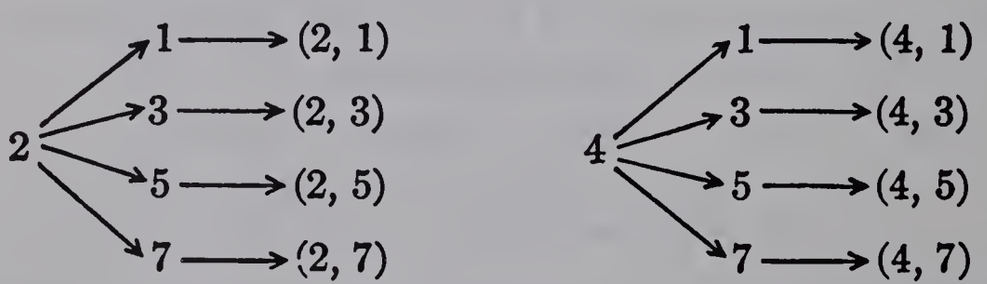
(v) R_5 "is greater than".

4. Given the set $B = \{0, \frac{1}{3}, \frac{1}{2}, 1, 2, 3\}$, list the set of ordered pairs of each of the following relations on B:

(i) P_1 "is less than" (ii) P_2 "is the reciprocal of"

(iii) P_3 "is one more than twice as great as".

3·2 Ordered pairs in cross-products. If $A = \{2, 4\}$ and $B = \{1, 3, 5, 7\}$, it is possible to form a set of eight ordered pairs whose first member is an element of A and whose second member is an element of B in the following manner:



The resulting set is $\{(2, 1), (2, 3), (2, 5), (2, 7), (4, 1), (4, 3), (4, 5), (4, 7)\}$ which is referred to as a *cross-product* of the two sets A and B. This cross-product is designated $A \times B$ (read, "A cross B") and, using set-builder notation, is defined as follows:

$$A \times B = \{(x, y) | x \in A, y \in B\}.$$

DEFINITION: *The cross-product, $A \times B$, of two sets A and B is the set of all ordered pairs whose first member is an element of A and whose second member is an element of B.*

The sets A and B also give rise to a second cross-product, $B \times A$, in which the first member of each ordered pair is from the set B and the second from the set A.

$$B \times A = \{(x, y) \mid x \in B, y \in A\}.$$

$$B \times A = \{(1, 2), (1, 4), (3, 2), (3, 4), (5, 2), (5, 4), (7, 2), (7, 4)\}.$$

If the set B is identical to the set A, then the cross-product

$$A \times A = \{(x, y) | x \in A, y \in A\} .$$

From the manner in which the ordered pairs are formed, it is seen that the number of ordered pairs in a cross-product is the product of the number of elements in each of the two sets.

Thus, the number of ordered pairs in $A \times B$ is $2 \cdot 4$ or 8, and in general:

If a set A has m elements and a set B has n elements, then the cross-product $A \times B$ has $m \cdot n$ or mn ordered pairs, and the cross-product $A \times A$ has m^2 ordered pairs.

Example. Given $A = \{1, 3, 5\}$.

(i) Determine the number of elements in $A \times A$.

(ii) List the ordered pairs of $A \times A$.

Solution. (i) Number of elements in $A \times A = 3 \cdot 3 = 9$.

(ii) $A \times A = \{(1, 1), (1, 3), (1, 5),$
$(3, 1), (3, 3), (3, 5),$
$(5, 1), (5, 3), (5, 5)\}$.

Write solutions to the following problems and compare them with those on page 407.

1. Given $A = \{1, 3, 5\}$, select the subset B of ordered pairs from $A \times A$ such that the second member is less than the first member.

2. Find the set of all ordered pairs of $N \times N$ (N is the set of natural numbers) such that the second member of each pair exceeds the first member by 2.

3. Given $A = \{8, 27, 64\}$, select the subset C of ordered pairs in $A \times A$ such that the second member of each pair is the cube of the first member.

Exercise 3-2

(A)

1. If A is the set of integers having only one digit and B is the set of odd integers having one digit, how many ordered pairs are there in $A \times B$?

2. If $A = \{1, 4, 7\}$, list the elements of $A \times A$.

3. Find $A \times B$ and $B \times A$ when $A = \{2, 5\}$ and $B = \{1, 3, 4\}$.

4. Under what condition is $A \times B = B \times A$?

5. Which ordered pair in $\{(1, 1), (3, 9), (2, 4), (4, 8), (6, 36), (5, 25)\}$ does not exhibit the same pattern as the other ordered pairs?

(B)

6. List $\{(x, y)|x, y \in A\}$ where $A = \{2, 3, 4\}$.
7. List the set of all ordered pairs in $\{5, 7\} \times \{4, 6, 8\}$.
8. If $A = \{1, 3, 5\}$ and $B = \{-1, -2\}$,
   (i) form the two products $B \times A$ and $A \times A$.
   (ii) How many elements are there in each of $A \times B$, $B \times A$, $A \times A$?
9. If two sets of real numbers, P and Q, have m and n elements respectively ($m, n \in {}^+I$), how many elements are there in each of the products $P \times Q$, $Q \times P$, $P \times P$, $Q \times Q$?
10. Classify as a finite or infinite set the cross-product $R \times R$, where R represents the set of real numbers.
11. $A = \{0, 2, 4, 6, 8\}$, $B = \{1, 3, 5, 9\}$
    (i) How many ordered pairs are there in $A \times B$? in $B \times A$? in $A \times A$?
    (ii) List the set of first elements in the ordered pairs of $A \times B$; of $B \times A$.
    (iii) How many ordered pairs of $A \times B$ have 6 as the first element?
12. List $\{m, n)|m, n \in P\}$, where $P = \{-2, 0, 2\}$.
13. If $A = \{1, 2, 3 \ldots, 9\}$, list the set of all ordered pairs whose second member exceeds its first member by 1.
14. List the elements of the set of all ordered pairs in $N \times N$ whose second member is double its first member.
15. If $A = \{1, 4, 7\}$, list the set of all ordered pairs in $A \times A$ in which the second member is the cube of the first member.
16. The Eastern Conference Football League consists of teams from Hamilton, Montreal, Ottawa, and Toronto. List ordered pairs to indicate all possible games, the second member of the pair being the visiting team.

3·3 **Graphing ordered pairs.** It has been noted previously that there is a one-to-one correspondence between the real numbers and the points on a real number line. *Fig. 3-1* is the graph of $\{-1.5, -\frac{1}{4}, \sqrt{2}, 2\}$. These real numbers are called the *coordinates* of the corresponding points.

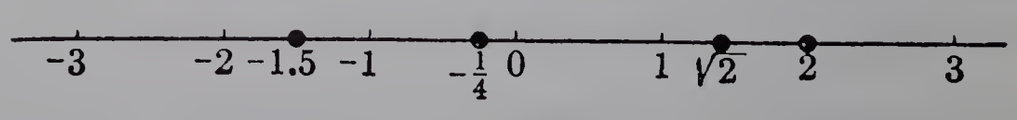


Fig. 3-1

Two real number lines drawn as illustrated in *Fig. 3-2* may be used as reference lines for indicating the position of any point in a plane. These number lines are called the *coordinate axes.*

The number line or axis drawn in the horizontal position is called the *x-axis*; the other, usually drawn vertically, is called the *y-axis.*

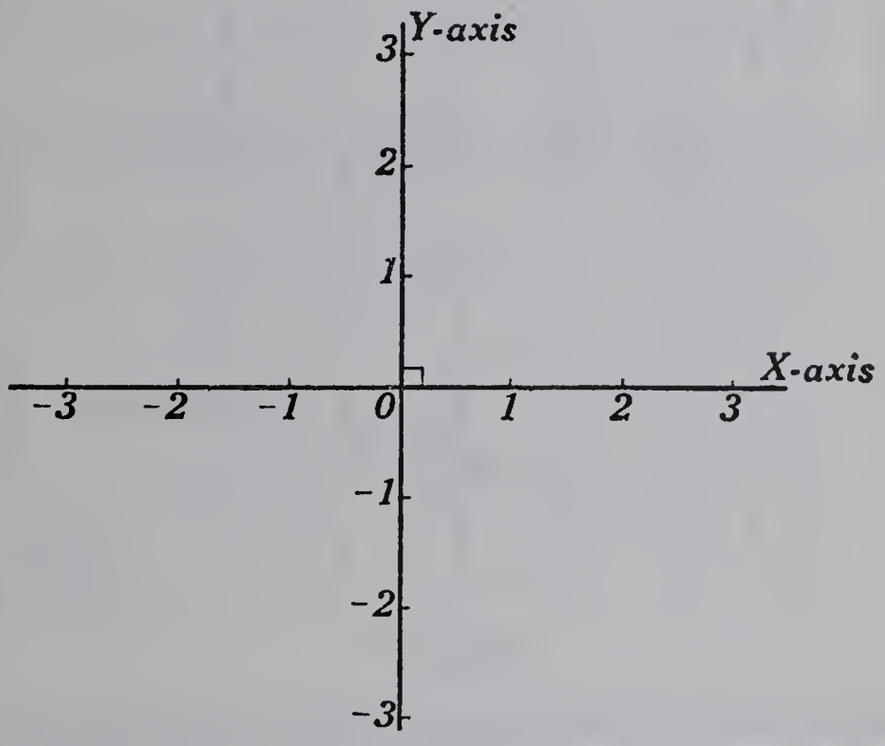


Fig. 3-2

If lines are drawn parallel to the x-axis through points on the y-axis whose coordinates are the integers

$$\ldots, -3, -2, -1, 1, 2, 3, \ldots,$$

and if lines are drawn parallel to the y-axis through points on the x-axis having integral coordinates, a *lattice* is formed (*Fig. 3-3*). The points of intersection of these lines are called *lattice points*, and the lines are called *lattice lines.* Each lattice point may be associated with an *ordered pair* of integers represented generally by (x, y) and called the *coordinates of the point.* The *x-coordinate* or *abscissa* is the directed distance from the y-axis to the point measured in a direction parallel to the x-axis; the *y-coordinate* or *ordinate* is the directed distance from the x-axis to the point measured in a direction parallel to the y-axis. In *Fig. 3-3*, the positions of the points $A(4, 2)$ and $B(2, 4)$ illustrate the effect of changing the order of the coordinates. Other lattice points with their coordinates are also shown in *Fig. 3-3.*

The number plane is divided into four *quadrants*, numbered as in *Fig. 3-3*.

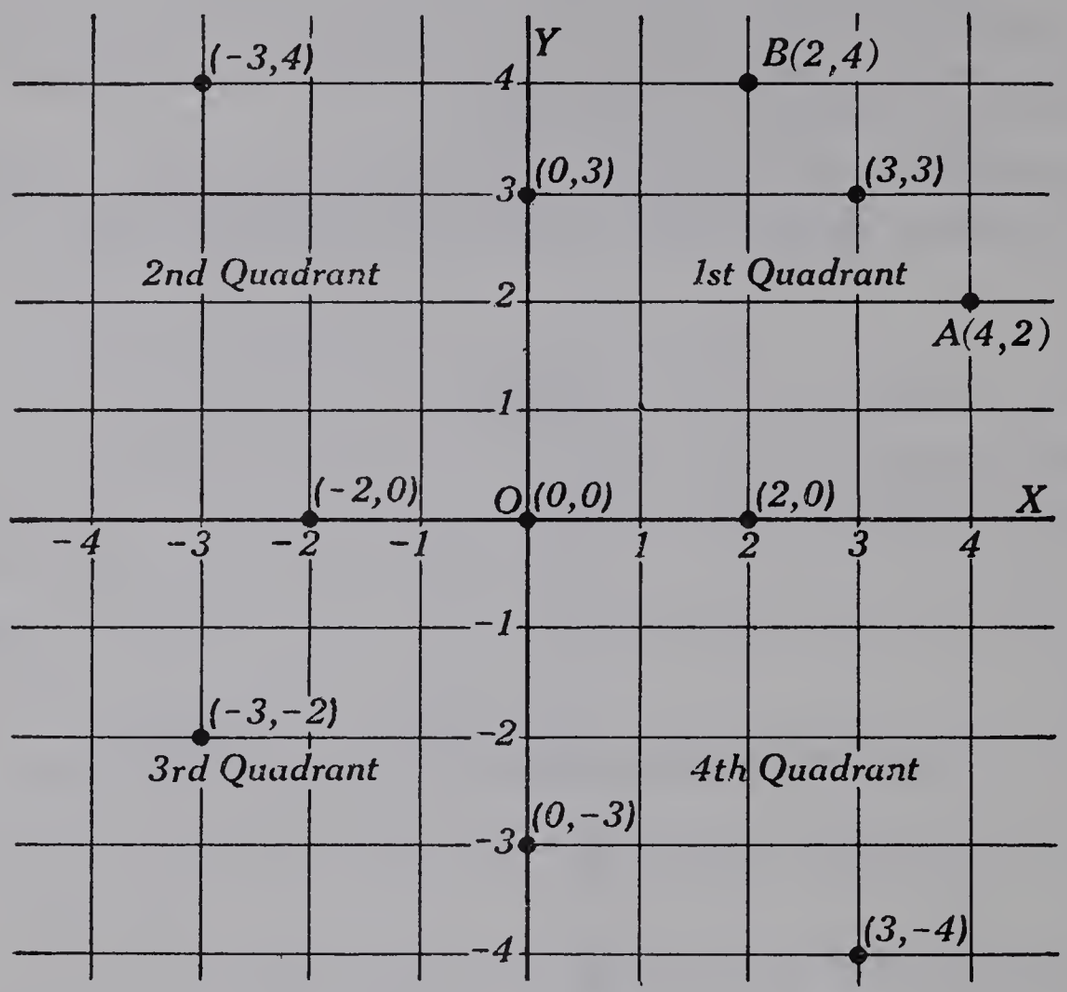


Fig. 3-3

Since the set of integers contains zero, lattice points appear on the axes as illustrated by the points whose coordinates are (2, 0), (0, 3), (− 2, 0), and (0, −3). The point of intersection of the axes has coordinates (0, 0) and is known as the *origin* (designated by *O*).

Lines parallel to the axes may be drawn through any points with coordinates $(x, 0)$ or $(0, y)$, $x, y \in R$, on the axes to meet in a unique point (x, y) on the plane. Thus, every point in a plane can be associated with a unique ordered pair of real numbers Conversely, every ordered pair of real numbers can be associated with a unique point in a plane.

The French mathematician and philosopher, René Descartes (1596-1650), is credited with introducing the above method of representing the position of a point in a plane with respect to number line axes of reference. In 1637 he published the first book or treatise on coordinate geometry. The fundamental assumption of this geometry is as follows:

There is a one-to-one correspondence between the set of all points in a plane and the set of all ordered pairs of real numbers.

The set of all ordered pairs of real numbers is the cross-product

$$\begin{aligned} R \times R &= \{(x, y) | x \in R, y \in R\} \\ &= \{(x, y) | x, y \in R\} . \end{aligned}$$

Cross-products are often referred to as *Cartesian products* in deference to Descartes.

Example 1. If $A = \{-3, 2, 4\}$, determine $A \times A$ and draw its graph.

Solution. $A \times A = \{(-3, -3), (-3, 2), (-3, 4), (2, -3), (2, 2), (2, 4), (4, -3), (4, 2), (4, 4)\}$.

The graph is shown in *Fig. 3-4*.

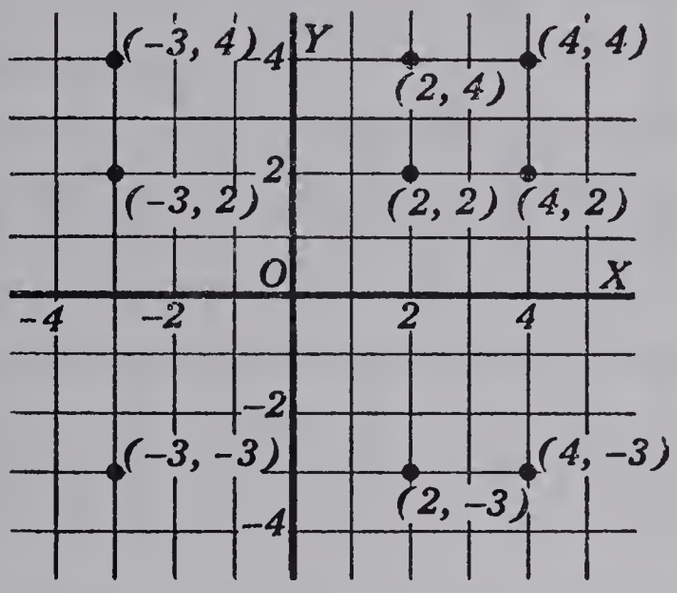


Fig. 3-4

Example 2. If $A = \{2, 3, 4\}$ and $B = \{1, 2, 3\}$, determine $A \times B$ and draw its graph.

Solution. $A \times B = \{(2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3), (4, 1), (4, 2), (4, 3)\}$.

The graph is shown in *Fig. 3-5*.

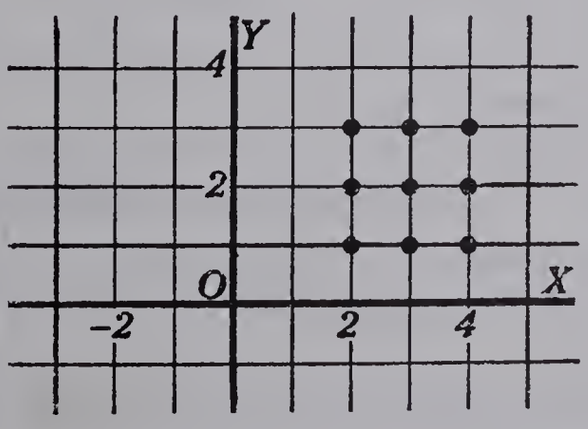
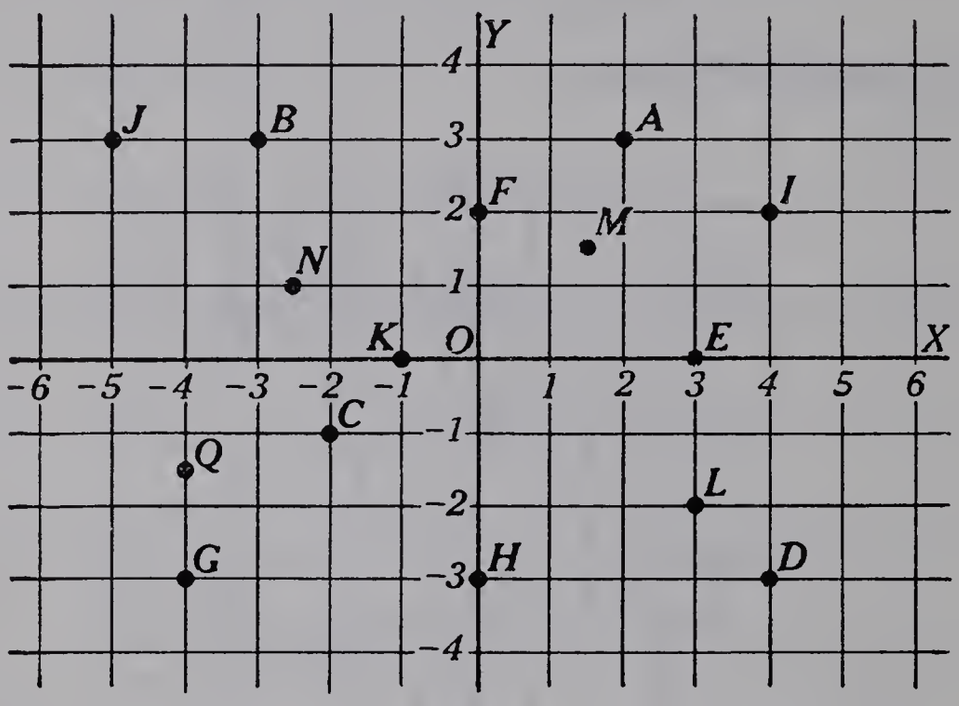


Fig. 3-5

Exercise 3-3

(A)

1. In the accompanying diagram, state the coordinates of the points marked:



(B)

2. Write the coordinates of the points indicated in the diagram of question 1.
3. Using convenient axes and scale, plot the points $P(10, 3)$, $Q(-6, 4)$, $R(0, 10)$, $S(-2, -8)$, $T(6, 0)$, and $V(2 -9)$.
4. (i) Plot the points:
   $A(-4, 0)$, $B(-2, 0)$, $C(0, 0)$, $D(2, 0)$, $E(4, 0)$, $F(6, 0)$.
   (ii) Of what line are these points a subset?
5. (i) Plot the points:
   $G(-2, -6)$, $H(-2, -3)$, $K(-2, 0)$, $L(-2, 3)$, $M(-2, 5)$.
   (ii) Describe the line of which these points are a subset.
6. (i) Plot the points:
   $N(-4, -4)$, $O(-1, -1)$, $P(0, 0)$, $Q(3, 3)$, $R(5, 5)$.
   (ii) Describe the line of which these points are a subset.
7. (i) Plot the points:
   $S(5, 0)$, $T(4, 3)$, $U(3, 4)$, $V(0, 5)$, $W(-3, 4)$, $X(-4, 3)$, $Y(-5, 0)$.
   (ii) Describe the curve on which these points appear to lie.
8. Draw the graph of
   $A = \{(-3, 2), (4, 6), (0, 1), (5, -7), (3, 0), (-6, 2)\}$.

9. $A = \{1, 2, 3, 4\}$

(i) List the ordered pairs of $A \times A$ and draw the graph.

(ii) If $A \times A = \{(x, y)|x, y \in A\}$, circle the points of the graph which correspond to the subset of $A \times A$ for which $y = x$.

(iii) Where in the graph are the points which correspond to the subset of $A \times A$ for which $y > x$; for which $y < x$?

10. If $M = \{-1, 1, 3\}$ and $N = \{-4, 0, 5)\}$:

(i) determine the Cartesian products $M \times N$ and $N \times M$;

(ii) draw the graphs of $M \times N$ and $N \times M$.

11. If $D = \{-2, 0, 2, 4\}$,

(i) list the ordered pairs of $D \times D = \{(x, y)|x, y \in D\}$;

(ii) select the subset $D_1 = \{(x, y)|y = x\}$ and draw its graph;

(iii) select the subset $D_2 = \{(x, y)|y > x\}$ and draw its graph;

(iv) select the subset $D_3 = \{(x, y)|y < x\}$ and draw its graph.

(v) How are the ordered pairs of D_1, D_2, D_3 related to $D \times D$?

***3·4* Relations in $U \times U$.** In question 11 of the previous exercise, the sentence

$$y > x; x, y \in D$$

is a rule of correspondence or association which, when applied on D,

$$D = \{-2, 0, 2, 4\}$$

determines the set of ordered pairs

$$\{(-2, 4), (0, 4), (2, 4), (-2, 2), (0, 2), (-2, 0)\},$$

which is a subset of $D \times D$.

Thus $D_1 = \{(x, y)|y > x, x, y \in D\}$

describes completely

a relation D_1

which is applied on D (*the universal set*)

and has defining sentence $y > x, x, y \in D$

and determines a set of ordered pairs (x, y) which is a subset of $D \times D$.

D_1 is said to be a relation *in* $D \times D$.

The set of *first elements* of the ordered pairs of D_1, that is,

$$\{-2, 0, 2\},$$

is called the *domain of the relation* D_1.

The set of *second elements* of the ordered pairs of D_1, that is,

$$\{0, 2, 4\},$$

is called the *range of the relation* D_1.

The relation D_1 is pictorially represented in *Fig. 3-6*.

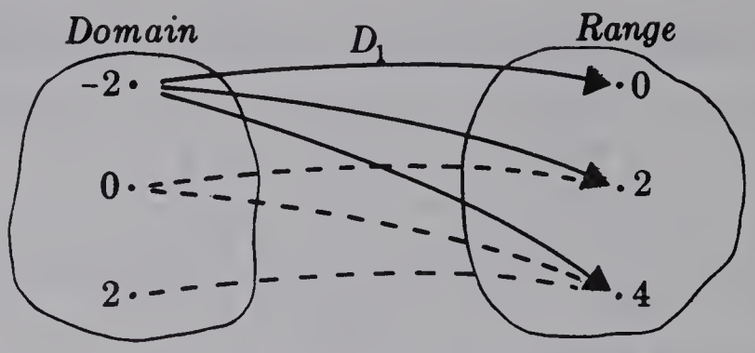


Fig. 3-6

In general, if with each element of set A there is associated one or more elements of set B, then this association is called a *relation* from A to B in $A \times B$. Since the association determines a set of ordered pairs, a relation in $A \times B$ determines a subset of $A \times B$. Also, a relation in $A \times A$ determines a subset of $A \times A$.

A relation P in $U \times U$ involves:

(i) a universal set U on which the relation is applied;

(ii) a defining sentence (rule of correspondence or association or mapping);

(iii) a set of ordered pairs determined by the defining sentence.

The relation P "is less than" applied on set U is conveniently expressed

$$P = \{(x, y) | y < x, x, y \in U\}$$

ordered pairs — universal set — defining inequation — defining sentence — P

which indicates that the relation P:

(i) is applied on U;

(ii) has the defining sentence $y < x, x, y \in U$;

(iii) determines a set of ordered pairs (x, y) which is a subset of $U \times U$.

The *domain* of a *relation* is the set of all *first* members of the ordered pairs of the relation.

The *range* of a *relation* is the set of all *second* members of the ordered pairs of the relation.

Example. Determine the ordered pairs of the following relation and state its domain and range:

$R = \{(x, y) | y = 2x, x, y \in U\}$, where $U = \{-2, -1, 0, 1, 2, 4\}$.

Solution. The *defining sentence* of this relation is $y = 2x,\ x,\ y \in U$. The relation is an equality relation defined on U and determines a subset of $U \times U$. Both x and y must be elements of U.

Thus, $R = \{(-1, -2), (0, 0), (1, 2), (2, 4)\}$.
The domain of R is $\{-1, 0, 1, 2\}$.
The range of R is $\{-2, 0, 2, 4\}$.

Write solutions to the following problems and compare them with those on page 407.

1. If $A = \{0, 1, 2, 3, 4, 5\}$, find the domain and range of the relation $B = \{(x, y) | y = x^2 + 1,\ x, y \in A\}$.

2. Find the domain and range of the relation $C = \{(x, y) | y = 2x - 1,\ x, y \in N_0\}$.

3. Find the domain and range of the relation $D = \{(x, y) | y = x,\ x, y \in R\}$.

4. Which of the ordered pairs
$(3, 4), (0, 0), (0 - 5), (4, -3), (5, 0), (0, 25), (4, 3), (6, -1)$
do not belong to the relation $E = \{(x, y) \mid x^2 + y^2 = 25,\ x, y \in I\}$?

Exercise 3-4

(A)

1. Find the domain and range of the relations:

(i) $R_1 = \{(x, y) | y = 2x + 1,\ x, y \in A\}$ where $A = \{1, 3, 5, 7\}$
(ii) $R_2 = \{(x, y) | y = 2x^2,\ x, y \in N_0\}$
(iii) $R_3 = \{(x, y) | 2x + y = 3,\ x, y \in N\}$
(iv) $R_4 = \{(x, y) | x + y = 0,\ x, y \in R\}$

2. List three ordered pairs of each of the following relations on the set I:

(i) $P_1 = \{(x, y) | y = x + 1\}$
(ii) $P_2 = \{(x, y) | x + y = 5\}$
(iii) $P_3 = \{(x, y) | y = x^2\}$
(iv) $P_4 = \{(x, y) | x = y^2\}$

(B)

3. Which of the ordered pairs $(1, 2), (2, 6), (18, 3), (0, 0), (4, 32)$ do not belong to the relation $Q = \{(x, y) | 2x^2 = y,\ x, y \in N\}$?

4. If $U = \{0, 1, 2, 3\}$ and (x, y) is an element of $U \times U$, determine the ordered pairs, and state the domain and range of each of the following relations; represent each association pictorially:

(i) $A = \{(x, y) | y = x + 1\}$
(ii) $B = \{(x, y) | x = 3\}$
(iii) $C = \{(x, y) | y > x\}$
(iv) $D = \{(x, y) | y = 4x\}$
(v) $E = \{(x, y) | y = 4\}$
(vi) $F = \{(x, y) | y = x^2\}$

5. If $A = \{-1, 0, 1\}$ and $B = \{-2, 0, 2\}$, find $A \times B$; select the ordered pairs for which the second member is double the first member, and state the domain and the range of this relation.

6. Find the domain and range of:
   (i) $F_1 = \{(x, y) | y = x^2, x, y \in R\}$
   (ii) $F_2 = \{(x, y) | x = -2, x, y \in R\}$
   (iii) $F_3 = \{(x, y) | y = 3, x, y \in R\}$
   (iv) $F_4 = \{(x, y) | x = 0, x, y \in R\}$

3·5 Relations and their graphs.

Example 1. Draw the graph of the relation

$$P_1 = \{(x, y) | y = x + 2, x, y \in I\} .$$

Solution. $P_1 = \{(x, y) | y = x + 2, x, y \in I\}$.

For P_1:

x	-3	-2	-1	0	1	2
y	-1	0	$+1$	$+2$	$+3$	$+4$

The partial graph of P_1 is shown in *Fig. 3-7*.

Domain is $\{x | x \in I\}$.
Range is $\{y | y \in I\}$.

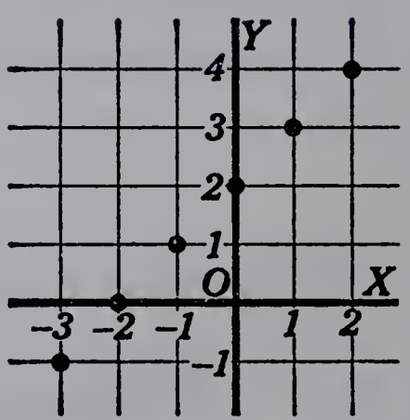


Fig. 3-7

DISCUSSION. This is an equality relation defined in $I \times I$. Both x and y must be selected from I. Since I is an infinite set, the relation determines an infinite set of ordered pairs. The graph consists of an infinite number of points. To obtain the *partial graph* of P_1, we arbitrarily select a convenient number of replacements for x from the set I, construct the corresponding table of values, and plot the corresponding points; these points are a partial graph of P_1. The complete graph cannot be drawn. The partial graph implies that the complete graph is an extension of this graph in the directions indicated by the plotted points.

The table does not state the domain and range, nor does it list all the ordered pairs. The domain and range are inferred from the nature of the graph or by the following argument.

Since $y = x + 2$, y is an integer for all replacements of x from I; $\therefore$ the domain is $\{x | x \in I\}$. Since $x = y - 2$, x is an integer for all replacements of y from I; $\therefore$ the range is $\{y | y \in I\}$.

Example 2. Draw the graph of the relation

$$P_2 = \{(x, y) | y = x + 2, x, y \in R\} .$$

Solution. $P_2 = \{(x, y) | y = x + 2, x, y \in R\}$.

For P_2:

x	-3	-2	-1	0	1	2
y	-1	0	1	2	3	4

Domain is $\{x | x \in R\}$.
Range is $\{y | y \in R\}$.

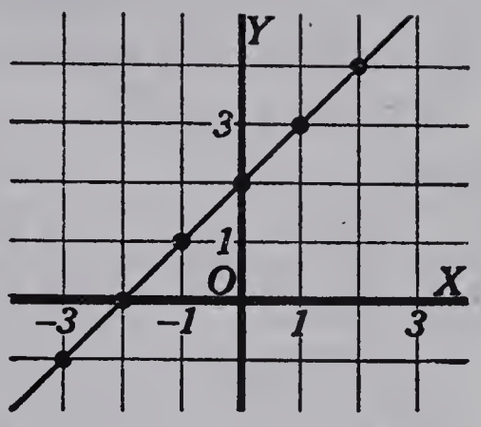


Fig. 3-8

DISCUSSION. Since the replacement set for x and y is an infinite set, the relation determines an infinite number of ordered pairs, and the graph is an infinite number of points. To obtain the graph, arbitrarily select a convenient number of replacements for x from R and construct the corresponding table of values. Plot the corresponding points. Since all real replacements for x determine a corresponding real number for y, it is a reasonable assumption that the graph is the continuous line or smooth curve drawn through the plotted points. Since these lie in a straight line, we say the graph is the straight line determined by these points.

Example 3. Draw the graph of the relation $P_3 = \{(x, y) | y = |x|, x, y \in R\}$.

Solution.

$P_3 = \{(x, y) | y = |x|, x, y \in R\}$.

For P_3:

x	-3	-2	-1	0	1	2	3
y	3	2	1	0	1	2	3

Domain is $\{x | x \in R\}$.
Range is $\{y | y \geqq 0, y \in R\}$.

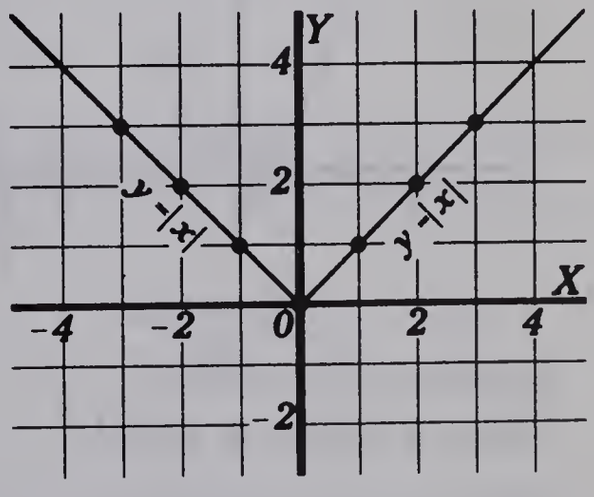


Fig. 3-9

DISCUSSION. The definition of absolute value suggests that a table of values for P_3 should include as replacements for x both positive and negative real numbers and zero. Both the plotted points and the defining sentence suggest that the graph is the union of the two rays from the origin partially indicated in *Fig. 3-9*. The graph also suggests that the domain of P_3 is $\{x | x \in R\}$, and the range is $\{y | y \geqq 0, y \in R\}$.

Example 4. Draw the graph of $P_2 = \{x, y | y = x^2, x, y \in R\}$ and state the domain and range of the relation.

Solution.

$P_2 = \{(x, y) | y = x^2, x, y \in R\}$.

$\because$ $x \in R$, then x^2 (or y) $\in R$.

$\therefore$ $P_2 = \{(x, x^2) | x \in R\}$.

For P_2:

x	-3	-2	-1.4	-1
y	9	4	1.96	1

$-.5$	0	.5	1	1.4	2	3
.25	0	.25	1	1.96	4	9

The partial graph is shown in *Fig. 3-10*.

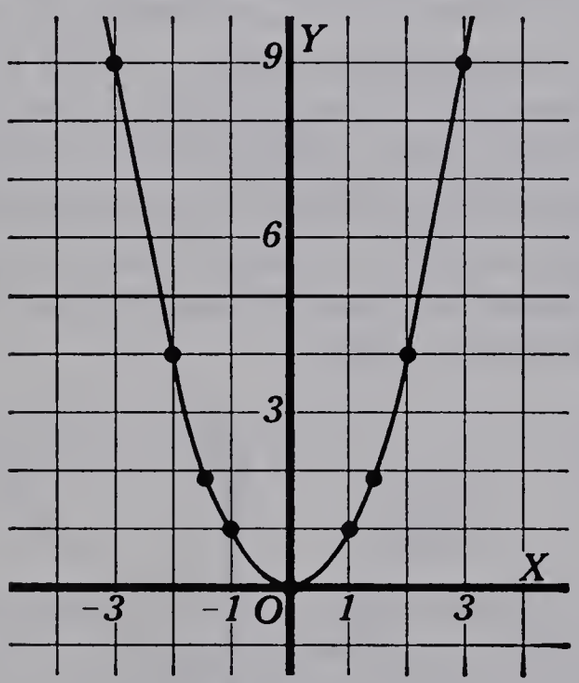


Fig. 3-10

Domain is $\{x | x \in R\}$.

Range is $\{y | y \geqq 0, y \in R\}$.

DISCUSSION. The table of values for P_1 in Example 4 is satisfactory for establishing the pattern of the graph of P_2. In the table given, a few more replacement values are used to illustrate that the smooth curve drawn through the plotted points really is a good approximation to the required graph.

Both the graph and the defining equation, $y = x^2$, suggest that as x assumes all real values in the interval -3 to $+3$, y assumes all real values in the range interval 0 to $+9$. Also, since x may be replaced by any real number, the curve extends indefinitely beyond the points whose coordinates are $(-3, 9)$ and $(3, 9)$.

The graph suggests that:
the domain is $\{x | x \in R\}$;
the range is $\{y | y \geqq 0, y \in R\}$.

Write solutions to the following problems and compare them with those on page 408.

1. Draw the graph of the relation
$M = \{(x, y) | 2x + y = 3, x, y \in R\}$, and state its domain and range.

2. Draw the graph of the relation
$N = \{(x, y) | x = 3, x, y \in R\}$, and state its domain and range.

Exercise 3-5

(A)

1. If $x, y \in R$, and the following are defining equations of relations, express the ordered pairs (x, y) of the relations in terms of the variable x:

(i) $3x = y$ (ii) $x - 2y = 0$ (iii) $x + y = 2$
(iv) $4x - y = 5$ (v) $3x + 4y = 0$ (vi) $2x + 3y = 4$
(vii) $5x + 2y + 6 = 0$ (viii) $ax - y = b$ (ix) $px = qy$
(x) $cx + dy = e$ (xi) $4x^2 - y = 0$ (xii) $x^2 + y - 2 = 0$

2. For $x, y \in I$, find three ordered pairs defined by the following equations:

(i) $2x + y = 3$ (ii) $3x - y = 5$ (iii) $x + 2y = 7$
(iv) $y = 2x^2$ (v) $xy = 12$ (vi) $y = x^2 - 2$

3. Determine which of the following ordered pairs satisfy the relation $A = \{(x, y) | x + 2y = 10, x, y \in I\}$:

(i) $(0, 5)$ (ii) $(2, 5)$ (iii) $(-3, 7)$
(iv) $(-2, 6)$ (v) $(4, 3)$ (vi) $(9, \frac{1}{2})$

(B)

In each of the following, list the ordered pairs determined by the relation and state its domain and range. $U = \{-1, 0, 1, 2, 3, 4\}$ *and* (x, y) *belongs to* $U \times U$.

4. $A = \{(x, y) | y = x\}$
5. $B = \{(x, y) | y = 2x\}$
6. $C = \{(x, y) | x < 2y\}$
7. $D = \{(x, y) | y = 4\}$
8. $E = \{(x, y) | y = x + 1\}$
9. $F = \{(x, y) | x = -1\}$

In each of the following, list four ordered pairs determined by the relation and state its domain and range, if $x, y \in I$:

10. $M = \{(x, y) | x = -3\}$
11. $N = \{(x, y) | y = 2\}$
12. $P = \{(x, y) | y = x - 1, 0 < x < 10\}$
13. $V = \{(x, y) | x = y^2\}$
14. $Q = \{(x, y) | y = 2|x|, -4 < x < 3\}$
15. $R = \{(x, y) | 3x + 2y = 6\}$
16. $S = \{(x, y) | y = 0\}$
17. $T = \{(x, y) | x = 0\}$

Draw the graph of each of the following relations; state the domain and range:

18. $A = \{(x, y)|x = 4, x, y \in I\}$ **19.** $B = \{(x, y)|y = 3, x, y \in R\}$

20. $C = \{(x, y)|y = x, x, y \in I\}$ **21.** $D = \{(x, y)|y = x + 1, x, y \in R\}$

22. $E = \{(x, y)|x + y = 2, x, y \in I\}$

23. $F = \{(x, y)|x + 2y = 3, x, y \in R\}$

24. $G = \{(x, y)|x = 3y - 2, 0 \leqq x \leqq 8, x \in R\}$

25. $H = \{(x, y)|3x - 2y = 6, x, y \in R\}$

26. $J = \{(x, y)|2x + 5y = 10, x, y \in R\}$

3·6 Linear relations. It has been observed that relations whose defining sentences are equations of the first degree in two variables, for example $2x - 3y = 6$, $x, y \in R$, appear to have graphs which are straight lines. We assume that this is true. Such relations are referred to as *linear relations.* Other relations, defined by inequations of the first degree in two variables, are also referred to as linear relations; the inequation is referred to as a *linear inequation.* Examples of inequality relations are discussed in section 3·9.

3·7 The x- and y-intercepts of a graph. The graph of the linear relation, $F = \{(x, y)|2x - 3y = 6, x, y \in R\}$, is illustrated in *Fig. 3-11*.

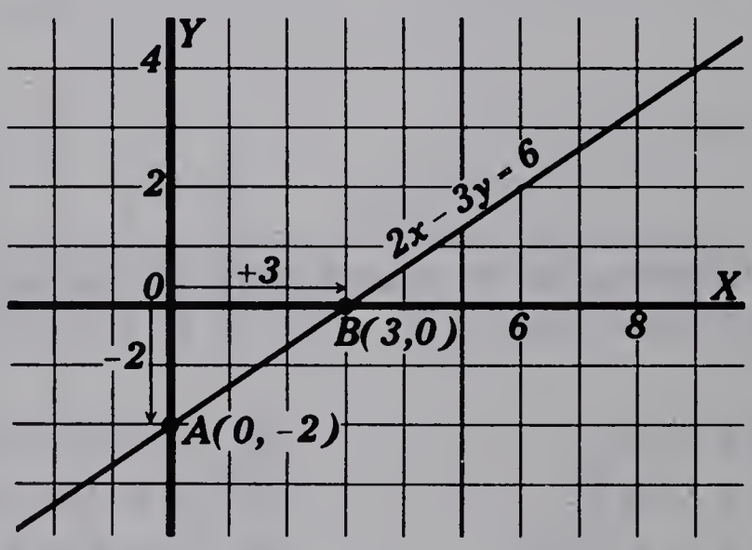


Fig. 3-11

The graph crosses the x-axis at the point B (3, 0). This point is a directed distance, + 3, from the origin. The graph crosses the y-axis at the point $A(0, -2)$. This point is a directed distance, − 2, from the origin. These directed distances are called the *x-intercept* and the *y-intercept*, respectively, of the graph. The points $B(3, 0)$ and $A(0, -2)$ are the points of intersection of the graph with the x- and y- axis, respectively.

To determine the x-intercept, let $y = 0$ in the defining sentence of the relation.

Thus, in this example, if $y = 0$, then $2x = 6$

$$\therefore x = 3$$

$\therefore$ the x-intercept is 3.

Similarly, if $x = 0$, then $-3y = 6$

$$\therefore y = -2$$

$\therefore$ the y-intercept is -2.

The position of a straight line graph may be determined by finding its x- and y-intercepts, provided the line does not pass through the origin.

For *any* relation, the *y-intercept(s)* of the graph are the *ordinates* of the point(s) of intersection of the graph with the y-axis. Similarly, the *x-intercept(s)* of the graph are the *abscissas* of the point(s) of intersection of the graph with the x-axis.

The x-intercept of the graph is obtained by letting $y = 0$ in the defining sentence and determining the corresponding x.

The y-intercept of the graph is obtained by letting $x = 0$ in the defining sentence and determining the corresponding y.

Write solutions for the following; compare them with those on page 409.

1. For the linear relations

(i) $L_1 = \{(x, y) \mid 2x + y = 4, x, y \in R\}$,

(ii) $L_2 = \{(x, y) \mid x = 2, x, y \in R\}$,

(iii) $L_3 = \{(x, y) \mid y = -1, x, y \in R\}$,

(a) determine the intercepts of the graph;

(b) make a table of values to include at least three ordered pairs and sketch the graph.

Exercise 3-6

(B)

For each of the following linear relations: (i) *determine the intercepts of the graph;* (ii) *make a table of values and sketch the graph.*

1. $A = \{(x, y) \mid y = -3x + 1, x, y \in R\}$

2. $B = \{(x, y) \mid x - 2y + 3 = 0, x, y \in R\}$

3. $C = \{(x, y) \mid 3x + 4y = 12, x, y \in R\}$

4. $D = \{(x, y) \mid 3x + 4y + 12 = 0, x, y \in R\}$

5. $E = \{(x, y) \mid y + 3 = 0, x, y \in R\}$

6. $F = \{(x, y) \mid 2x + 1 = 0, x, y \in R\}$

7. $G = \{(x, y) \mid 2x + 3y = 5, -5 \leqq x \leqq 6, x \in I, y \in Q\}$

8. $H = \{(u, v) \mid v = 2u - 1, -2 \leqq u \leqq 3, u, v \in R\}$

3·8 Algebraic determination of domain and range of a relation.

Two important aspects of any relation, its *domain* and its *range*, are introduced in Section 3·4.

DEFINITION: *The domain of a relation A on the set of real numbers is the set of all real numbers x which are the first elements of the ordered pairs (x, y) of A.*

In set-builder notation,

$$\text{Domain of relation } A = \{x \mid (x, y) \in A\}.$$

DEFINITION: *The range of a relation A on the set of real numbers is the set of all real numbers y which are the second elements of the ordered pairs (x, y) of A.*

In set-builder notation,

$$\text{Range of relation } A = \{y \mid (x, y) \in A\}.$$

For many relations the domain and range *must be determined by an analysis of the defining sentence.* The type of analysis involved is illustrated in the following examples.

Example 1. For the linear relation $L = \{(x, y) \mid y - x = 3, x, y \in R\}$:

(i) determine the intercepts of the graph;

(ii) determine the domain and range of the relation;

(iii) make a table of values and sketch the graph.

Solution.

(i) *x-intercepts.* Let $y = 0$ in $y - x = 3$.

$\therefore -x = 3,$

$\therefore x = -3.$

$\therefore$ the x-intercept is -3.

y-intercepts. Let $x = 0$ in $y - x = 3$.

$\therefore y = 3.$

$\therefore$ the y-intercept is 3.

(ii) *Domain.*

$$y - x = 3$$
$$\leftrightarrow \quad y = 3 + x.$$
$$\therefore \; y \in R \leftrightarrow (x + 3) \in R.$$

But $(x + 3) \in R$ for all $x \in R$.

$\therefore$ the domain of the relation L is $\{x \mid x \in R\}$.

Range.

$$y - x = 3$$
$$\leftrightarrow \quad x = y - 3.$$
$$\therefore \; x \in R \leftrightarrow (y - 3) \in R.$$

But $(y - 3) \in R$ for all $y \in R$.

$\therefore$ the range of the relation L is $\{y \mid y \in R\}$.

(iii) *Table of values.*

x	0	1	2	-1	-2	-3
y(or $x + 3$)	3	4	5	2	1	0

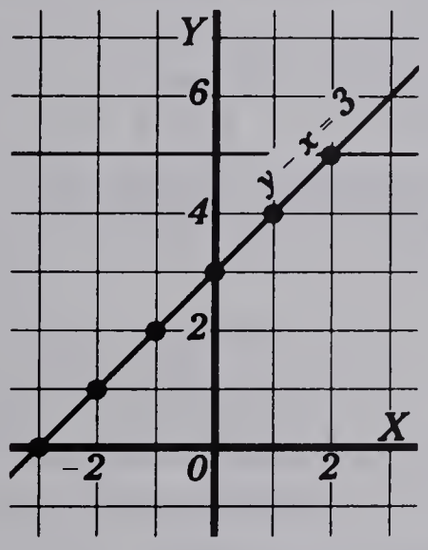


Fig. 3-12

Example 2. For the relation $B = \{(u, v) \mid u^2 + v^2 = 25, u, v \in R\}$:

(i) determine the intercepts of the graph;

(ii) determine the domain and range of the relation;

(iii) make a table of values and sketch the graph.

Solution.

(i) *u-intercepts.* Let $v = 0$ in $u^2 + v^2 = 25$.

$$\therefore \; u^2 = 25,$$
$$\therefore \; u = \pm 5.$$

$\therefore$ the u-intercepts are ± 5.

v-intercepts. Let $u = 0$ in $u^2 + v^2 = 25$.

$$\therefore\ v^2 = 25,$$

$$\therefore\ v = \pm 5.$$

$\therefore$ the v-intercepts are ± 5.

(ii) *Domain.*

$$u^2 + v^2 = 25$$

$$\leftrightarrow \quad v^2 = 25 - u^2$$

$$\leftrightarrow \quad v = \pm\sqrt{25 - u^2}.$$

$$\therefore\ v \in R \leftrightarrow 25 - u^2 \geqq 0$$

$$\leftrightarrow \quad u^2 \leqq 25$$

$$\leftrightarrow \quad |u| \leqq 5$$

$$\leftrightarrow \quad -5 \leqq u \leqq 5.$$

$\therefore$ the domain of B is $\{u \mid -5 \leqq u \leqq 5,\ u \in R\}$.

Range.

$$u^2 + v^2 = 25$$

$$\leftrightarrow \quad u^2 = 25 - v^2$$

$$\leftrightarrow \quad u = \pm\sqrt{25 - v^2}.$$

It follows that the range of B is $\{v \mid -5 \leqq v \leqq 5,\ v \in R\}$.

Since the domain of B is $\{u \mid -5 \leqq u \leqq 5\}$, the graph of B must be on or between the vertical lines defined by $u = -5$ and $u = 5$.

Since the range of B is $\{v \mid -5 \leqq v \leqq 5\}$, the graph of B must be on or between the horizontal lines defined by $v = -5$ and $v = 5$.

It follows that the graph of B is a subset of the square region outlined by broken lines in *Fig. 3-13*.

(c) *Table of values.*

u	0	3	4	5	−3	−4	−5
v	± 5	± 4	± 3	0	± 4	± 3	0

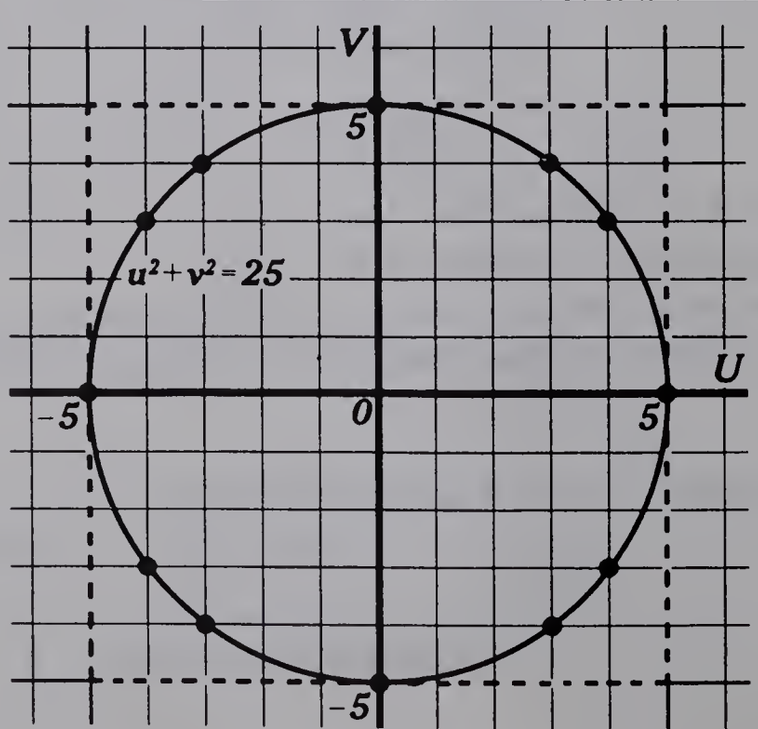


Fig. 3-13

Exercise 3-7

(A)

1. What is the domain of a relation?
2. What is the range of a relation?

(B)

For each of the following relations, find the intercepts of the graph, the domain, and the range of the relation from an analysis of the defining conditions. Sketch the graph.

3. $A = \{(u, v) \mid u^2 + v^2 = 16, u, v \in R\}$
4. $B = \{(r, s) \mid r^2 + s^2 = 4, r, s \in R\}$
5. $C = \{(x, y) \mid 3x - y = 6, x, y \in R\}$
6. $D = \{(x, y) \mid y - 2 = 0, x, y \in R\}$
7. $E = \{(u, v) \mid u = v, -2 \leqq u \leqq 3, u, v \in R\}$
8. $F = \{(p, q) \mid 2p - q + 1 = 0, -3 \leqq p \leqq 3, p, q \in R\}$
9. $G = \{(x, y) \mid y = -2x - 3, x, y \in I\}$
10. $H = \{(x, y) \mid y = |x|, -4 \leqq x \leqq 4, x, y \in R\}$
11. $R = \{(x, y) \mid (x - 1)^2 + (y + 1)^2 = 25, x, y \in R\}$

3·9 Analysis of relations defined by inequations.

Example 1. For the relation

$$A = \{(x, y) \mid y \geqq 2x + 3, x, y \in R\}:$$

(i) sketch the graph;

(ii) from the graph determine the intercepts of the graph and the domain and range of the relation.

Solution.

(i) $y \geqq 2x + 3$

The graph of $y = 2x + 3$ is a straight line with x-intercept $-\frac{3}{2}$, y-intercept 3.

The graph of A is the set of all points with coordinates (x, y) on and above the line defined by $y = 2x + 3$ (*Fig. 3-14*).

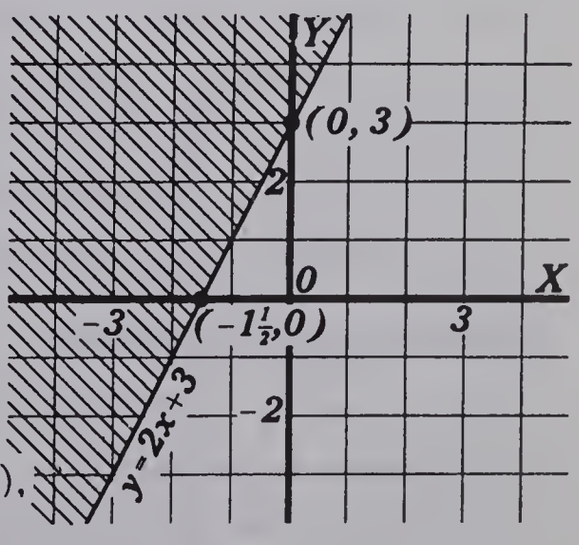


Fig. 3-14

(ii) *x-intercepts.* Every $x \in R$ such that $x \leqq -\frac{3}{2}$ is an x-intercept, that is, each member of $\{x \mid x \leqq -\frac{3}{2}, x \in R\}$ is an x-intercept.

y-intercepts. Every $y \in R$ such that $y \geqq 3$ is a y-intercept, that is, each member of $\{y \mid y \geqq 3, y \in R\}$ is a y-intercept.

Domain. The domain of A is $\{x \mid x \in R\}$.

Range. The range of A is $\{y \mid y \in R\}$.

Example 2. For the relation

$$C = \{(x, y) \mid y > x, -3 < x \leqq 3, x, y \in R\}:$$

(i) sketch the graph;

(ii) from the graph determine the intercepts of the graph and the domain and range of the relation.

Solution. (i) $y > x$

The graph of $y = x$ for $-3 < x \leqq 3$, $x, y \in R$, is a segment of a straight line open at the end for which $x = -3$, and which lies on the origin (0, 0) and the point with coordinates (1, 1).

The graph of C *Fig. 3-15* is the set of all points with coordinates (x, y) above the line segment. The broken lines (*Fig. 3-15*) indicate that the points of these boundaries are not included in the graph.

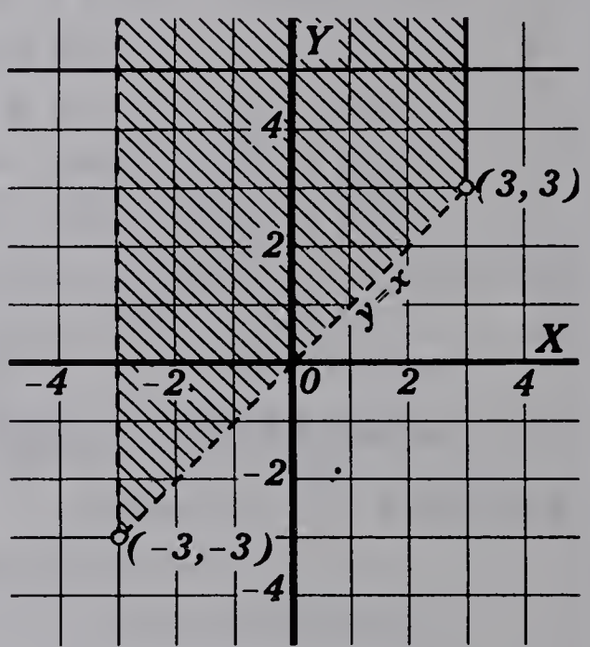


Fig. 3-15

(ii) *x-intercepts.* Every $x \in R$ such that $-3 < x < 0$ is an x-intercept, that is every element of $\{x \mid -3 < x < 0, x \in R\}$ is an x-intercept.

y-intercepts. Every $y \in R$ such that $y > 0$ is a y-intercept, that is every element of $\{y \mid y > 0, y \in R\}$ is a y-intercept.

Domain. The domain of C is $\{x \mid -3 < x \leqq 3, x \in R\}$.

Range. The range of C is $\{y \mid y > -3, y \in R\}$.

Example 3. For the relation

$$D = \{(r, s) \mid r \geqq |s - 1|, r, s \in R\}:$$

(i) sketch the graph;

(ii) from the graph determine the intercepts of the graph and the domain and range of the relation.

The absolute value of a real number x is defined as follows:

$$\text{if } x \geqq 0, \text{ then } |x| = x,$$

$$\text{if } x < 0, \text{ then } |x| = -x.$$

Solution. Since $r \geqq |s - 1|$, two cases are to be considered.

Case 1.

If $s - 1 \geqq 0$, that is $s \geqq 1$, then $|s - 1| = s - 1$ and D is defined in part by the sentence

$$r \geqq s - 1,\ s \geqq 1,$$

$$\text{or } s \leqq r + 1,\ s \geqq 1.$$

The graph of the part of D so defined is the set of all points on and below the line whose equation is $r = s - 1$ (s-intercept 1, r-intercept -1) and for which $s \geqq 1$ (*Fig. 3-16*).

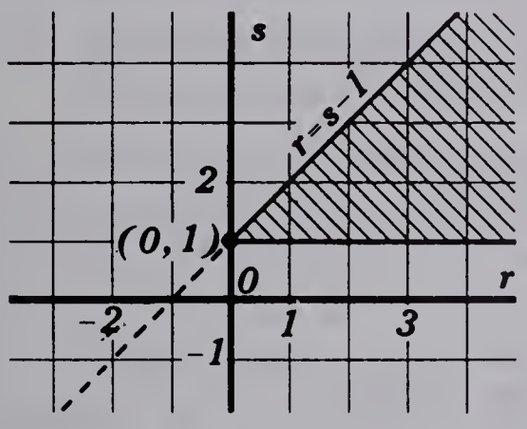


Fig. 3-16

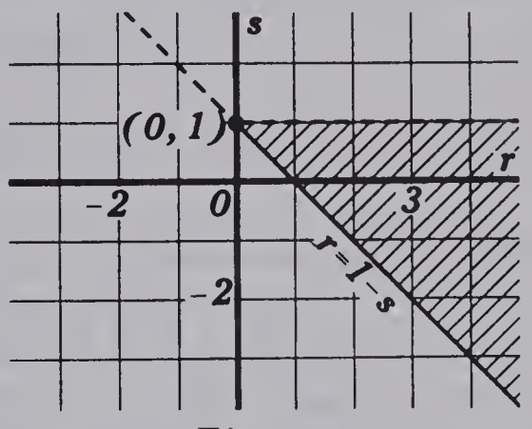


Fig. 3-17

Case 2.

If $s - 1 < 0$, that is $s < 1$, then $|s - 1| = 1 - s$ and D is defined in part by the sentence

$$r \geqq 1 - s,\ s < 1,$$

$$\text{or } s \geqq 1 - r,\ s < 1.$$

(i) The graph of the part of D so defined is the set of all points on and above the line with equation $r = 1 - s$ (s-intercept 1, r-intercept 1) for which $s < 1$ (*Fig. 3-17*).

The graph of D is the union of these two sets and is illustrated in *Fig. 3-18*.

(ii) *r-intercepts.* Every $r \in R$ such that $r \geqq 1$ is an r-intercept, that is, every element of $\{r \mid r \geqq 1,\ r \in R\}$ is an r-intercept.

s-intercepts. The s-intercept is 1.

Domain. The domain of D is $\{r \mid r \geqq 0,\ r \in R\}$.

Range. The range of D is $\{s \mid s \in R\}$.

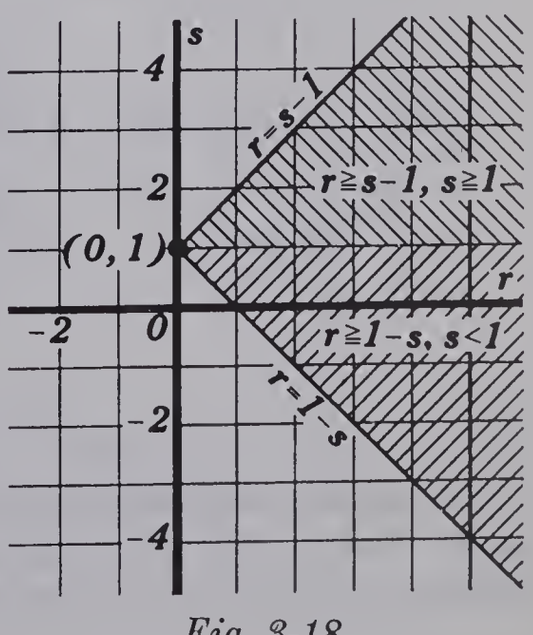


Fig. 3-18

Exercise 3-8

(B)

For each of the following relations:

(i) *sketch the graph;*

(ii) *from the graph, determine the intercepts of the graph and the domain and range of the relation.*

1. $A = \{(x, y) \mid y < x, -3 < x < 3, x, y \in R\}$

2. $B = \{(x, y) \mid y \geqq 3x - 2, x, y \in R\}$

3. $C = \{(u, v) \mid u \leqq v + 2, u, v \in R\}$

4. $D = \{(r, s) \mid 1 \leqq r + s, r, s \in R\}$

5. $E = \{(x, y) \mid y > |x|, -3 \leqq x \leqq 4, x, y \in R\}$

6. $F = \{(x, y) \mid |y| \geqq x, x, y \in R\}$

(C)

7. $G = \{(u, v) \mid |v| = u, u, v \in I\}$

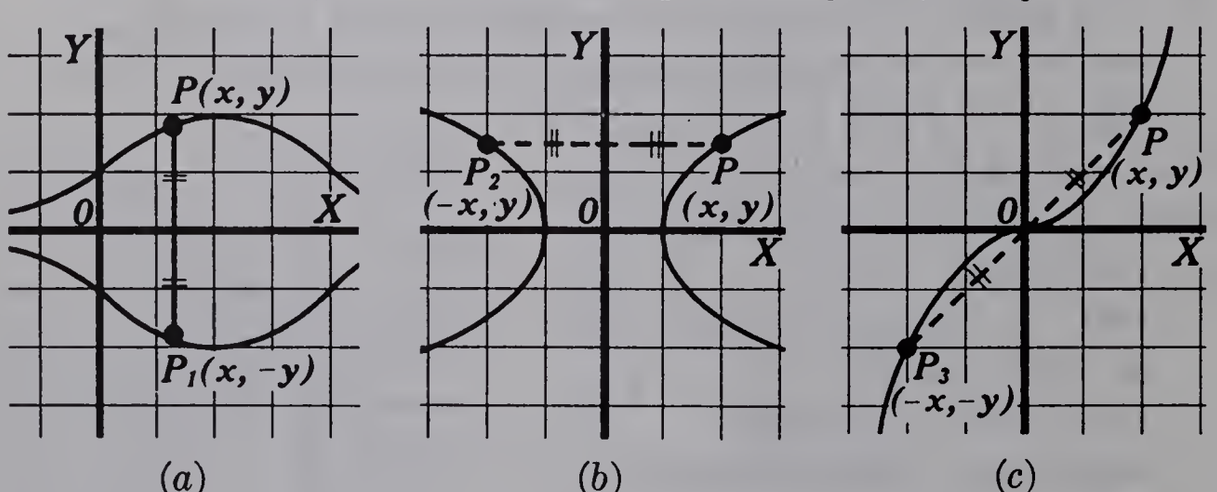
8. $H = \{(x, y) \mid |y| \leqq x, x, y \in R\}$

9. $K = \{(r, s) \mid s < |r|, -4 < r < 4, r, s \in R\}$

10. $L = \{(x, y) \mid |x - y| \leqq 1, x, y \in R\}$

11. $M = \{(x, y) \mid y^2 = x - 1, x, y \in R\}$

3·10 **Relations; symmetry with respect to the axes and origin.** Certain relations have graphs which possess properties of symmetry with respect to one or more straight lines or with respect to a point, . (*Fig 3-19*).



(*a*) (*b*) (*c*)

Fig. 3-19

The graph in *Fig. 3-19* (*a*) is said to be symmetric with respect to the x-axis, that in *Fig. 3-19* (*b*) to be symmetric with respect to the y-axis, and that in *Fig. 3-19* (*c*) to be symmetric with respect to the origin. Using

geometric language, the first graph has the property that if $P(x, y)$ is any point on the graph, then the *reflection of this point in the x-axis* (that is, the point $P_1(x, -y)$) is also on the graph. Similarly, the second graph has the property that the *reflection* $P_2(-x, y)$, *in the y-axis* of any point $P(x, y)$ on the graph, is also on the graph. In the case of the third graph (symmetry with respect to the origin), the geometric property involved is that the *reflection* $P_3(-x, -y)$, *through the origin* of any point $P(x, y)$ on the graph is also on the graph.

DEFINITION: The graph of a relation is:

(a) *symmetric with respect to the x-axis* if the point $P_1(x, -y)$ is on the graph whenever the point $P(x, y)$ is on the graph;

(b) *symmetric with respect to the y-axis* if the point $P_2(-x, y)$ is on the graph whenever the point $P(x, y)$ is on the graph;

(c) *symmetric with respect to the origin* if the point $P_3(-x, -y)$ is on the graph whenever the point $P(x, y)$ is on the graph.

It is clear that if a graph is symmetric with respect to *both* the x-axis and the y-axis, then it is also symmetric with respect to the origin. This was the case in *Fig. 3-19* (*b*), for example. The converse of this, however, is not true in general, as can be seen from *Fig. 3-19* (*c*).

The following *tests for symmetry* are simple to apply and should be remembered by the student. They are immediate consequents of the definitions (a), (b), (c) given above. *The graph of a relation is*

(i) *symmetric with respect to the x-axis if the defining equation (or other defining conditions) of the relation remain unchanged when y is replaced by* $(-y)$;

(ii) *symmetric with respect to the y-axis if the defining conditions of the relation remain unchanged when x is replaced by* $(-x)$;

(iii) *symmetric with respect to the origin if the defining conditions of the relation remain unchanged when both x and y are replaced by* $(-x)$ *and* $(-y)$, *respectively.*

Example 1. Discuss symmetry with respect to the axes and the origin of the graph of each of the relations

(i) $A = \{(x, y \mid x^2 + y^2 = 25, x, y \in R\}$,

(ii) $B = \{(x, y) \mid y = x^2, x, y \in R\}$.

Solution.

(i) The defining equation is $x^2 + y^2 = 25$.
If x is replaced by $(-x)$, the equation becomes $(-x)^2 + y^2 = 25$ or $x^2 + y^2 = 25$. Since the equation is unchanged the graph of A is symmetric with respect to the y-axis.

If y is replaced by $-y$ the equation becomes $x^2 + (-y)^2 = 25$ or $x^2 + y^2 = 25$. The graph of A is symmetric with respect to the x-axis.

Since the graph is symmetric with respect to both axes it is symmetric with respect to the origin. (The graph of A is a circle with centre the origin and radius 5, *Fig. 3-13* page 58.)

(ii) The defining equation is $y = x^2$.

Substituting $-x$ for x, the equation becomes $y = (-x)^2$ or $y = x^2$.

Substituting $-y$ for y, the equation becomes $-y = x^2$ or $y = -x^2$.

$\therefore$ the graph is symmetric with respect to the y-axis. (The graph is shown in *Fig. 3-20*. The curve is a parabola.)

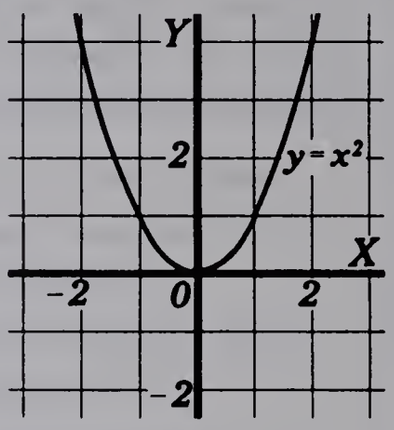


Fig. 3-20

Do the following problems and compare your solutions with those on page 410.

Discuss symmetry with respect to the axes and the origin of the graph of each of the following relations:

1. $C = \{(x, y) \mid |y| = x,\ x, y \in R\}$.

2. $D = \{(x, y) \mid y^3 = x,\ x, y \in R\}$.

3. For the relation $A = \{(u, v) \mid u = v^2,\ u, v \in R\}$:

(i) determine the intercepts of the graph;

(ii) determine the domain and range of the relation;

(iii) discuss the symmetry of the graph;

(iv) make a table of values and sketch the graph.

Exercise 3-9

(A)

Discuss the symmetry, with respect to the axes and the origin, of the graphs of each of the following relations:

1. $A = \{(x, y) \mid y = 4x^2,\ x, y \in R\}$

2. $B = \{(x, y) \mid y^2 = 2x + 1,\ x, y \in R\}$

3. $C = \{(x, y) \mid y^2 - x^2 = 1,\ x, y \in R\}$

4. $D = \{(x, y) \mid 3x^2 + 4y^2 = 12,\ x, y \in R\}$

5. $E = \{(x, y) \mid y = x^3,\ x, y \in R\}$

6. $F = \{(x, y) \mid x^2 + y^2 - 2x = 0,\ x, y \in R\}$

(B)

For each of the following relations, determine the intercepts of the graph, the domain and range of the relation; discuss the symmetry, and sketch the graph:

7. $P = \{(u, v) \mid v = 2u^2,\ u, v \in R\}$

8. $Q = \{(x, y) \mid y = x^2 + 1,\ -3 \leqq x \leqq 3,\ x, y \in R\}$

9. $R = \{(r, s) \mid s = r^2 - 1,\ -3 \leqq r \leqq 3,\ r, s \in I\}$

10. $S = \{(x, y) \mid 4y^2 = x,\ x, y \in R\}$

11. $T = \{(u, v) \mid v^2 \leqq u,\ u, v \in R\}$

12. $L = \{(x, y) \mid x^2 + y^2 = 1,\ x, y \in R\}$

13. $M = \{(x, y) \mid y = |x|,\ x, y \in R\}$

(C)

14. $A = \{(x, y) \mid y = 8x^3,\ x, y \in R\}$

15. $B = \{(p, q) \mid |q| = |p|,\ p, q \in I\}$

16. $C = \{(x, y) \mid |y| \geqq |x|,\ x, y \in R\}$

17. $D = \{(x, y) \mid |x| + |y| = 2,\ x, y \in R\}$

Chapter IV

FUNCTIONS

4·1 Discovery of the concept of a function.

Write solutions for the following problems and compare your solutions with those on page 411.

1. For the relation

$$R_1 = \{(x, y) \mid y = 2x - 1, -1 \leqq x \leqq 2, x, y \in I\}:$$

(i) list the ordered pairs determined by R_1;

(ii) state the domain and range of R_1;

(iii) how many elements of the range correspond to each element of the domain?

(iv) draw the graph of the relation;

(v) if a line drawn parallel to the y-axis intersects the graph, in how many points does it do so?

2. For the relation

$$R_2 = \{(x, y) \mid y^2 = 4x, x \leqq 4, x, y \in I\}:$$

(i) list the ordered pairs determined by the relation;

(ii) state the domain and range of R_2;

(iii) what elements of the range correspond to the number 1 of the domain?

(iv) how many elements of the range correspond to each element of the domain?

(v) draw the graph of R_2;

(vi) if a line drawn parallel to the y-axis intersects the graph, in how many points does it do so?

3. For the relation

$$R_3 = \{(x, y) \mid y = x^2 - 2, |x| \leqq 3, x, y \in R\}:$$

(i) list six ordered pairs determined by the relation;

(ii) determine the intercepts of the graph;

(iii) plot the points corresponding to the ordered pairs and the intercepts;

(iv) draw a smooth curve through the points;

(v) from the graph, state the domain and range of the relation;

(vi) from the graph, determine whether there is a unique element of the range for each element of the domain;

(vii) if a line drawn parallel to the y-axis intersects the graph, in how many points does it do so?

4. For the relation

$$R_4 = \{(x, y) \mid x = 2, |y| \leqq 4, x, y \in I\}:$$

(i) list the ordered pairs determined by the relation;

(ii) state the domain and range of the relation;

(iii) how many elements of the range correspond to the element 2 of the domain?

(iv) If a line drawn parallel to the y-axis intersects the graph, in how many points does it do so?

5. A relation in which each element of the domain is associated with one and only one element of the range is called a *function.*

(i) Which of the relations R_1, R_2, R_3, and R_4 in problems 1 to 4 are functions?

(ii) If a line drawn parallel to the y-axis intersects the graph of a function, in how many points does it do so?

DEFINITION: *A function on a set A is a relation on A such that for every element of the domain there corresponds a unique element of the range.*

4·2 A function as a mapping. The word function is used to represent a special kind of *association* or *correspondence* between the members of two sets; one set is the domain of the function, and the second set is the range of the function.

If with each element of the domain D there is associated in some way a unique element of the range R, then this association is called a *function from D to R.*

Fig. 4-1 illustrates such an association. D is represented by the set of three points A_1, A_2, A_3 and R is represented by the set of two points B_1, B_2.

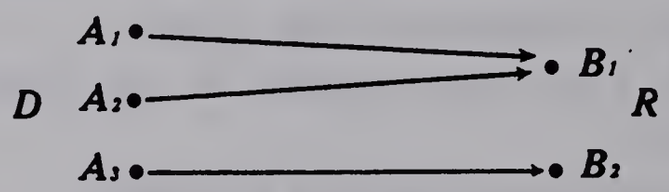


Fig. 4-1

We sometimes say the function *maps D* onto R. In *Fig. 4-1* the function is represented by the arrows drawn from the points representing elements of the domain to the points representing elements of the range; A_1 is mapped into B_1, A_2 into B_1, and A_3 into B_2. The ordered pairs of the function are (A_1, B_1), (A_2, B_1), (A_3, B_2).

Consider the association illustrated in *Fig. 4-2* and state why the association of the four points A, B, C, D of (1) onto the four points E, F, G, H of (2) is not a function.

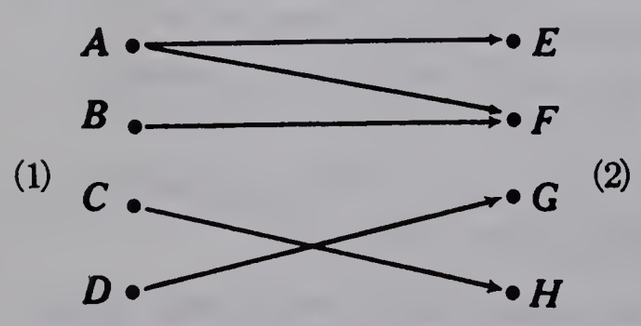


Fig. 4-2

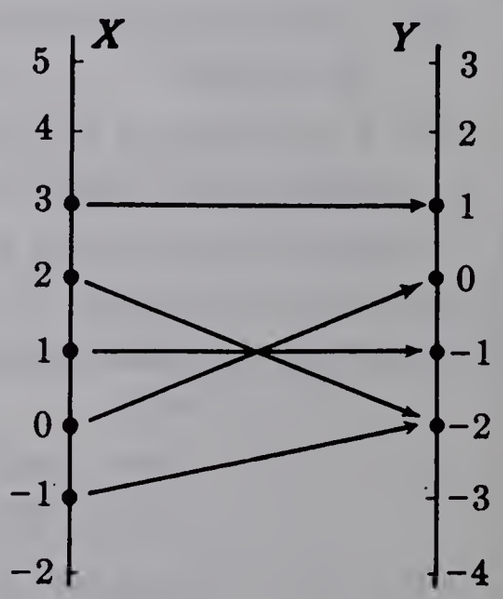


Fig. 4-3

In *Fig. 4-3* the two sets of real numbers $\{-1, 0, 1, 2, 3\}$ and $\{-2, -1, 0, -1\}$ are represented by particular points on two number lines. Tell why the mapping represented by the arrows is a function. State the domain and range of the function and list the ordered pairs of the function.

The function $h = \{(x, y) \mid y = 2x, x \in I\}$,

or $h = \{(x, 2x) \mid x \in I\}$

is a mapping of the set I of all integers *onto* the set E of all even integers. This mapping is partially illustrated in *Fig. 4-4.*

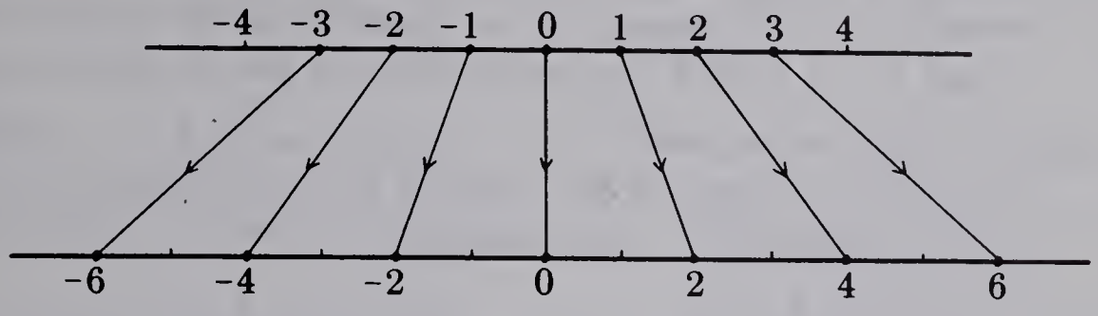


Fig. 4-4

This function (or mapping) is said to *map* each element $x \in I$ onto its *image* $2x$. That is, $2x$ is said to be the image of x under the mapping h. This is usually indicated by writing

$$h : x \to 2x, x \in I .$$

Notice that this notation suffices to completely specify the function h, since it specifies the *domain* I of h and tells us that for each $x \in I$ the corresponding element (or image of x) in the range of h is $2x$. The range of h is E.

Similarly, the mapping defined by

$$f : x \to |x|, x \in R$$

is the function whose domain is R and which maps each $x \in R$ onto its image $|x|$. That is, in terms of ordered pairs,

$$f = \{(x, y) \mid y = |x|, x \in R\} .$$

In this case, f is a mapping of the set of all real numbers onto the set of all non-negative real numbers.

Functions or mappings are classified as either *one-to-one* or *many-to-one.* A function f is said to be *one-to-one* if each element of the range of f is the image under f of exactly one element of the domain of f. If, however, there is at least one element of the range of f which is the image under f of two or more elements of the domain of f, then f is said to be a *many-to-one* mapping.

Example 1. For each of the following, state whether the function is one-to-one or many-to-one:

(i) $h : x \to 2x, x \in I$;

(ii) $f : x \to |x|, x \in R$;

(iii) $g : x \to 2, x \in Q$.

Solution.

(i) h is one-to-one. Each element $2x \in E$ of the range of h is the image of the single element $x \in I$ under the mapping h.

Another method of showing this: Note that if $2x_1$ and $2x_2$ are elements of the range of h, then $2x_1 = 2x_2 \leftrightarrow x_1 = x_2$. This states that the images of two elements of the domain of h are equal if and only if the two elements of the domain are equal.

(ii) f is many-to-one, since $|x_1| = |x_2| \leftrightarrow x_2 = \pm x_1$. Hence, each (non-zero) element of the range of f is the image of *two* elements of the domain of f. For example, 1 is the image under f of both 1 and -1.

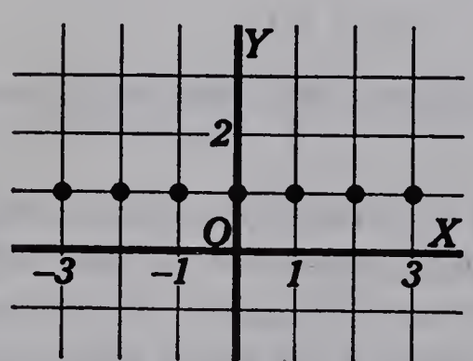
(iii) g is many-to-one, since the image under g of *every* element of the domain Q is the same element, namely 2. This type of many-to-one function is called a *constant function.*

Exercise 4-1

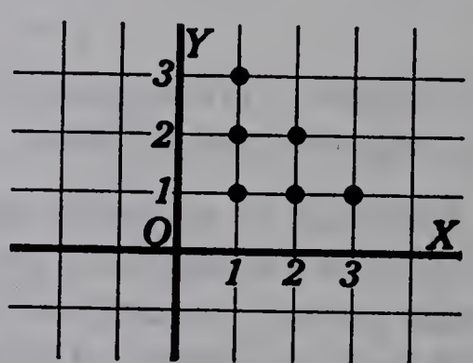
(A)

State which of the following are graphs of functions and classify each function as one-to-one, or many-to-one:

1.



2.



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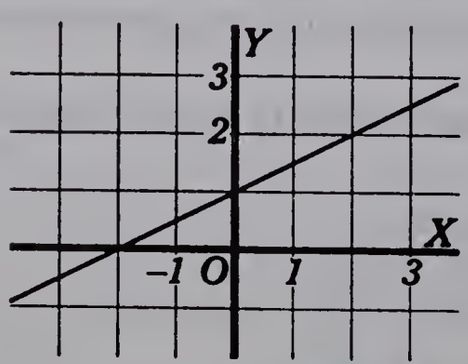

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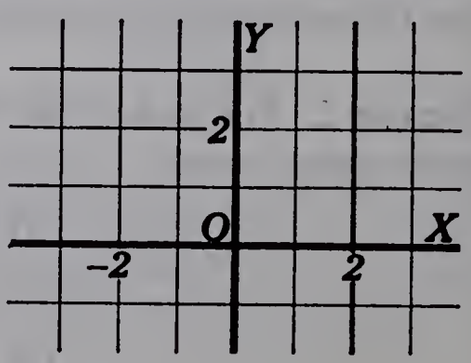

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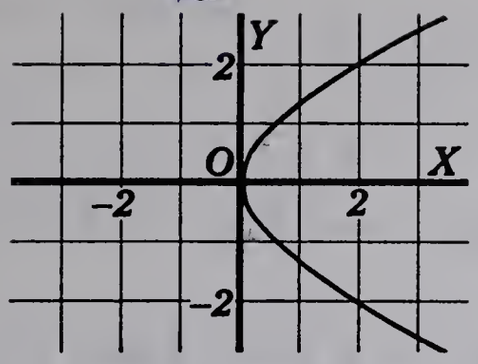

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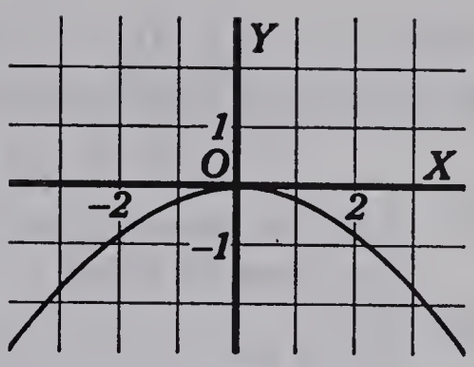
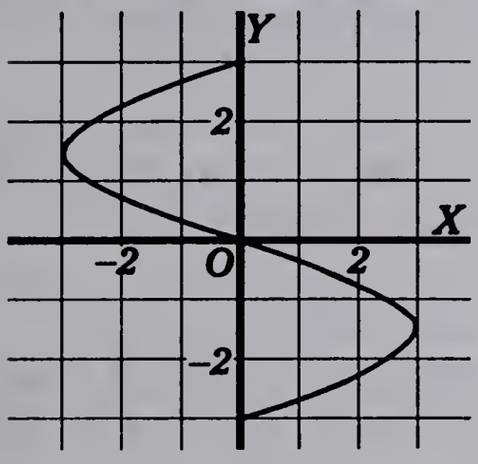
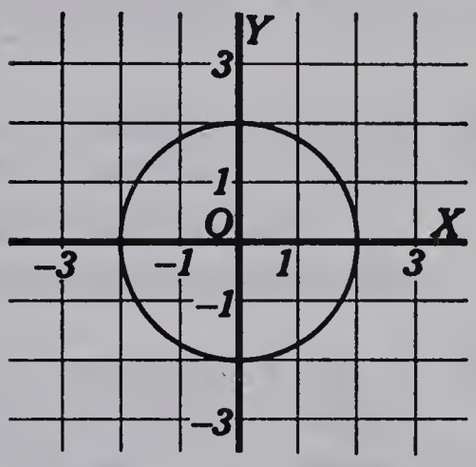


7.



8.



9. For each of the following, state whether the function is one-to-one, or many-to-one:

(i) $g: x \to 5 - x, x \in R$
(ii) $k: x \to x, x \in R$
(iii) $p: x \to 3, x \in I$
(iv) $q: x \to -x, x \in Q$
(v) $a: x \to \sqrt{x}, x \in {}^{+}R$
(vi) $f: x \to 2^x, x \in R$
(vii) $d: x \to x^2, x \in Q$
(viii) $\{(x, y) \mid y = 2, x \in R\}$
(ix) $m: x \to 3 \mid x \mid, x \in I$
(x) $n: x \to x^2 - 2x, x \in R$

State which of the following relations are functions and express those which are in the form $f : x \to ?, x \in D$, when D is the domain of f:

(B)

10. $S = \{(p, q) \mid q = 5p - 7, p, q \in R\}$
11. $M = \{(x, y) \mid x^2 = -5y, x, y \in I\}$
12. $K = \{(x, y) \mid y^2 = -5x, x, y \in R\}$
13. $R = \{(x, y) \mid y = 4, x, y \in R\}$
14. $U = \{(m,n) \mid m > n, m, n \in I\}$
15. $N = \{(t, s) \mid s = 10t + 15t^2, t \geqq 0, t, s \in R\}$
16. $C = \{(x, y) \mid x^2 + y^2 = 9, x, y \in I\}$
17. $H = \{(x, y) \mid x^2 - y^2 = 16, x, y \in R\}$
18. $Q = \{(1, 1), (2, 2), (3, 3), (4, 4)\}$

19. $G = \{(-8, 4), (-4, 1), (0, 0), (4, 1), (8, 4)\}$

20. Draw the graph of the function
$$F = \{(x, y) \mid 2y = 5x - 3, \quad -1 \leqq x \leqq 1, x, y \in I\}.$$

21. List the elements of $F = \{(x, y) \mid y = x^2 + 1, \quad |x| \leqq 2, x, y \in I\}$ and draw its graph.

22. Write an algebraic sentence in x and y which defines the function
$$A = \{(-2, -1), (-1, 0), (0, 1), (1, 2)\}.$$

23. For the function
$B = \{(-3, 9), (-2, 4), (-1, 1), (0, 0), (1, 1), (2, 4), (3, 9)\}$:
(i) state the domain and range of the function;
(ii) write a defining sentence.

24. A set of rectangles is such that each has a width of 5 units. The lengths vary and are represented by x units, where $x \in R$. The largest rectangle has a length of 25 units.
(i) Define the relation between the area (y square units) of the rectangles and the given units.
(ii) Determine whether the relation is a function.
(iii) State the domain and range of the relation.

25. (i) Define the relation between the area of a circle (A square units) and its radius (r units).
(ii) State the domain and range of this relation.
(iii) Determine whether the relation is a function.

26. A jet-plane has an average speed of 600 miles per hour. It can carry sufficient fuel for a 6-hour run.
(i) Define the relation between distance (d miles) and time (t hours) for this plane.
(ii) State the domain and range of the relation.
(iii) Is this relation a function?

27. A boy takes a job for eight weeks during the summer. He opens an account in the bank and deposits \$25 a week.
(i) Define the relation between his total deposit (d dollars) and the numbers of weeks (n) worked.
(ii) State the domain and range of the relation.
(iii) Is the relation a function?

4·3 Inverse function.

Answer the following questions; compare your solutions with those on page 413.

1. Consider the function

$$f = \{(x, y) \mid y = 4x, 0 \leqq x \leqq 4, x \in R\}:$$

(i) find the ordered pairs of the function for $x \in \{0, 1, 2, 3, 4\}$;

(ii) plot the corresponding points and draw the smooth curve through them;

(iii) make a new set of ordered pairs from those in (i) by interchanging the elements of each pair; that is each x becomes the corresponding y and each y the corresponding x;

(iv) plot the corresponding points to the ordered pairs of (iii) and draw the smooth curve through them;

(v) draw the line defined by the equation $y = x$, $x \in R$;

(vi) by testing the plotted points determine whether each curve is the image (reflection) of the other in the line of (v); that is, determine whether the two curves are symmetrically situated with respect to the line defined by $y = x$, $x \in R$;

(vii) write a defining sentence for the curve of (iv); (recall how the ordered pairs were formed);

(viii) name the relation so defined f^{-1} and express it in set-builder notation. Is this relation a function? State its domain and range and compare these with the domain and range of the given function f.

These two functions are an example of a special class of functions called *inverse functions*. f^{-1} is the inverse function to f and f is the inverse function to f^{-1}.

The function f^{-1} is obtained from f by making three changes:

(i) the variables in the defining equation are interchanged;

(ii) the domain of f becomes the range of f^{-1};

(iii) the range of f becomes the domain of f^{-1}.

The graphs of f and f^{-1} are always symmetrically situated with respect to the line defined by $y = x$, $x \in R$.

The following may be observed from the solution of the previous example:

(i) $f = \{(x, y) \mid y = 4x, 0 \leqq x \leqq 4, x \in R\}$.
The range of f is $\{y \mid 0 \leqq y \leqq 16, y \in R\}$.

(ii) f maps x into $4x$ where $0 \leqq x \leqq 4, x \in R$, or
f describes the operation of *multiplying* a real number in the domain $0 \leqq x \leqq 4$ by 4 ;

(iii) $f^{-1} = \{(x, y) \mid x = 4y, 0 \leqq y \leqq 4, y \in R\}$ or,

$$f^{-1} = \left\{(x, y) \mid y = \frac{x}{4}, 0 \leqq x \leqq 16, x \in R\right\}.$$

The range of f^{-1} is $\{y \mid 0 \leqq y \leqq 4, y \in R\}$.

f^{-1} maps x into $\frac{x}{4}$ where $0 \leqq x \leqq 16, x \in R$, or

f^{-1} describes the operation of *dividing* a real number in the domain $0 \leqq x \leqq 16, x \in R$ by 4 .

The operation (division) described in f^{-1} is the inverse operation to the operation (multiplication) described in f.

Thus f^{-1} is called the *inverse function* to f and conversely.

2. *For the function*

$$g = \{(x, y) \mid y = x^2, 0 \leqq x \leqq 4, x \in R\},$$

answer the questions listed in question 1 *and compare your solutions with those on page* 413.

Exercise 4-2

(B)

1. The ordered pairs of a function h are

$$(0, 1), (1, 3), (2, 6), (3, 10).$$

(i) State the domain and range of the function.

(ii) Write the ordered pairs of the inverse function h^{-1}.

(iii) State the domain and range of the inverse function.

(iv) Draw the graphs of the two functions with reference to the same axes and illustrate that the graph is symmetrical about the line defined by $y = x, x \in R$.

2. For the linear function

$$f = \{(x, y) \mid y = 2x + 2, x, y \in R\} :$$

(i) write the inverse function f^{-1};

(ii) draw the graphs of the two functions and illustrate that they are symmetrically situated about the line defined by $y = x, x \in R$.

Each of the following sets of ordered pairs are the ordered pairs of a function. For each:

(i) *draw the graph of the ordered pairs;*

(ii) *construct the ordered pairs of the inverse function;*

(iii) *draw the graph of the ordered pairs of* (ii);

(iv) *test that the graphs of the two functions are symmetrical about the line defined by* $y = x$, $x \in R$.

3. $\{(0, 0), (-1, 3), (-2, 6), (-3, 9)\}$
4. $\{(0, 0), (-2, 1), (-4, 2), (-6, 3)\}$
5. $\{(0, 0), (-2, -1), (-4, -2), (-6, -3)\}$
6. $\{(0, -1), (-1, -4), (-2, -7), (-3, -10)\}$
7. $\{(-1, -1), (-3, -3), (-5, -5), (-7, -7)\}$
8. $\{(2, 0), (4, -2), (6, -4), (8, -6)\}$

4·4 Further examples of functions. Since to each x in the domain of a function, there corresponds a unique real y in its range, a function is completely determined if its domain is given, and a rule is given which determines this corresponding value of y for each x in the domain. For example,

$$f_1 = \{(x, y) \mid y = x + 2, x \in I\}$$
$$= \{(x, x + 2) \mid x \in I\},$$

is a well-defined function since its domain is given as I, and for each $x \in I$ the rule for determining the corresponding value of y is "add 2 to x".

When this method of defining a function is used (that is, when the defining equation gives an explicit formula for y in terms of x, together with the domain) it is always to be assumed that the additional condition, or conditions, involving x give the domain of the function.

For example,

$$f_2 = \{(x, y) \mid y = \tfrac{1}{2}\sqrt{x} - 1, x \geqq 0, x \in R\},$$

is a function with domain $\{x \mid x \geqq 0, x \in R\}$.

Example 1. For the function

$$f = \{(x, y) \mid y = 4x^2 - 1, \ -1 \leqq x \leqq 1, x \in R\}:$$

(i) determine the intercepts of the graph;

(ii) determine the domain, and range of the function;

(iii) discuss the symmetry of the graph;

(iv) make a table of values, and sketch the graph.

Solution.

(i) *x-intercepts.* Let $y = 0$ in $y = 4x^2 - 1$.

$$\therefore\ x^2 = \tfrac{1}{4},$$

$$\therefore\ x = \pm \tfrac{1}{2}.$$

$\therefore$ the x-intercepts are $\pm \frac{1}{2}$.

y-intercepts. Let $x = 0$, then $y = -1$.

$\therefore$ the y-intercept is -1.

(ii) *Domain.* The domain is given as $\{x \mid -1 \leqq x \leqq 1, x \in R\}$.

Range.

$$y = 4x^2 - 1$$

$$\leftrightarrow \quad x^2 = \frac{1 + y}{4}$$

$$\leftrightarrow \quad x = \frac{\pm\sqrt{1 + y}}{2}.$$

$\therefore\ x \in R \leftrightarrow 1 + y \geqq 0, y \in R.$

$1 + y \geqq 0, y \in R \leftrightarrow y \geqq -1, y \in R.$

Also $-1 \leqq x \leqq 1$ or $|x| \leqq 1$.

$$\therefore\ \left|\pm\frac{\sqrt{1 + y}}{2}\right| \leqq 1$$

$$\leftrightarrow\ \sqrt{1 + y} \leqq 2.$$

$$\therefore\ 1 + y \leqq 4$$

$$\text{or}\quad y \leqq 3.$$

Thus the range of f is $\{y \mid -1 \leqq y \leqq 3, y \in R\}$.

(iii) *Symmetry.* Replace x by $(-x)$ in $y = 4x^2 - 1$.

$\because (-x)^2 = x^2$ for $-1 \leqq x \leqq 1$, the equation is unchanged,

$\therefore$ the graph is symmetric with respect to the y-axis.

It is not symmetric with respect to either the x-axis or the origin.

(iv) *Table of values.*

x	0	$\frac{1}{2}$	1
y (or $4x^2 - 1$)	-1	0	3

The part of the graph in *Fig. 4-5* to the left of the y-axis is obtained from that to the right by reflecting in the y-axis.

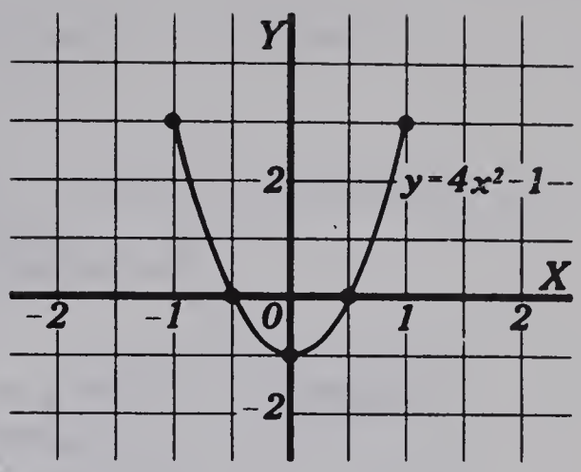


Fig. 4-5

Example 2. Let $f_3 = \{(p, q) \mid p^2 + q^2 = 4, q \leqq 0, p, q \in R\}$.

(i) Justify the statement: the relation f_3 is a function.

(ii) Determine the intercept of the graph of f_3; the domain, and range of f_3; discuss the symmetry of the graph and sketch the graph.

(iii) Give an alternative definition of f_3 by stating its domain and giving a rule for determining the value of q for each p in the domain.

Solution.

(i) The defining conditions

$$p^2 + q^2 = 4 \text{ and } q \leqq 0$$

$$\leftrightarrow \quad q^2 = 4 - p^2 \text{ and } q \leqq 0$$

$$\leftrightarrow \quad q = \pm\sqrt{4 - p^2} \text{ and } q \leqq 0$$

$$\leftrightarrow \quad q = -\sqrt{4 - p^2} \quad (\text{since } q \leqq 0).$$

$\therefore$ f_3 is a function, since to each p there corresponds at most one q for which $(p, q) \in f_3$.

(ii) *p-intercepts.* Let $q = 0$ in $q = -\sqrt{4 - p^2}$.

$$\therefore 4 - p^2 = 0$$

$$\text{or} \qquad p = \pm 2.$$

$\therefore$ the p-intercepts are ± 2.

q-intercepts. Let $p = 0$, then $q = -\sqrt{4}$

$$\therefore q = -2.$$

$\therefore$ the q-intercept is -2.

Domain. Since $q = -\sqrt{4 - p^2}$

$q \in R \leftrightarrow 4 - p^2 \geqq 0,\ p \in R$

$\leftrightarrow \quad p^2 \leqq 4,\ p \in R$

$\leftrightarrow \quad |p| \leqq 2,\ p \in R.$

$\therefore$ the domain of f_3 is $\{p \mid |p| \leqq 2,\ p \in R\}$

or $\{p \mid -2 \leqq p \leqq 2,\ p \in R\}$.

Range. Since $q = -\sqrt{4 - p^2}$ where $-2 \leqq p \leqq 2,\ p \in R$.

$\therefore$ q takes the value -2 (for $p = 0$)

and the value 0 for $p = \pm 2$

and all real values between 0 and -2.

$\therefore$ the range of f_3 is $\{q \mid -2 \leqq q \leqq 0,\ q \in R\}$.

Symmetry. Replace p by $(-p)$ in $q = -\sqrt{4 - p^2}$.

$\because (-p)^2 = p^2$, the equation is unchanged.

$\therefore$ the graph is symmetric with respect to the q-axis.

It is not symmetric with respect to either the p-axis or the origin since $q \leqq 0$ for every point $(p, q) \in f_3$.

Table of values.

p	0	1	2
$q(\text{or } -\sqrt{4 - p^2})$	-2	$-\sqrt{3}$	0

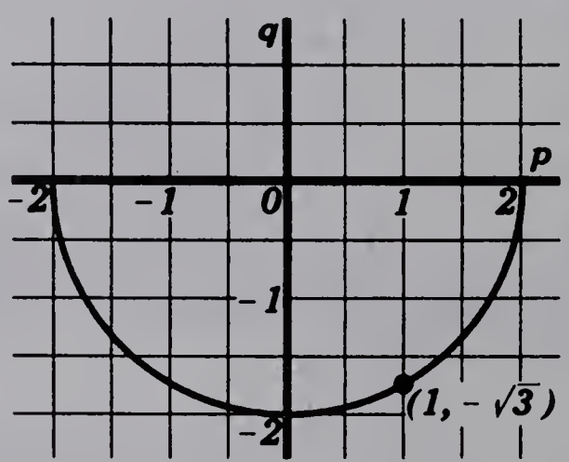


Fig. 4-6

Note, *Fig. 4-6* that it is sufficient to compute the coordinates of points only in the fourth quadrant, since the part of the graph in the third quadrant can be obtained from this by a reflection in the q-axis.

(iii) From parts (i) and (ii),

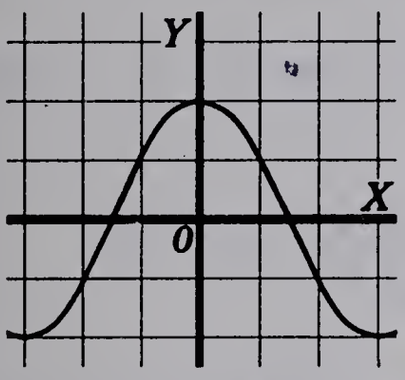
$$f_3 = \{(p, q) \mid q = -\sqrt{4 - p^2},\ |p| \leqq 2,\ p \in R\},$$

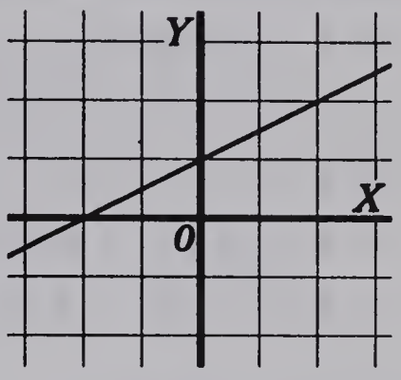
or $f_3 = \{(p, -\sqrt{4 - p^2}) \mid |p| \leqq 2,\ p \in R\}$.

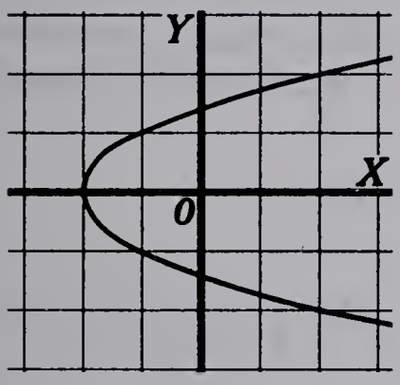
Exercise 4-3

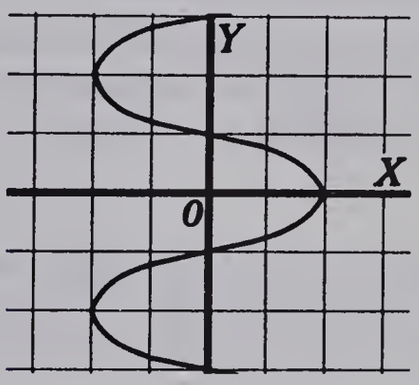
(A)

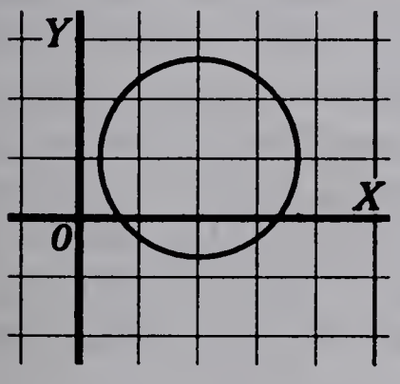
Each of the following figures is the graph of a relation. State whether the relation is a function or not; justify your answer.

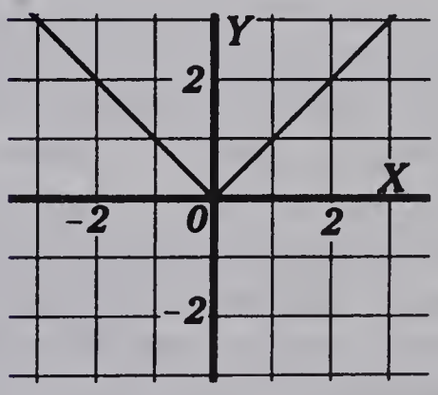
1. 

2. 

3. 

4. 

5. 

6. 

State the domain and range of each of the following relations, and determine which of these relations are also functions:

7. $F = \{(0, 0), (1, 2), (2, 4), (-1, 0), (0, -1)\}$
8. $G = \{(-2, 1), (-1, 1), (0, 2), (1, 2), (2, \sqrt{3})\}$
9. $f = \{(1, 2), (1, -2), (\sqrt{2}, 3), (\sqrt{3}, 4)\}$
10. $g = \{(\frac{1}{4}, \frac{1}{2}), (\frac{1}{2}, 1), (1, 2), (2, 4)\}$

(B)

Determine the intercepts of the graph, domain, and range of the relation; discuss the symmetry of the graph, and sketch the graph of each of the following relations. In case f is a function, give an alternative definition by stating its domain and giving an explicit formula for y in terms of x:

11. $f = \{(x, y) \mid 2x + y = 5, x \in {}^{+}I, y \in I\}$

12. $f = \{(x, y) \mid y^2 = x, y \geqq 0, x, y \in R\}$

13. $f = \{(x, y) \mid x^2 + y^2 = 9, x, y \in R\}$

14. $f = \{(x, y) \mid x^2 - y + 1 = 0, |x| \leqq 2, x, y \in R\}$

15. $f = \{(x, y) \mid |x| + y = 0, |x| \leqq 3, x, y \in R\}$

***4·5* Function notation.** Because exactly one value y in the range of a function f corresponds to each x in its domain, this unique corresponding value of y is often denoted by $f(x)$, read "f at x". $f(x)$ is called the *value of the function at* x. Thus, if

$$f = \{(x, y) \mid y = x^2 - 2, x \in R\},$$

then f is a function (with domain R), and

$$f(0) = 0^2 - 2 = -2, \qquad f(\sqrt{2}) = (\sqrt{2})^2 - 2 = 0, \qquad f(-2) = (-2)^2 - 2 = 2,$$

and in general, $f(x) = x^2 - 2$ for each $x \in R$. Sometimes one speaks of the "function $(x^2 - 2)$", meaning by this the function f defined above.

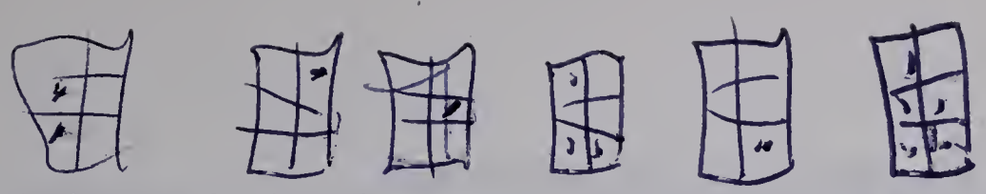
The function notation just introduced provides another method of defining a specific function. To illustrate, the above function could be defined as follows: *let f be the function defined by $f(x) = x^2 - 2$, $x \in R$.* This method is essentially the same as that introduced in Section 4·4, since it consists of *prescribing the value of the function at each x in the domain, together with the domain.* As a further illustration, the function g defined by

$$g(x) = \sqrt{x - 2} - 5x, \quad x \geqq 2, x \in R,$$

is a well-defined function with domain $\{x \mid x \geqq 2, x \in R\}$.

In terms of the concept of a function as a mapping, introduced in Section 4·2, $f(x)$ is the image of x under the mapping f. If, for each x in the domain D of a function f, a formula or rule is given for determining $f(x)$, then the mapping f is completely specified by writing

$$f : x \to f(x), x \in D .$$



Example 1. For the function f defined by

$$f(x) = \frac{x-1}{x^2+1},\ x \in R,$$

compute (i) $f(0)$, (ii) $f(1)$, (iii) $f(2)$, (iv) $f(a^2)$, where $a \in R$, (v) $f(a+h)$, where $a, h \in R$.

Solution. (i) $f(0) = \dfrac{0-1}{0^2+1}$

$= -1.$

(ii) $f(1) = \dfrac{1-1}{1^2+1}$

$= 0.$

(iii) $f(2) = \dfrac{2-1}{2^2+1}$

$= \dfrac{1}{5}.$

(iv) $f(a^2) = \dfrac{a^2-1}{a^4+1}.$

(v) $f(a+h) = \dfrac{(a+h)-1}{(a+h)^2+1}$

$= \dfrac{a+h-1}{a^2+2ah+h^2+1}.$

Example 2. Let f be a function whose domain is R, such that $f(a-b) = f(a) \cdot f(b)$ for all $a, b \in R$. Prove that the range of f must be a proper subset of $\{0, 1\}$.

Solution. $\because\ f(a-b) = f(a) \cdot f(b)$ for all $a, b \in R$,

$\therefore\ f(a) = f(a) \cdot f(0)$ for all $a \in R$ (setting $b = 0$).

Either $f(0) = 0$, or $f(0) \neq 0$.

Case 1. $f(0) = 0$. In this case, $f(a) = f(a) \cdot f(0) = 0$ for all $a \in R$, so the range of f is $\{0\}$, proving the result in this case.

Case 2. $f(0) \neq 0$. In this case, $f(0) = f(0) \cdot f(0)$.

$\therefore\ 1 = f(0)$ (dividing by $f(0) \neq 0$).

Setting $b = a \in R$,

$f(0) = [f(a)]^2$ for all $a \in R$.

$\therefore\ 1 = [f(a)]^2$ for all $a \in R$.

$\therefore\ f(a) = \pm 1$ for all $a \in R$.

Setting $b = \dfrac{a}{2}$,

$$f\left(\frac{a}{2}\right) = f(a) \cdot f\left(\frac{a}{2}\right) \text{ for all } a \in R.$$

$\therefore 1 = f(a)$ for all $a \in R$ (dividing by $f\left(\dfrac{a}{2}\right) = \pm 1 \neq 0$).

$\therefore$ the range of f is $\{1\}$ in this case.
$\therefore$ the range of f is a proper subset of $\{0, 1\}$ in all cases.

Write solutions for the following; compare your solutions with those on page 414.

1. For the functions
$f = \{(x, y) \mid y = x + 2, x \in R\}$, $g = \{(x, y) \mid y = \sqrt{x^2 + 4}, x \in R\}$,
compute (i) $f[g(-2)]$, (ii) $g[f(-2)]$, (iii) $f[g(x)]$ for $x \in R$.

2. If f is the function defined by

$$f(u) = u + \frac{1}{u},\ u \in R,\ u \neq 0,$$

determine (i) $f(x^2)$, (ii) $f(u + 2)$, (iii) $f\left(\frac{1}{u}\right)$, (iv) $f[f(x)]$.

Exercise 4-4

(A)

If $f = \{(x, y) \mid y = x^2 - 4, x \in R\}$, and $g = \{(x, y) \mid y = x(x - 2), x \in R\}$, *compute:*

1. $f(0)$	**2.** $f(1)$	**3.** $f(-1)$	**4.** $f(\sqrt{2})$
5. $f(-2)$	**6.** $f(\sqrt{3})$	**7.** $g(0)$	**8.** $g(-2)$
9. $g(2)$	**10.** $g(5)$	**11.** $f[g(0)]$	**12.** $g[f(0)]$

(B)

Let f *and* g *be the functions defined by*

$$f(x) = \sqrt{x},\ x \geqq 0,\ x \in R;\ g(x) = x^2 + 4x - 5,\ x \in R,$$

respectively. Compute:

13. $f(0), f(1), f(4), f(6), f(9)$
14. $g(0), g(1), g(-2), g(-5)$
15. $f[g(1)], f[g(3)], g[f(1)], g[f(3)]$
16. $f[(-3)^2], f(a^2)$ for $a \in R$
17. $g[f(x)]$ for $x \geqq 0, x \in R$

(C)

18. Let f be the function defined by $f(u) = \dfrac{u}{u^2 + 1},\ u \in R$.

Find: (i) $f\left(\frac{1}{u}\right)$, for $u \neq 0$ (ii) $f(2u)$ (iii) $f(u - 1)$ (iv) $f[f(x)]$.

19. Determine the range of the function g of the preceding questions 14-17. (Hint: note that $x^2 + 4x - 5 = (x + 2)^2 - 9$ for each $x \in R$.)

20. If f and g are the functions of questions 13-17, for what x is $f[g(x)]$ defined?

21. Let f be a function with domain R, such that $f(a + b) = f(a) - f(b)$ for all $a, b \in R$. Prove that $f(b) = 0$ for all $b \in R$. (Hint: First show that $f(0) = 0$.)

22. Let g be a function with domain R, such that $g(x + 1) = g(x)$ for all $x \in R$. Prove that $g(0) = g(1)$, and $g(-1) = g(1)$.

Chapter V

THE LINEAR FUNCTION AND ITS APPLICATIONS

***5•1* The general linear function.** The cost of entertaining guests at a social event depends partly on certain fixed costs and partly on the number of guests. If

(i) the fixed costs are \$25;

(ii) other costs are \$$3x$ where $x \in N_0$ and represents the number of guests;

(iii) \$$y$ represents the total cost;

then $$y = 3x + 25,\ x \in N_0$$

is the defining sentence of a *function*

$$C = \{(x, y) \mid y = 3x + 25,\ x \in N_0\}$$

defining the cost.

This relation is a *well-defined function*, since to each x in the domain, which is given, there corresponds one and only one y in the range. Since the defining sentence is an equation of the first degree, this relation is a linear relation. Since the relation is also a function, it is called a *linear function.*

The graph of the function C is illustrated in *Fig. 5-1.* It consists of a series of isolated points which appear to lie on a straight line. The range of the function is $\{y \mid y \geqq 25,\ y \in N_0\}$. In the practical application of

this function to the cost of entertainment some limitation must be placed on the number of guests and thus on the cost.

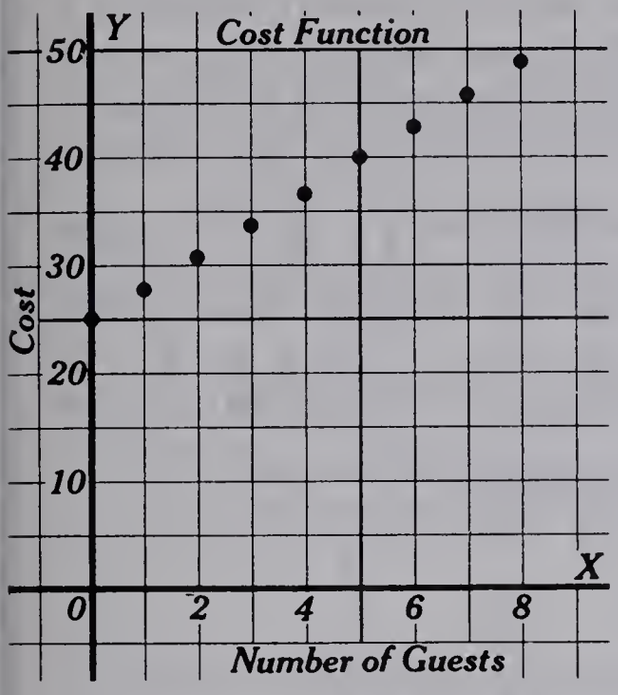


Fig. 5-1

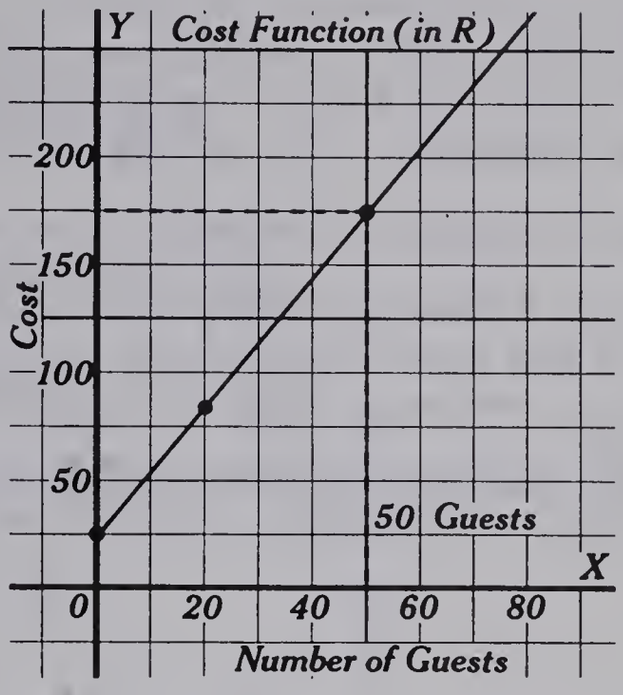


Fig. 5-2

The application of linear (and other) functions is often made graphically. The graph of the function is drawn and the cost, for example, of a particular number of guests is read directly from the graph.

We have seen in earlier work that the graph of a linear relation in N or I is a set of points which lie on a straight line, and the graph of a linear relation in R is a straight line.

To apply the function C graphically it would be practical to draw the graph of the linear function

$$C_1 = \{(x, y) \mid y = 3x + 25, x \in R, x \geqq 0\},$$

and read corresponding whole number values from it as indicated in *Fig. 5-2*.

Some other examples of linear functions are:

(i) $D = \{(t, d) \mid d = vt, t \in R, t \geqq 0\}$,
the function defining the distance-time relation for a constant speed v ft./sec.;

(ii) $S = \{(T, V) \mid V = 1089 + 2T, T \in R, T \geqq 0\}$,
the function defining the speed (feet per second)-temperature relation of the speed of sound in air at T degrees centigrade;

(iii) $T = \{(F, C) \mid C = \frac{5}{9}(F - 32), F \in R\}$,
the function defining the Centigrade-Fahrenheit temperature relation.

The general linear relation in R is any relation of the form

$$L = \{(x, y) \mid Ax + By + C = 0,\ x,\ y \in R\},$$

where A, B, C denote real numbers with not both A, B zero. Its graph is a straight line. With two exceptions, both the domain and the range of any linear relation on R is R. The first exception occurs when $A = 0$, that is, for the relation $L_1 = \{(x, y) \mid By + C = 0,\ x,\ y \in R\}$. In this case, the domain is still R, but the range is the one-element set $\left\{-\frac{C}{B}\right\}$. The graph of L_1 is a horizontal line (*Fig. 5-3*). The second exception occurs when $B = 0$, that is, for the relation $L_2 = \{(x, y) \mid Ax + C = 0,\ x,\ y \in R\}$. In this case, the range is still R, but the domain is the one-element set $\left\{-\frac{C}{A}\right\}$. The graph of L_2 is a vertical line (*Fig. 5-4*).

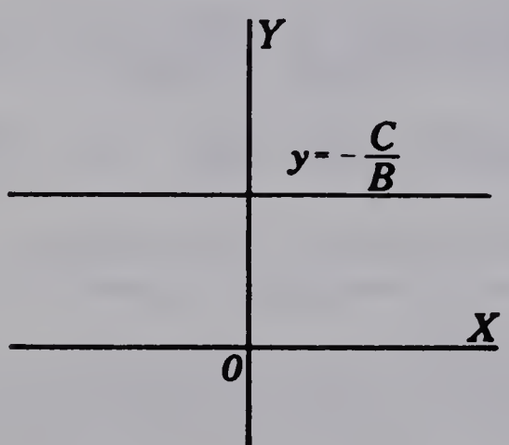


Fig. 5-3

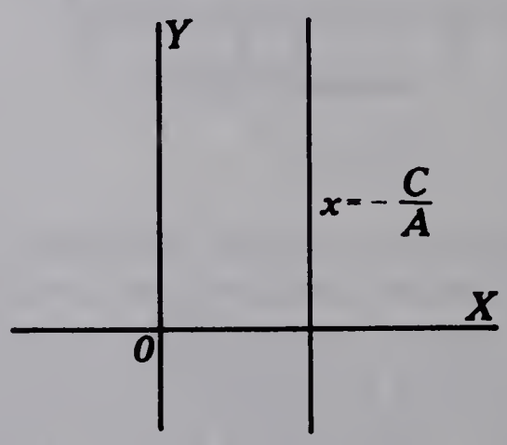


Fig. 5-4

Thus it is seen that in all cases except when $B = 0$ the relation L is a function. Thus the general linear function in R is

$$l = \{(x, y) \mid Ax + By + C = 0,\ x \in R\},$$

where A, B, $C \in R$, $B \neq 0$.

Exercise 5-1

(A)

1. For the linear function $f = \{(x, y) \mid 3x + 2y - 3 = 0,\ x,\ y \in R\}$, state:

(i) by comparison with the general equation $Ax + By + C = 0$, the values of A, B, and C;

(ii) the domain; (iii) the range;

(iv) the x- and y-intercepts of the graph of the function.

2. For the general linear function

$$l = \{(x, y) \mid Ax + By + C = 0,\ x, y \in R\}$$

if $A, B \neq 0$, state:

(i) the range; (ii) the domain;
(iii) the x- and y-intercepts of the graph.

If $A, B, C \in R$, state:

(iv) the condition that the range be $\left\{-\frac{C}{B}\right\}$;

(v) the condition that the domain be $\left\{-\frac{C}{A}\right\}$;

(vi) the condition that the graph be a line parallel to the y-axis;
(vii) the condition that the graph be a line parallel to the x-axis.

(B)

3. For the distance-time function

$$D = \{(t, d) \mid d = vt,\ 0 \leqq t \leqq 20,\ t \in R\},$$

where d denotes the distance in miles, t the time in hours, and v the speed in miles per hour:

(i) if the speed is 60 m.p.h., rewrite the function to indicate this;
(ii) state the domain of the function;
(iii) determine the range of the function;
(iv) find $D(3)$, $D(13)$, $D(17)$, and draw the graph of the function. Is the graph a line, a line segment, or a ray?
(v) From the graph read $D(3\frac{1}{4})$, $D(7\frac{1}{8})$, $D(15\frac{3}{4})$.

4. For the speed-temperature function for sound in air

$$S = \{(T, V) \mid V = 1089 + 2T,\ -20 \leqq T \leqq 60,\ T \in R\}$$

(speed in feet per second and T in centigrade degrees):

(i) state the domain of the function;
(ii) determine the range of the function;
(iii) rewrite the function with the defining equation in the form $Ax + By + C = 0$; for this equation state the numbers corresponding to A, B, and C ;
(iv) determine $S(0)$, $S(10)$, $S(50)$ and draw the graph of the function;
(v) from the graph read $S(-18)$, $S(-2)$, $S(16)$, $S(57)$; translate the results into the speed of sound at the given temperature.

5. For the Centigrade-Fahrenheit function

$$T = \{(F, C) \mid C = \tfrac{5}{9}(F - 32),\ -20 \leqq F \leqq 120,\ F \in R\}:$$

(i) state the domain;

(ii) determine the range;

(iii) rewrite the function with the defining equation in the form $Ax + By + C = 0$; for this equation state the numbers corresponding to A, B, and C;

(iv) determine $T(-4)$, $T(5)$, $T(113)$, and draw the graph of the function;

(v) from the graph obtain the Centigrade temperatures corresponding to the following Fahrenheit temperatures: $-10°$, $0°$, $65°$, $105°$;

(vi) from the graph obtain the Fahrenheit temperatures corresponding to the following Centigrade temperatures: $-10°$, $-4°$, $30°$, $42°$.

5·2 Slope of a line. In *Fig. 5-5*, $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ are any two points on the line l, and $\triangle P_1MP_2$ is a right triangle with sides parallel to the axes. The change in x from P_1 to P_2 is $x_2 - x_1$ and is represented by the directed segment P_1M(the run from P_1 to P_2). Similarly, the change in y from P_1 to P_2 is $y_2 - y_1$ and is represented by the directed segment MP_2 (the rise from P_1 to P_2). The change in x is called the *increment* of x and is represented by $\triangle x$ (delta x).

Thus $\triangle x = x_2 - x_1$.

Similarly the change in y, $\triangle y$, is called the *increment of* y, and $\triangle y = y_2 - y_1$.

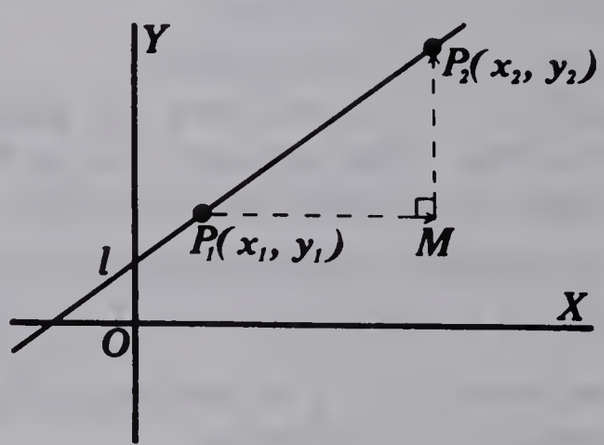


Fig. 5-5

In *Fig. 5-6*, $A(-1, -1)$, $B(1, 3)$, $C(2, 5)$, and $D(3, 7)$ are points of the line l.

For the directed segment AC:

$\triangle x = 2 - (-1) = 3$,

$\triangle y = 5 - (-1) = 6$,

$\therefore\ \triangle y : \triangle x = 6 : 3 = 2 : 1$.

For the directed segment BD:

$\triangle x = 3 - 1 = 2$,

$\triangle y = 7 - 3 = 4$,

$\therefore\ \triangle y : \triangle x = 4 : 2 = 2 : 1$.

Thus, for the directed segments AC and BD of line l, $\triangle y : \triangle x$ is a constant ratio.

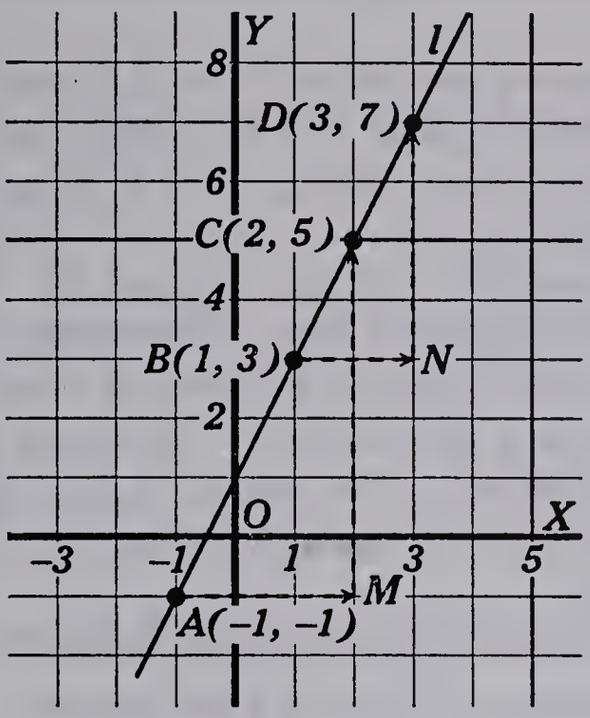


Fig. 5-6

It can be shown that:

$\triangle y : \triangle x$ is a constant ratio for all pairs of distinct points of a straight line.

The constant ratio $\triangle y : \triangle x$ for a given line is called the *slope* of the line. We will assume this property without proof and refer to it as the *constant slope property* of a line.

If $\triangle x \neq 0$, then the slope $\triangle y : \triangle x$ is the real number $\dfrac{\triangle y}{\triangle x}$. If this real number is represented by m, then

$$m = \frac{\triangle y}{\triangle x} = \frac{y_2 - y_1}{x_2 - x_1}.$$

Two particular cases should be noted:

(i) if the line is parallel to the x-axis, then $\triangle y = 0$, and the slope of the line is 0;

(ii) if the line is parallel to the y-axis, then $\triangle x = 0$, and the slope is not defined.

The constant slope property of a straight line leads to the following test for the collinearity of three points:

If three points P_1, P_2, and P_3 are such that the slope of the line segment P_1P_2 is equal to the slope of the line segment P_2P_3, or is equal to the slope of the line segment P_1P_3, then P_1, P_2, and P_3 are collinear.

Write solutions for the following problems and compare them with those on page 415.

1. (i) Find the slope of the line on $P_1(-2, 1)$ and $P_2(5, 4)$.
(ii) What sign has the slope of a line which rises toward the right?

2. (i) For the line segment having $R(-1, 2)$ and $S(4, -2)$ as end points, calculate:
(a) the slope of RS (b) the slope of SR
Compare the slopes and draw a conclusion.
(ii) What sign has the slope of a line which rises to the left?

3. (i) Find the slope of the line on $A(-2, 3)$ and $B(3, 3)$.
(ii) What is the slope of a line parallel to the x-axis?

4. (i) Find the slope of the line segment having end points $M(3, -2)$ and $N(3, 4)$.
(ii) What is the slope of a line segment parallel to the y-axis?

5. (i) When calculating the slope of a line segment, is the order in which the points are named of any significance?
(ii) State the sign of the slope of a line which rises to the right.
(iii) State the sign of the slope of a line which rises to the left.
(iv) State the slope of a line parallel to the x-axis.
(v) What may be said of the slope of a line parallel to the y-axis?

6. Using the test for collinearity, determine whether the points $A(2, -3)$, $B(-1, -1)$, and $C(-4, 1)$ are collinear.

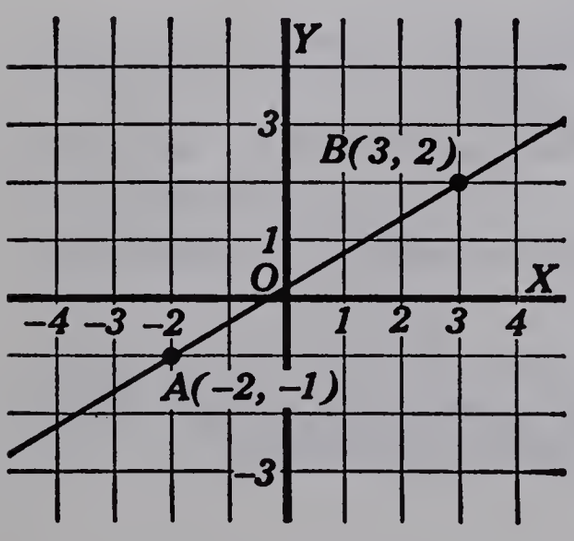
7. Find the real number x, such that the points $P(x, 2)$, $Q(-1, 3)$, and $R(3, -2)$ are collinear.

Exercise 5-2

(A)

In each of the following, find $\triangle x$ and $\triangle y$ from A to B:

1.



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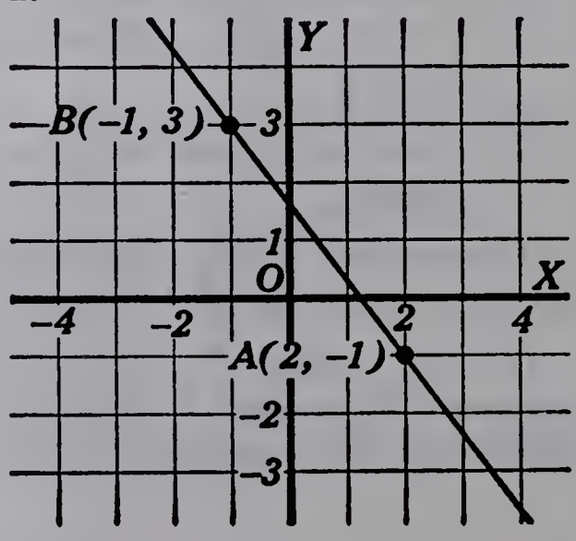
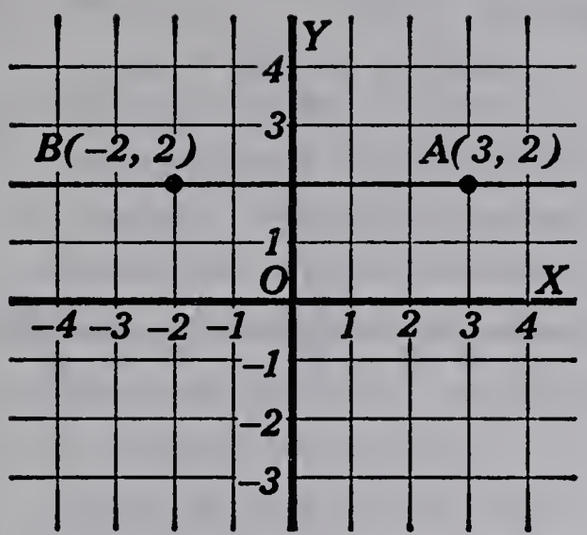


3.



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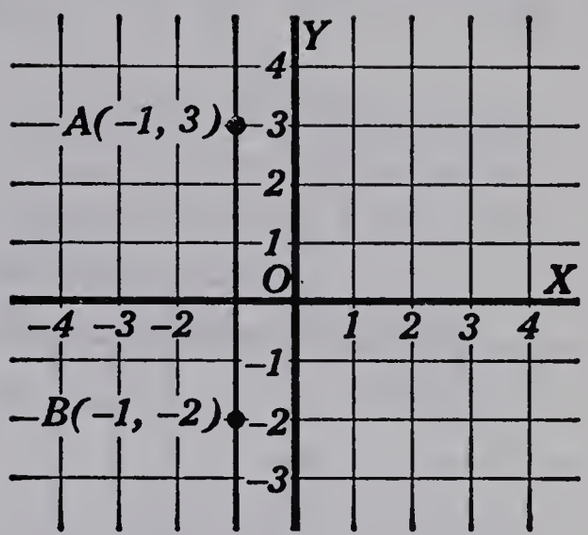
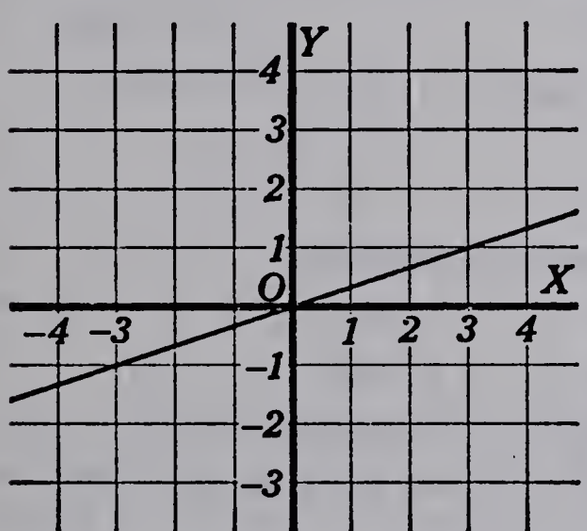
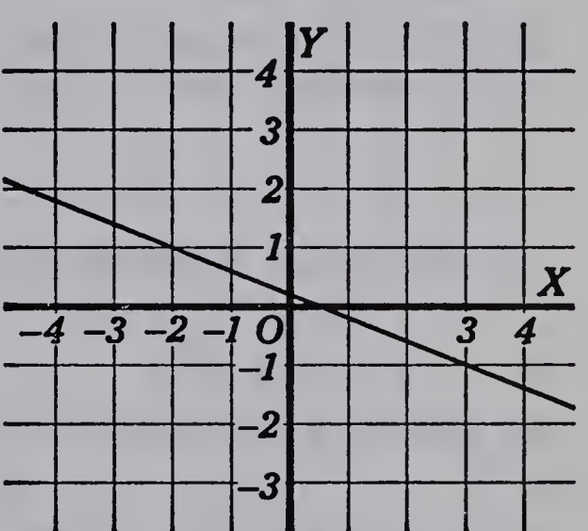


State the slope of the line in each of the following graphs:

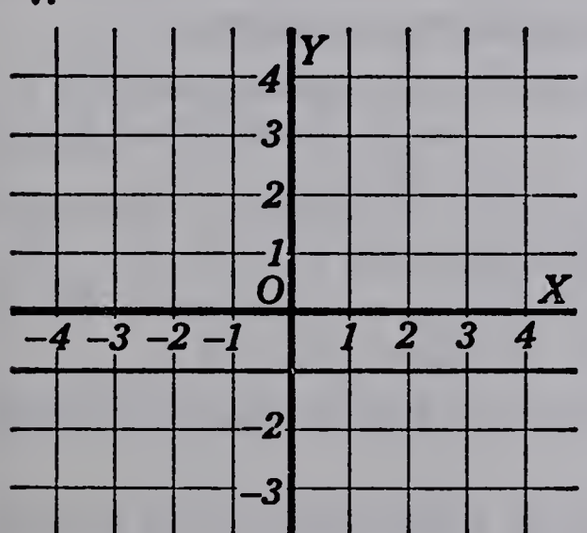
5.



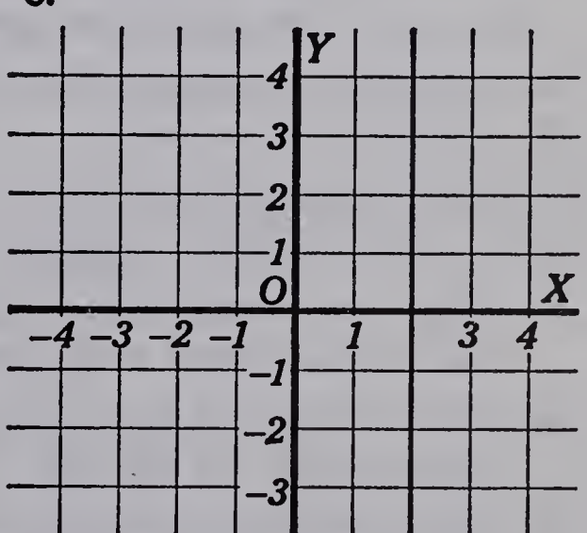
6.



7.



8.



Find the slopes of the lines on the pairs of points having coordinates as follows:

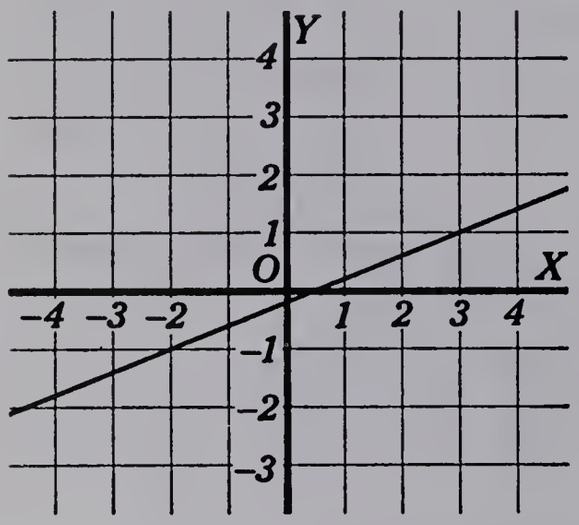
9. (7, 0) and (− 2, 0)

10. (3, 6) and (3, − 2)

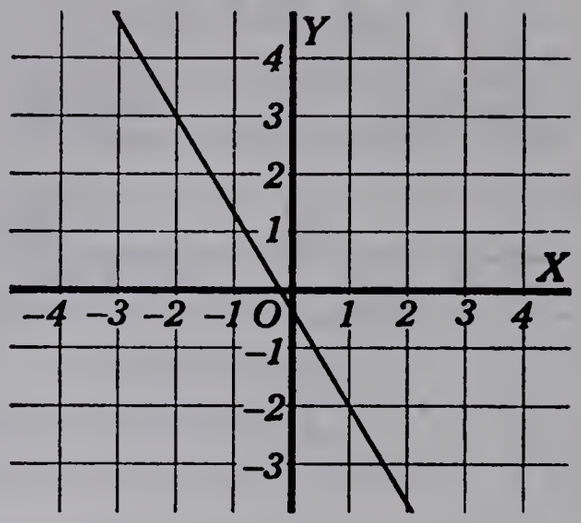
11. (− 2, 7) and (− 2, 2)

State the slope of the line in each of the following graphs:

12.



13.



14. State a characteristic of the slope when a line rises upward:
(i) to the right (ii) to the left

15. What is the slope of a line parallel:
(i) to the x-axis (ii) to the y-axis

(B)

Find the slopes of lines on the points having coordinates:

16. $(5, -2)$ and $(3, -4)$

17. $(-5, 4)$ and $(2, -3)$

18. $(-4, 2)$ and $(3, 2)$

19. $(-1, 0)$ and $(-1, -4)$

20. $(2a, 3a)$ and $(6a, 5a)$

21. $(0, -1)$ and $(0, -3)$

22. $(-5, -1)$ and $(-3, -4)$

23. $(-p, -2p)$ and $(6p, -2p)$

24. (a, b) and $(-2a, -2b)$

Determine if the points with coordinates as follows are collinear:

25. $(3, -7)$, $(1, -1)$, $(0, 2)$

26. $(5, 7)$, $(3, -4)$, $(-5, -7)$

27. $(3, 7)$, $(3, 1)$, $(-3, 1)$

28. $(-5, 4)$, $(-1, 1)$, $(3, -2)$

29. $(2, 5)$, $(2, -3)$, $(-2, 4)$

30. $(3, 7)$, $(3, 0)$, $(3, -7)$

31. $(-8, 0)$, $(0, 6)$, $(-4, 9)$

32. $(-3, -2)$, $(0, -2)$, $(5, -2)$

33. What real number does a represent, if the line on $A(a, 0)$ and $B(0, 3)$ has the same slope as the line on $C(-5, 4)$ and $D(3, -2)$?

34. Given the points $A(-3, -4)$, $O(0, 0)$, and $B(3, 4)$, find the slopes of the two lines AO and OB.

35. Find b for which the slope of the line on $L(0, b)$ and $M(3, 2)$ is equal to the slope of the line on M and $N(5, 7)$.

36. Write and simplify the algebraic sentence which states that the slope of the line on $P(x, y)$ and $Q(-3, 7)$ is equal to the slope of the line on Q and $R(4, -3)$.

37. Find y such that $P(-2, 1)$, $Q(3, -4)$, and $R(-3, y)$ are collinear.

38. Determine if points having coordinates $(a, 3b)$, $(2a, 6b)$, and $(-3a, -9b)$ are collinear.

39. Find the condition that $A(x, y)$, $B(2, -3)$, and $C(-5, 3)$ are collinear. (A *condition* is an algebraic statement (equation or inequation) expressing a relationship between the elements involved in a situation: in this case, the coordinates of the three given points.)

40. Find x such that $P(x, -2)$, $Q(0, -4)$, and $R(3, 0)$ are collinear.

5·3 The point-slope form of the equation of a line.

1. In *Fig. 5-7*, the line l :

(i) is on the point $A(3, 1)$;

(ii) has slope $\frac{5}{2}$.

By the constant slope property for $P(x, y)$ any point of the line other than $A(3, 1)$,

$$\text{slope of } PA = \tfrac{5}{2}$$

$$\leftrightarrow \quad \frac{y-1}{x-3} = \frac{5}{2}$$

$$\leftrightarrow \quad 5x - 15 = 2y - 2$$

$$\leftrightarrow \quad 5x - 2y - 13 = 0 .$$

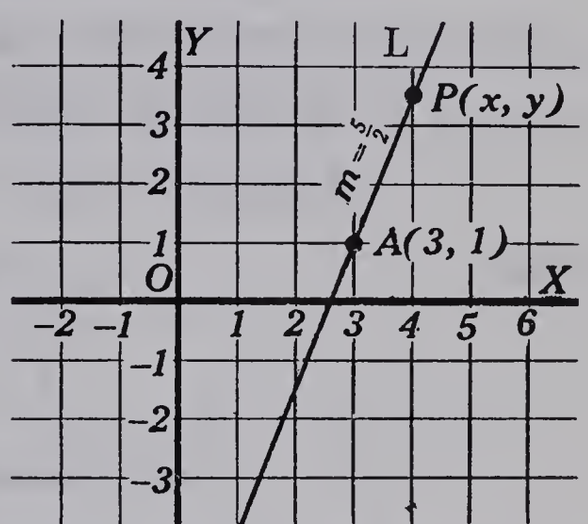


Fig. 5-7

Since this equation is satisfied by the coordinates of A, it is satisfied by the coordinates of every point of the line.

Since the steps in the development are reversible, it follows that every point whose coordinates satisfy the equation is a point of the line.

$\therefore$ the equation of the line is $5x - 2y - 13 = 0$.

A similar procedure may be used to develop the equation of a line on $P_1(x_1, y_1)$ and having slope m.

Hypothesis: l is a line on a point $P_1(x_1, y_1)$ and having slope m ($m \in R$; that is, l is not parallel to the y-axis).

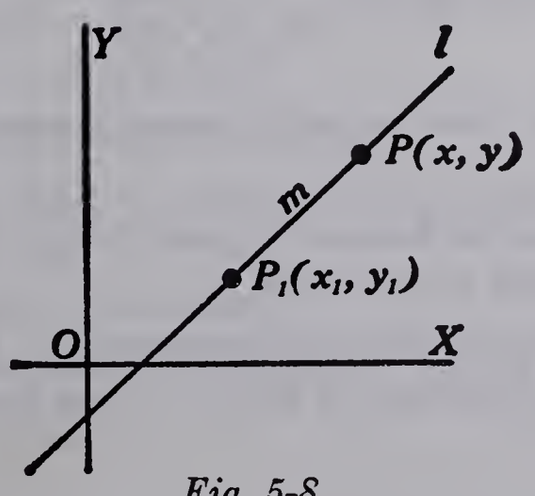


Fig. 5-8

Solution. For $P(x, y)$ any point of the line other than $P_1(x_1, y_1)$,

$$\text{slope of } P_1P = m \quad \text{(constant slope property of a line)}$$

$$\leftrightarrow \quad \frac{y - y_1}{x - x_1} = m$$

$$\leftrightarrow \quad y - y_1 = m(x - x_1) \ .$$

Since this equation is satisfied by the coordinates of P_1, it is satisfied by the coordinates of every point of the line.

Since the steps in the development are reversible, it follows that every point whose coordinates satisfy the equation is a point of the line.

$\therefore$ the equation of the line is $y - y_1 = m(x - x_1)$.

This form of the equation of a line is called the *point-slope form.*

Corollary 1: The equation of the line parallel to the x-axis and on $P_1(x_1, y_1)$ is $y = y_1$.

Proof:

$$l \parallel x\text{-axis}$$

$$\leftrightarrow \quad m = 0$$

$$\leftrightarrow \quad y - y_1 = 0$$

$$\leftrightarrow \quad y = y_1 \ .$$

$\therefore$ the equation of the line is $y = y_1$.

Corollary 2: The equation of the line parallel to the y-axis and on $P_1(x_1, y_1)$ is $x = x_1$.

Proof:

$$l \parallel y\text{-axis}$$

$$\leftrightarrow \quad \triangle x = 0$$

$$\leftrightarrow \quad x - x_1 = 0$$

$$\leftrightarrow \quad x = x_1 \ .$$

$\therefore$ the equation of the line is $x = x_1$.

Corollary 3: The equation of a line on the origin is $y = mx$.

Proof: If $P_1(x_1, y_1)$ is the origin, then $x_1 = 0$, $y_1 = 0$.

$\therefore$ the equation is $y - 0 = m(x - 0)$

$$\leftrightarrow y = mx.$$

Write solutions for the following problems and compare them with those on page 416.

1. Find the equation of the line on the point $A(4, -3)$ and (i) parallel to the x-axis (ii) parallel to the y-axis.
2. Find the equation of the line on the points $A(-2, 3)$ and $B(4, -1)$.
3. Find the equation of the line on $A(2, -1)$ and having a slope $-\sqrt{3}$.

Exercise 5-3

(A)

State, in simplified form, the slope of the line determined by each pair of points having coordinates as follows:

1. $(2, 3), (4, 7)$
2. $(-3, 5), (2, -7)$
3. $(-4, -6), (-1, -2)$
4. $(0, 0), (-3, 5)$
5. $(0, -3), (4, 0)$
6. $(-2, -2), (2, 2)$
7. $(2, 3), (-1, 3)$
8. $(-3, 4), (-5, -6)$
9. $(-3, 3), (3, -3)$

In each of the following, the slope of a line and the coordinates of a point on it are given. State the equation of the line in unsimplified form:

10. $\frac{2}{3}$; $(-4, 1)$
11. $-\frac{4}{3}$; $(3, -2)$
12. 1; $(3, 4)$
13. -1; $(2, -2)$
14. $-\frac{5}{6}$; $(-1, 0)$
15. $\frac{7}{2}$; $(0, -5)$
16. 0; $(1, 4)$
17. 0; $(0, 0)$
18. $-\frac{1}{4}$; $(-3, 2)$

(B)

19. Using the information below, determine, in simplified form, the equation of each of the following lines and show each line on a diagram:
(i) on $A(-2, 7)$ and having slope $\frac{4}{5}$
(ii) on $A(0, 0)$ and $B(3, 3)$
(iii) on $Q(-5, 4)$ and $R(1, 4)$
(iv) having slope -1 and on $(-3, 7)$
(v) on $M(4, -3)$ and having slope -1
(vi) having slope 0 and on $A(2, -4)$
(vii) on $A(1, -3)$ and $B(1, 7)$
(viii) having slope $-\frac{3}{2}$ and on $A(-\frac{2}{3}, \frac{3}{4})$
(ix) on $A(\frac{3}{2}, 0)$ and $B(0, -\frac{4}{3})$
(x) having slope $-\frac{2}{3}$ and on $A(\frac{1}{2}, -\frac{1}{2})$

20. What is the equation of a straight line parallel to the x-axis and on $A(5, -4)$?

21. Find the equation of the x-axis.

22. A straight line is on the point $A(3, 4)$ and is parallel to the y-axis. Find its equation.

23. Find the equation of the y-axis.

24. Find the equation of the straight line on P_1 (a, b) and $O(0, 0)$. Does this equation have any special feature? Is this characteristic of the equations of all lines passing through the origin?

5·4 Other forms of the equation of a line.

Discovery Exercise 5-4

(B)

Using the two given conditions in each of the following, find the equation of the line and compare your solution with that on page 417.

1. Write the equation of line l in *Fig. 5-9*.

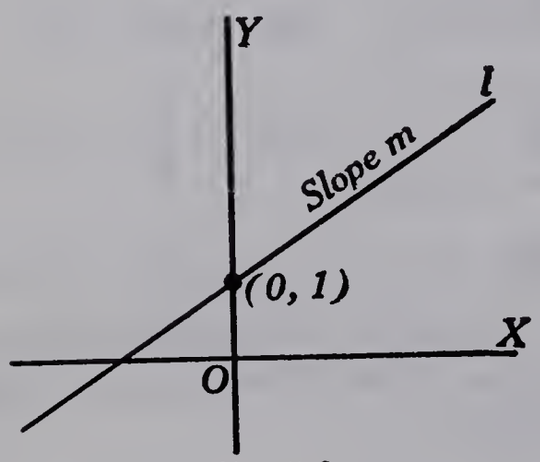


Fig. 5-9

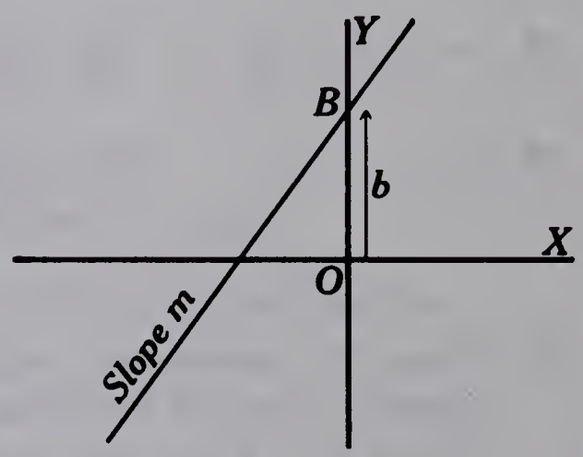


Fig. 5-10

2. The line in *Fig. 5-10* has slope m and y-intercept b. Find its equation.

3. In *Fig. 5-11*, the line has slope m and x-intercept a. Find its equation.

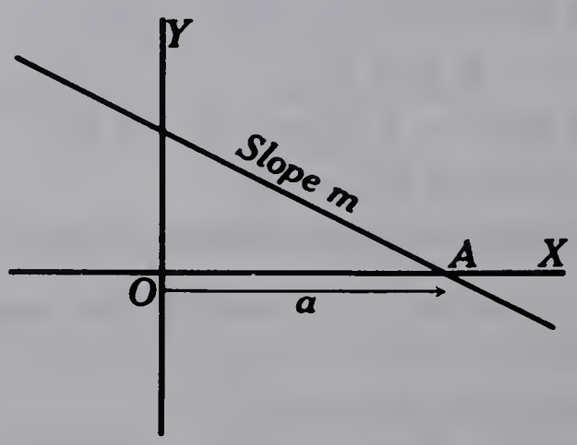


Fig. 5-11

Summary. Forms of the equation of a line.

GIVEN CONDITIONS	EQUATION OF THE LINE
Slope m Point $P_1(x_1, y_1)$	$y - y_1 = m(x - x_1)$
Slope m y-intercept b	$y = mx + b$
Slope m x-intercept a	$y = m(x - a)$

Write solutions for the following problems and compare them with those on page 417.

1. Find the equation of the line determined by each of the following:
   (i) slope $\sqrt{3}$ and y-intercept -2;
   (ii) x-intercept 3 and slope $-\frac{3}{4}$.
2. By transforming the equation $2x - 3y + 6 = 0$ to the form $y = mx + b$, find the slope and y-intercept and draw the graph of the line defined by the equation.
3. Find the slope of the line with defining sentence $Ax + By + C = 0$, $B \neq 0$, $x, y \in R$, A, B, C represents real constants.
4. Find the equation of a line having x-intercept 3 and y-intercept 4.
5. Find the equation of a line having x-intercept a and y-intercept b.

Exercise 5-5

(A)

1. State in unsimplified form the equation of the line determined by each of the following:
   (i) x-intercept 3, slope $-\frac{1}{2}$
   (ii) slope $-\frac{1}{2}$, y-intercept $3\frac{1}{2}$
   (iii) $A(3, 0)$ and $B(0, -2)$
   (iv) slope -1, y-intercept -2
   (v) $A(0, 5)$, slope $-\sqrt{3}$

(B)

2. Find the equation of the line with slope $\frac{1}{\sqrt{3}}$ and x-intercept $3\frac{1}{2}$.

3. A line has x-intercept -2 and slope $\frac{3}{2}$. Find its equation.
4. Find the equations of lines, segments of which form the sides of the rectangle of which three vertices are $A(-3, 2)$, $B(-3, -1)$, and $C(4, -1)$.
5. (i) Find the y-intercept of the line with equation $2x - y + 4 = 0$.
   (ii) Find the equation of the line having the same y-intercept as the line in (i) and having slope $-\frac{1}{\sqrt{3}}$.
6. By transforming the equation $x + 2y - 5 = 0$ to the form $y = mx + b$, find the slope and y-intercept of the line defined by the, equation.
7. For the triangle with vertices $A(-1, 5)$, $B(-6, -3)$, and $C(4, 1)$ find the equation of the line of which the median from A is a segment.

5·5 Conditions for parallelism and perpendicularity.

a. *Parallelism.*

In the particular case illustrated in *Fig.* 5-12,

(i) $l_1 \parallel l_2$;

(ii) h, a, k, and b are the lengths of PN, QN, RM, and SM, respectively.

If m_1 and m_2 are the slopes of l_1 and l_2, respectively, then $m_1 = \frac{h}{a}$ and $m_2 = \frac{k}{b}$.

Since $l_1 \parallel l_2$, it may be shown that

$\triangle PQN \sim \triangle RSM$.

$\because \frac{h}{a} = \frac{k}{b}$.

$\therefore m_1 = m_2$.

Conversely, if $m_1 = m_2$, then $l_1 \parallel l_2$.

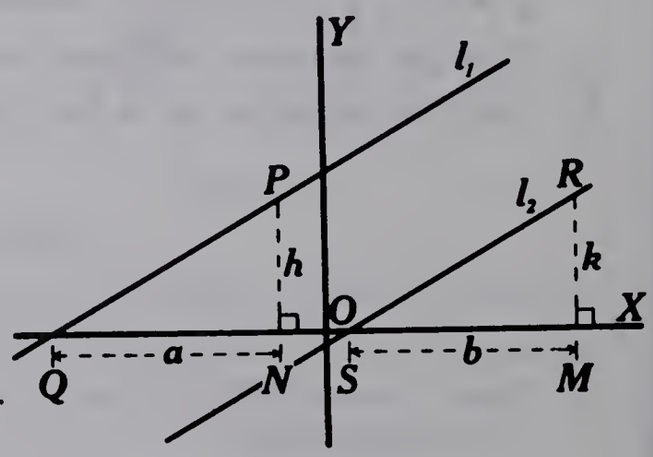


Fig. 5-12

This discussion suggests the Theorem:

Two distinct lines are parallel if and only if the slopes of the lines are equal.

or

If m_1 and m_2 are the slopes of two distinct lines l_1 and l_2 respectively, then

$$l_1 \parallel l_2 \leftrightarrow m_1 = m_2 .$$

b. *Perpendicularity.*

In the particular case illustrated in *Fig. 5-13*:

(i) $l_1 \perp l_2$;

(ii) h, a, and b are the lengths of PS, QS, and SR respectively.

If m_1 and m_2, $(m_1, m_2 \neq 0)$, are the slopes of l_1 and l_2 respectively, then

$$m_1 = \frac{h}{a}, \quad m_2 = \frac{h}{-b}.$$

Since $l_1 \perp l_2$ and $PS \perp QR$,
it may be shown that

$$\triangle PQS \sim \triangle RPS .$$

$$\therefore \quad \frac{h}{a} = \frac{b}{h}.$$

$$\therefore \quad m_1 = \frac{b}{h}$$

$$\therefore \quad m_1 m_2 = \frac{b}{h} \times \frac{h}{-b} = -1$$

$$\therefore \quad m_1 = \frac{-1}{m_2} \qquad (m_1 \text{ and } m_2 \text{ are } \textit{negative reciprocals.})$$

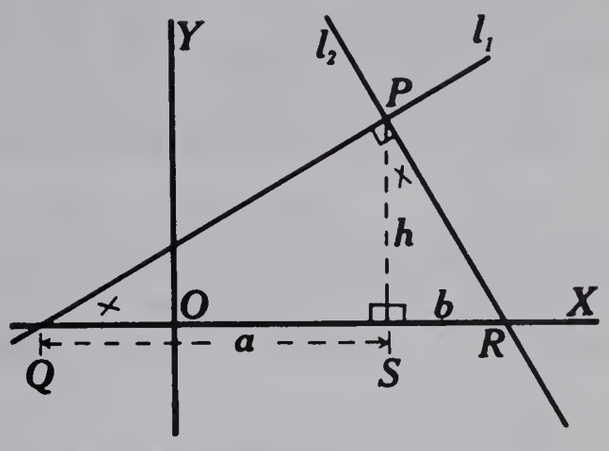


Fig. 5-13

Conversely, if $m_1 = -\frac{1}{m_2}$ then $l_1 \perp l_2$.

This discussion suggests the Theorem:

If m_1 and m_2 $(m_1, m_2 \neq 0)$ are the slopes of two lines l_1 and l_2 respectively, then

$$l_1 \perp l_2 \leftrightarrow m_1 = -\frac{1}{m_2} \text{ or } m_1 m_2 = -1 .$$

Write solutions for the following examples and compare them with those on page 418.

1. Prove that the quadrilateral with vertices $A(-3, 3)$, $B(-4, -3)$, $C(4, -2)$, and $D(5, 4)$ is a parallelogram.
2. Prove that the line on $A(-5, 3)$ and $B(7, 1)$ is perpendicular to the line on $C(4, 6)$ and $D(3, 0)$.
3. Prove that the triangle with vertices $A(-1, 6)$, $B(-3, 3)$, and $C(5, 2)$ is right-angled at A.

Exercise 5-6

(A)

1. State the slope of lines perpendicular to lines with slope:

(i) -2 (ii) $\frac{3}{5}$ (iii) $\frac{p}{q}$ (iv) a (v) 0

2. State the slope of lines parallel to the lines with equations:

(i) $y = x + 3$ (ii) $y = \sqrt{3}x + 7$ (iii) $y = 8$

(iv) $\sqrt{3}y = x - 2$ (v) $x + y = 7$

3. State the slope of lines perpendicular to lines with slope:

(i) $\frac{1}{\sqrt{3}}$ (ii) $3 : 0$ (iii) -1 (iv) $-\sqrt{3}$ (v) 1

(B)

4. Find the coordinates of a point P in the y-axis such that the line on P and $A(-7, -3)$ is parallel to the line on $C(-4, 3)$ and $D(4, 7)$.

5. Determine whether the triangle with vertices $A(2, 4)$, $B(0, -4)$, and $C(5, -1)$ is right-angled.

6. Find the equation of the line on $A(2, -3)$ and parallel to the line on $C(-4, 0)$ and $D(2, 5)$.

7. Find the equation of the line on $A(-3, 2)$ which is perpendicular to the line on A and $B(1, 5)$.

8. Find the abscissa of the point A with ordinate 2, such that the line on A and the origin is perpendicular to the line on the origin and $B(2, -6)$.

9. $A(1, -4)$, $B(-3, 0)$, and $C(1, -5)$ are three vertices of parallelogram $ABCD$. If D is in the fourth quadrant, find the equations of the lines of which AD and CD are segments.

10. Find the slope of a line which is parallel to the line with equation $3x - 4y + 7 = 0$.

(C)

11. Find the slope of lines perpendicular to the line with equation $Ax + By + C = 0$.

12. Two sides of a parallelogram are segments of lines with equations $4x + 3y - 12 = 0$ and $x - 3y = 0$. One of the vertices is $(2, 7)$. Find the equations of the lines of which the remaining two sides are segments.

13. Find the coordinates of the point of intersection of the diagonals of a quadrilateral having vertices $A(-1, 3)$, $B(-3, -2)$, $C(1, -1)$, and $D(4, 2)$.

5·6 The linear equation $y = mx + b$, $x \in R$.

The sentence $y = mx + b$, $x \in R$ is the defining sentence of a linear function

$$l_1 = \{(x, y) \mid y = mx + b, x \in R\}$$

for all m and $b \in R$.

Since this particular form of the linear equation is the defining sentence of a function for all real replacements of the coefficient, the function l_1 is sometimes referred to as the general linear function. This function may be expressed

$$l_1 = \{(x, mx + b) \mid x \in R\},$$
$$l_1 = \{(x, l_1(x)) \mid l_1(x) = mx + b, x \in R\}.$$

The graph of the function for any real m and b is a straight line with slope m and y-intercept b. Note that there is no real replacement of m and b for which this equation can represent a line parallel to the y-axis,

5·7 The role of m in $y = mx + b$, $x \in R$. Consider the linear function

$$f = \{(x, y) \mid y = mx + 2, x \in R\}.$$

The sentence $y = mx + 2$ has one undetermined constant m which may be any real number and represents the slope of a line.

Write solutions to the following questions; compare your solutions with those on page 419.

1. Copy and complete the following table.

m	$y = mx + 2$	DESCRIPTION OF THE LINE
1	$y = x + 2$	Slope 1, y-intercept 2
-2		
-3		
$+2$		
-4		
$+\frac{1}{5}$		
$-\frac{1}{4}$		
$+\frac{1}{3}$		
0		

2. What common characteristic do each of the lines have?

3. How does each line differ from the others?

4. If all such lines are considered, what do they have in common?

5. How are such lines described?

6. Draw a diagram representing the information in the table.

Such a set of lines is called a *family* or *system* of lines. Each member of the family has a common characteristic, in this case the same y-intercept 2. Each member differs in some respect; in this case the slope of each line is different.

In this example "m" may be referred to as the family variable although it is a constant or fixed number for each member. Such a variable is called a *parameter*.

7. Find the equation of the line with y-intercept 3 and on the point $A\ (3, 2)$.
8. Find the equation of the line with y-intercept -6 and perpendicular to the line $y = 5x + 4$.

5·8 The role of b in $y = mx + b$, $x \in R$. Consider the linear function

$$g = \{(x, y \mid y = 3x + b, x \in R\}.$$

The sentence $y = 3x + b$ has one undetermined constant b which may be any real number and represents the y-intercept of the line.

Write solutions for the following problems; compare your solutions with those on page 420.

1. Copy and complete the following table.

b	$y = 3x + b$	DESCRIPTION OF THE LINE
-3	$y = 3x - 3$	Slope 3, y-intercept -3
-1		
0		
2		
4		
6		

2. What common characteristic do each of the lines have?
3. How does each line differ from the others?
4. If all such lines are considered, what do they have in common?
5. How are such lines described?
6. Draw a diagram representing the information in the table.
7. What does the equation $y = 3x + b$, $b \in R$ represent geometrically?
8. In such an equation what name is given to b?

Exercise 5-7

(A)

1. State the equation of the family of lines determined by each of the following conditions:

(i) slope 2 (ii) slope -1
(iii) y-intercept 3 (iv) y-intercept -2
(v) slope 1 (vi) on the point (0, 6)
(vii) parallel to a line with equation $y = 3x + 2$
(viii) slope $\sqrt{3}$

2. State the equations of *two* families of lines to which each of the lines defined by the following equations belongs:

(i) $y = 3x + 2$ (ii) $y = -2x + 3$
(iii) $y = -x - 4$ (iv) $y = \frac{1}{5}x - 5$

3. (i) State the condition that the linear relation

$$L = \{(x, y) \mid ax + by + c = 0, x, y \in R\} .$$

be a linear function.

(ii) What is the graph of this relation if (a) $a = 0$, (b) $b = 0$?

4. If $m, b \in R$, why is the relation

$$L_1 = \{(x, y) \mid y = mx + b, x, y \in R\}$$

always a function?

(B)

5. A line l has the same y-intercept as the line with equation $y = 3x + 4$.
(i) Write the equation of a family of lines to which l belongs.
(ii) Write the equation of the member of the family whose slope is -2.

6. A line p is perpendicular to the family of lines with equation $y = -2x + b$.
(i) What is the slope of a family of lines to which p belongs?
(ii) Write the equation of this family.
(iii) What does the parameter in the equation represent?
(iv) Find the equation of the member on the point $A\ (0, \frac{5}{3})$.

7. (i) Write the equation of the family of lines parallel to the line with equation $y = 2x - 3$.
(ii) Find the equation of the member of the family on the point $P_1(2, 3)$.
(iii) Find the equation of the member of the family on the point $P_1(x_1, y_1)$.

8. (i) Write the equation of the family of lines perpendicular to the line with equation $y = 3x + 4$.
(ii) Find the equation of the member of the family on the point $A(3, -2)$.
(iii) Find the equation of the member of the family on the point $P_1(x_1, y_1)$.

9. A line is on the point A (2, 3); its slope is 3. Determine its equation.

10. A line with slope -2 is on the point B (3, 0). Determine its equation.

5·9 Comparison of $y = mx + b$ and $Ax + By + C = 0$, $x \in R$. In Section 5·1 the sentence

$$Ax + By + C = 0,\ x \in R,\ B \neq 0$$

was described as the defining sentence of the general linear function.

$$Ax + By + C = 0 \tag{1}$$

$$\leftrightarrow \quad By = -Ax - C \tag{2}$$

$$\leftrightarrow \quad y = -\frac{A}{B}x - \frac{C}{B},\ B \neq 0. \tag{3}$$

If we compare equation (3) with the form

$$y = mx + b$$

we see that the lines whose equations are $Ax + By + C = 0$, $B \neq 0$ have

(i) slope $m = -\dfrac{A}{B}$,

(ii) y-intercept $= -\dfrac{C}{B}$.

Exercise 5-8

(A)

1. State each of the following equations in the form $y = mx + b$:

(i) $x + y + 3 = 0$ (ii) $3x - y + 2 = 0$

(iii) $-x + 3y - 2 = 0$ (iv) $5x - 3y + 6 = 0$

2. Under what condition can the equation $ax + by + c = 0$ not be expressed in the form $y = mx + b$?

3. State the equation of the line determined by each of the following pairs of conditions:

(i) slope 2, y-intercept -3 ;

(ii) slope -3, on point $A(3, 4)$;

(iii) slope $\frac{1}{2}$, x-intercept 4 ;

(iv) on the two points $B(3, 4)$, $C(2, 3)$.

4. State the geometric conditions implied by the form of each of the following equations of lines:

(i) $y = 3x + 6$ (ii) $y - 3 = 2(x - 2)$

(iii) $y = 3(x - 3)$ (iv) $y - 3 = \dfrac{3 - 2}{2 - 1}(x - 2)$

5. Using the form of equation best suited, write equations for each of the following families of lines:

(i) slope $-\frac{1}{5}$ (ii) y-intercept -3

(iii) on the point $A(2, 3)$ (iv) x-intercept -5

6. State the slope, the y-intercept, and the x-intercept of the lines represented by each of the following:

(i) $3x - 2y + 6 = 0$ (ii) $5x + 3y - 10 = 0$

(iii) $-x - 3y + 15 = 0$ (iv) $\sqrt{3}x + \sqrt{2}y + \sqrt{5} = 0$

(v) $5x - 10 = 0$ (vi) $3y - 7 = 0$

(B)

7. Determine the equations of the following lines in the form $ax + by + c = 0$:

(i) slope $\frac{1}{5}$, y-intercept $\frac{1}{6}$;

(ii) on the points $A(4, 5)$, $B(3, -1)$;

(iii) slope $\frac{1}{3}$, x-intercept -2 ;

(iv) slope $-\frac{3}{5}$, on the point $C(3, -2)$;

(v) parallel to the line with equation $3x - 2y + 3 = 0$ and on the point $D(3, 5)$;

(vi) perpendicular to the line with equation $5x + 3y - 2 = 0$ and on the point $E(-2, -3)$;

(vii) slope $\frac{1}{\sqrt{3}}$ and with the same y-intercept as the line with equation $3x + 2y - 6 = 0$;

(viii) x-intercept 4, y-intercept 2 .

8. Write the equation of the family of lines on $A(2, -6)$. Select the member of the family satisfying each condition:

(i) x-intercept 5 ;

(ii) equal intercepts on the axes ;

(iii) y-intercept 3 times x-intercept .

9. Determine the equation of the family of lines:

(i) parallel to line $3x - 2y + 6 = 0$;

(ii) parallel to line $Ax + By + C = 0$;

(iii) perpendicular to line $3x - 2y + 6 = 0$;

(iv) perpendicular to line $Ax + By + C = 0$.

5·10 **General expressions for lengths of line segments parallel to the axes.**

(i) Length of a line segment parallel to the x-axis and with end points $P_1(x_1, y_1)$ and $P_2(x_2, y_1)$ is $|\triangle x|$

$$|\triangle x| = |x_2 - x_1| .$$

(ii) Length of a line segment parallel to the y-axis and with end points $P_1(x_1, y_1)$ and $P_2(x_1, y_2)$ is $|\triangle y|$

$$|\triangle y| = |y_2 - y_1|.$$

Write a solution for the following problem and compare it with that on page 420.

1. Calculate the lengths of the line segments determined by the following pairs of end points:

(i) $A(-2, 0), B(-8, 0)$ (ii) $P(4, -2), Q(4, 0)$
(iii) $C(0, 0), A(0, -5)$ (iv) $R(-2, -3), S(0, -3)$
(v) $M(a, -1), N(a, -4)$ (vi) $W(p, q), Z(r, q)$
(vii) $A(a_1, b_1), B(a_1, b_2)$ (viii) $C(0, y_1), D(0, y_2)$

5·11 The method of calculating the length of the line segment determined by any two points (distance between any two points).

Discovery Exercise 5-9

(B)

Write solutions for the following problems and compare them with those on page 421.

1. In *Fig. 5-14*, PR and QR are parallel to the axes.

(i) Calculate PR and RQ.
(ii) Using the Pythagorean Theorem, calculate PQ.

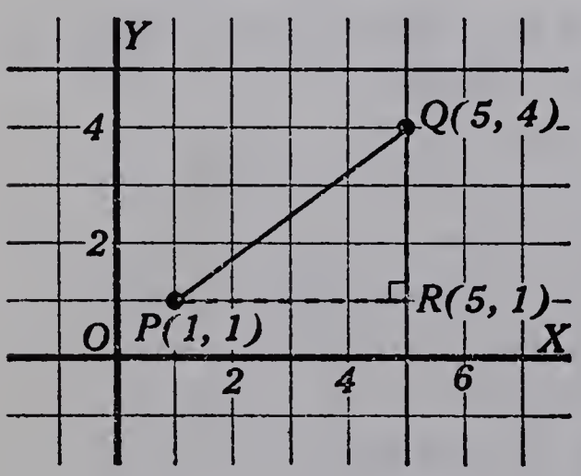


Fig. 5-14

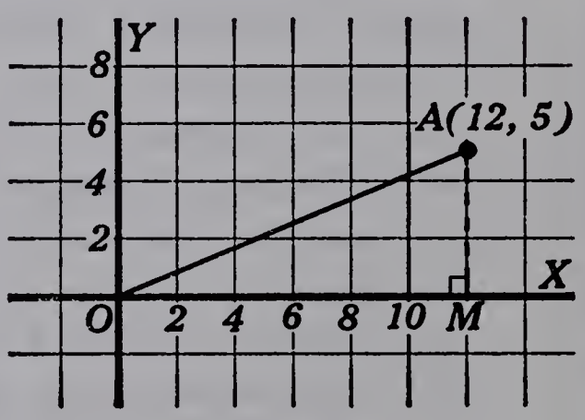


Fig. 5-15

2. With reference to *Fig. 5-15*:

(i) determine the coordinates of M
(ii) calculate OM, AM, and OA.

3. In *Fig. 5-16*, AC and BC are drawn parallel to the axes.

(i) Find the coordinates of C.

(ii) Calculate CB, AC, and AB.

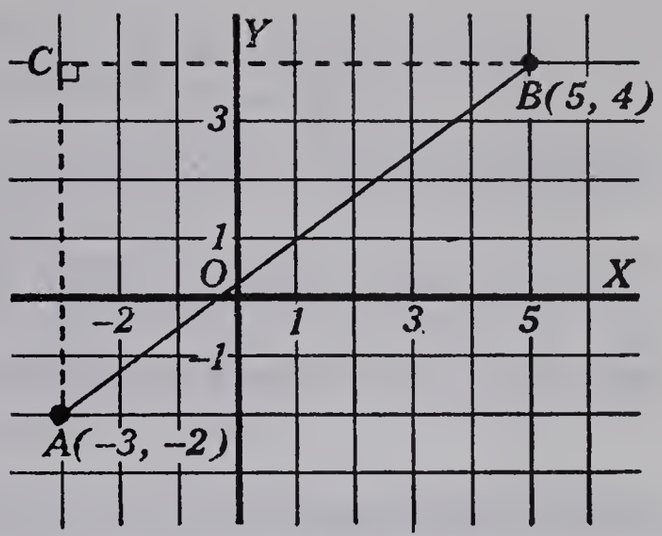


Fig. 5-16

4. Calculate the lengths of the line segments determined by the following pairs of end points:

(i) $P(-2, 3)$, $Q(3, -9)$ (ii) $A(1, 1)$, $B(5, 3)$

(iii) $P_1(-3, -4)$, $P_2(3, 4)$ (iv) $R(7, -1)$, $S(-8, 7)$

(v) $M(a, b)$, $N(c, d)$ (vi) $P_1(x_1, y_1)$, $P_2(x_2, y_2)$

5·12 General expression for the distance between any two points.

Hypothesis: A line segment joining any two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$

Required: To find the length of P_1P_2

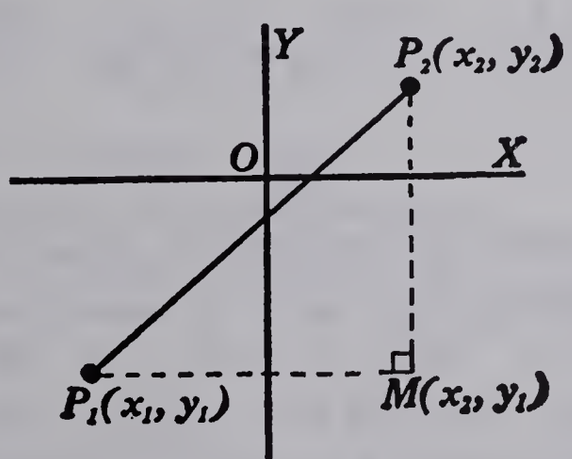


Fig. 5-17

Line segments P_1M and P_2M parallel to the axes meet on M (*Fig. 5-17*). Since the coordinates of M are (x_2, y_1),

$$\therefore \quad P_1M = |x_2 - x_1|$$
$$\text{and} \quad MP_2 = |y_2 - y_1|.$$

By the Pythagorean Theorem:

$$P_1P_2^2 = P_1M^2 + M^2P_2$$
$$\leftrightarrow P_1P_2^2 = |x_2 - x_1|^2 + |y_2 - y_1|^2.$$

$$\leftrightarrow P_1P_2 = \sqrt{|x_2 - x_1|^2 + |y_2 - y_1|^2} \quad (\because P_1P_2 \in {}^+R)$$
$$\leftrightarrow P_1P_2 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}\,.$$

Corollary: If $P_2(x_2, y_2)$ is the origin, then $x_2 = 0$, $y_2 = 0$, and

$$P_1O = \sqrt{x_1^2 + y_1^2}\,.$$

The completion postulate states that B is between A and C if and only if $AB + BC = AC$. Thus, A, B, and C are collinear if $AB + BC = AC$. This is a second test for the collinearity of three points.

Example. Show that $A(0, 3)$, $B(-2, 6)$, $C(4, -3)$ are collinear.

Solution.

$BA = \sqrt{(-2-0)^2 + (6-3)^2} = \sqrt{4+9} = \sqrt{13}$,

$BC = \sqrt{(4-(-2))^2 + (-3-6)^2} = \sqrt{36+81} = \sqrt{117} = 3\sqrt{13}$,

$AC = \sqrt{(4-0)^2 + (-3-3)^2} = \sqrt{16+36} = \sqrt{52} = 2\sqrt{13}$.

Since $BA + AC = BC$, $\therefore$ B, A, C are collinear.

Exercise 5-10

(A)

1. State expressions for the distance between each of the following pairs of points:

(i) $(0, 0)$, $(-3, 4)$ (ii) $(9, -12)$, $(0, 0)$ (iii) $(2, 4)$, $(10, 10)$
(iv) $(-3, -5)$, $(2, 7)$ (v) $(-5, 4)$, $(4, 16)$ (vi) $(1, -4)$, $(9, 14)$

(B)

2. Calculate the lengths of line segments whose end points have coordinates as follows:

(i) $(-4, 3)$, $(0, 0)$ (ii) $(-4, 7)$, $(3, -2)$ (iii) $(-1, -4)$, $(4, 6)$
(iv) $(3, 3)$, $(-1, -1)$ (v) $(2, 4)$, $(-6, 4)$ (vi) $(1, -4)$, $(1, -6)$

3. Find x so that the length of the segment with end points the origin and $A(x, 8)$ is 10.

4. Find the perimeters of the figures determined by:

(i) $A(4, 0)$, $B(0, 3)$, $C(-4, 0)$
(ii) $P(2, -4)$, $Q(2, 6)$, $R(-2, 6)$, $S(-2, -4)$

5. Determine the type of triangle having vertices:

(i) $A(-4, 2)$, $B(7, 2)$, $C(-4, 6)$
(ii) $P(-2, -2)$, $Q(-2, 8)$, $R(7, 3)$

6. Prove that $A(3, 4)$, $B(-4, 3)$, $C(-3, 4)$, $D(5, 0)$, and $E(0, -5)$ are points of a circle with centre $O(0, 0)$, the origin.
7. Find the real number y so that the point $M(-2, y)$ is equidistant from $P(-3, 2)$ and $Q(5, -4)$.
8. Find the coordinates of a point (i) on the x-axis (ii) on the y-axis, which is equidistant from $A(-1, 2)$ and $B(3, -2)$.
9. Show that $M(1, 0)$ is the centre of the circle circumscribing the triangle with vertices $A(6, 0)$, $B(4, 4)$, and $C(-3, 3)$.
10. Show that $B(-3, 1)$ is the midpoint of the line segment whose end points are $A(-8, 7)$ and $C(2, -5)$.
11. The distance from the point $P(x, y)$ to $Q(-2, 4)$ is 5. Write and simplify the algebraic equation corresponding to this statement.
12. Prove that $A(-2, 3)$, $B(2, -1)$, and $C(4, -3)$ are collinear.
13. Determine whether the points $P(3, -2)$, $Q(-4, 3)$, and $R(-10, 8)$ are collinear.
14. Find the real number x so that $A(-3, 2)$, $B(3, -4)$, and $C(x, -8)$ are collinear.

5·13 Method of calculating the coordinates of the midpoint of a line segment.

Discovery Exercise 5-11

(B)

Write solutions for the following problems and compare them with those on page 421.

1. In *Fig. 5-18*, M is the midpoint of P_1P_2.
   (i) Find, by inspection, the coordinates of M.
   (ii) Verify your conclusion by proving that $P_1M = MP_2 = \frac{1}{2}P_1P_2$.
   (iii) How may the abscissa of M be obtained from the abscissas of P_1 and P_2?

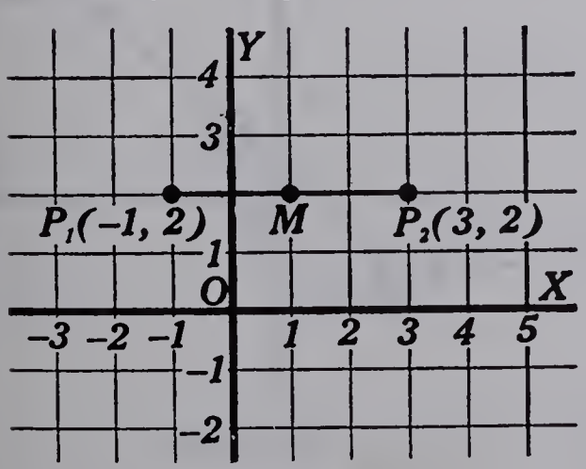


Fig. 5-18

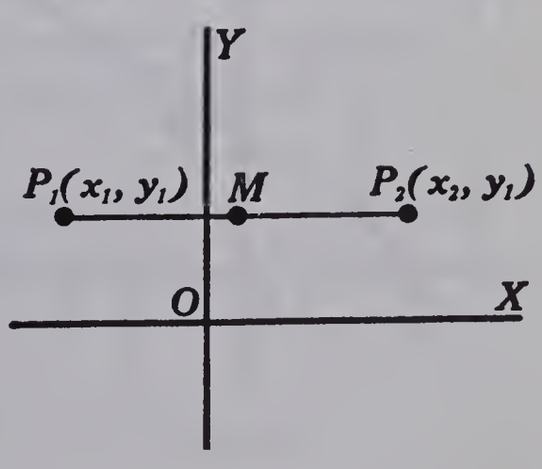


Fig. 5-19

2. In *Fig. 5-19*, M is the midpoint of P_1P_2.

(i) Predict the coordinates of M.

(ii) Verify your conclusion by proving that $P_1M = MP_2 = \frac{1}{2}P_1P_2$.

3. In *Fig. 5-20*, M is the midpoint of P_1P_2.

(i) Find, by inspection, the coordinates of M.

(ii) Verify your conclusion by proving that $P_1M = MP_2 = \frac{1}{2}P_1P_2$

(iii) How may the ordinate of M be obtained from the ordinates of P_1 and P_2?

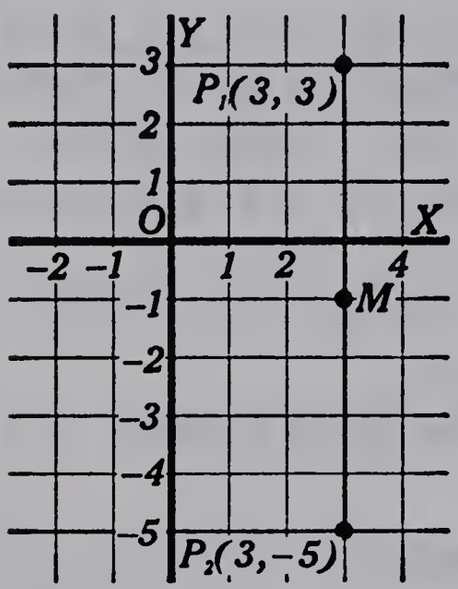


Fig. 5-20

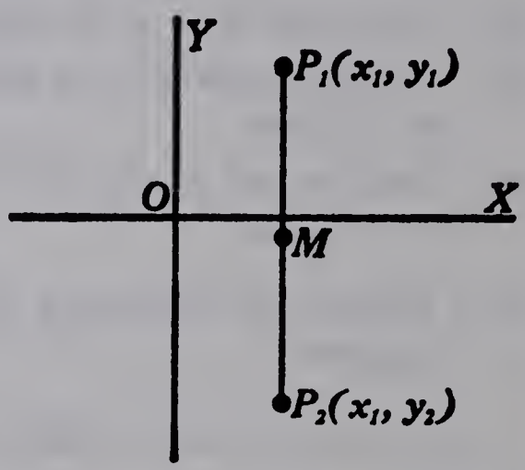


Fig. 5-21

4. In *Fig. 5-21*, M is the midpoint of P_1P_2.

(i) Predict the coordinates of M.

(ii) Verify your conclusion by proving that $P_1M = MP_2 = \frac{1}{2}P_1P_2$.

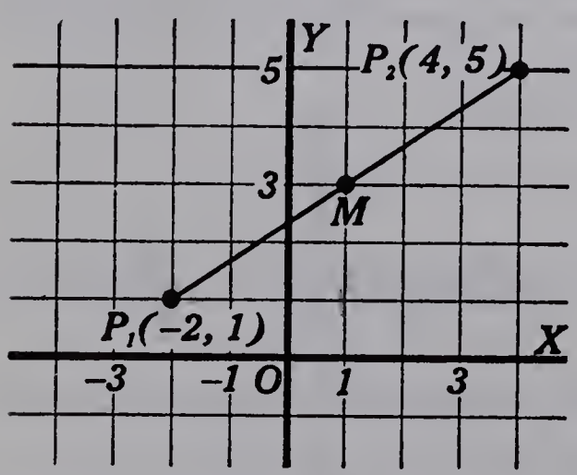


Fig. 5-22

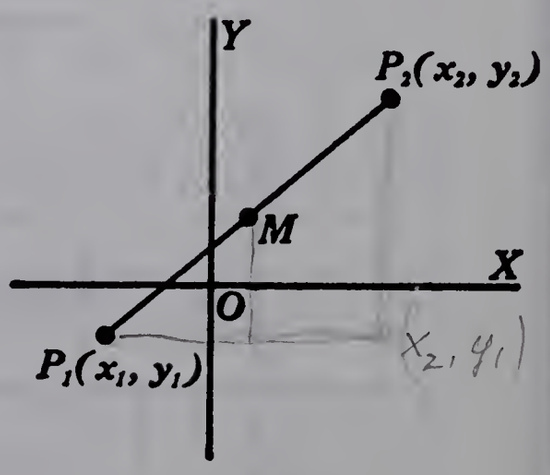


Fig. 5-23

5. In *Fig. 5-22*, M is the midpoint of P_1P_2.

(i) Find, by inspection, the coordinates of M.

(ii) Verify your conclusion by proving that $P_1M = MP_2 = \frac{1}{2} P_1 P_2$.

(iii) How may the coordinates of M be found from the coordinates of P_1 and P_2?

6. In *Fig. 5-23*, M is the midpoint of P_1P_2.

(i) Predict the coordinates of M.

(ii) Verify your conclusion by proving that $P_1M = MP_2 = \frac{1}{2} P_1P_2$.

(iii) How may the coordinates of M be found from the coordinates of P_1 and P_2?

5·14 General expression for the coordinates of the midpoint of the line segment determined by any two points.

Hypothesis: M is the midpoint of the line segment joining $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$.

Conclusion: The coordinates of M are $\left(\frac{x_2 + x_1}{2}, \frac{y_2 + y_1}{2}\right)$

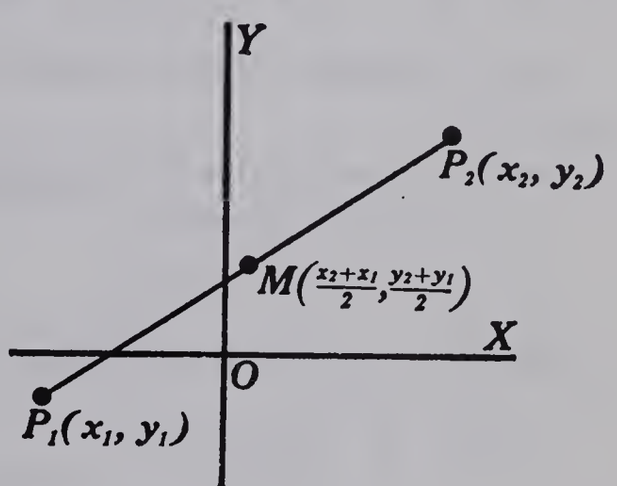


Fig. 5-24

Proof:

$$P_1M = \sqrt{\left(\frac{x_2 + x_1}{2} - x_1\right)^2 + \left(\frac{y_2 + y_1}{2} - y_1\right)^2}$$

$$= \sqrt{\left(\frac{x_2 + x_1 - 2x_1}{2}\right)^2 + \left(\frac{y_2 + y_1 - 2y_1}{2}\right)^2}$$

$$= \sqrt{\left(\frac{x_2 - x_1}{2}\right)^2 + \left(\frac{y_2 - y_1}{2}\right)^2}$$

$$= \frac{1}{2}\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \quad .$$

$$MP_2 = \sqrt{\left(x_2 - \frac{x_2 + x_1}{2}\right)^2 + \left(y_2 - \frac{y_2 + y_1}{2}\right)^2}$$

$$= \sqrt{\left(\frac{2x_2 - x_2 - x_1}{2}\right)^2 + \left(\frac{2y_2 - y_2 - y_1}{2}\right)^2}$$

$$= \sqrt{\left(\frac{x_2 - x_1}{2}\right)^2 + \left(\frac{y_2 - y_1}{2}\right)^2}$$

$$= \frac{1}{2}\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \ .$$

$\because\ P_1P_2 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$\therefore\ P_1M = MP_2 = \frac{1}{2} P_1P_2$

$\therefore\ M\left(\frac{x_2 + x_1}{2}, \frac{y_2 + y_1}{2}\right)$ *or* $M\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$ **is the midpoint of** P_1P_2.

Write a solution for the following problem and compare it with that on page 423.

1. Find the coordinates of the midpoints of line segments determined by the following end points:

(i) $A(-2, -3)$, $B(4, 2)$ (ii) $P(2, -5)$, $Q(-3, -1)$

(iii) $W(-4, -5)$, $Z(-6, -7)$ (iv) $C(0, -1)$, $D(1, 1)$

(v) $R(-3, -2)$, $S(3, 2)$ (vi) $M\left(\frac{a}{2}, \frac{b}{2}\right)$, $N\left(\frac{c}{2}, \frac{d}{2}\right)$

Exercise 5-12

(A)

1. Find the coordinates of the midpoints of the line segments having the following end points:

(i) (2, 3) and (− 2, − 3) (ii) (− 6, 5) and (3, − 4)

(iii) (− 4, − 7) and (− 4, 3) (iv) (− 2, 3) and (6, 3)

(v) (0, 0) and (− 3, 2) (vi) (− 1, − 2) and (− 5, − 6)

(B)

2. For the triangle $A(-7, 4)$, $B(3, -2)$, $C(1, 2)$, find the lengths of the three medians.

3. $C(1, 0)$ is a point of trisection of the line segment with end points $A(-1, -2)$ and $B(5, 4)$. Find the coordinates of the other point of trisection.

4. For the triangle with vertices $A(1, 5)$, $B(-1, -1)$, $C(2, -2)$, prove that the midpoint of AC is equidistant from the three vertices.
5. Show that the y-axis bisects the line segment with end points $P(-5, 7)$ and $Q(5, 2)$.
6. Find the coordinates of the centre of the circle for which the ends of a diameter are $R(-3, 2)$ and $S(6, -7)$.
7. Determine whether the diagonals of the figure with vertices $A(-5, 3)$, $B(-4, -2)$, $C(7, -1)$, and $D(6, 4)$ bisect each other.
8. For the triangle with vertices $A(-2, -3)$, $B(6, -3)$, and $C(4, 5)$, prove that the length of the line segment with end points the midpoints of the sides AC and BC is one-half the length of AB.
9. If M is the midpoint of the side AB in question 8, prove that $AC^2 + BC^2 = 2\,AM^2 + 2\,CM^2$.
10. A quadrilateral has vertices $A(3, 2)$, $B(-2, 1)$, $C(-3, -5)$, and $D(5, -3)$. E, F, G, and H are the midpoints of AB, BC, CD, and DA, respectively. Prove that $FE = GH$.

Review Exercise 5-13

(A)

1. Find the lengths of the line segments determined by points with the following coordinates:

   (i) $(0, 0)$, $(-4, 3)$ (ii) $(2, 5)$, $(-4, 5)$

   (iii) $(4, -1)$, $(4, -7)$ (iv) $(-2, 4)$, $(10, 9)$

   (v) $(7, -1)$, $(-8, 7)$ (vi) $(2, -3)$, $(5, -4)$

2. State the coordinates of the midpoints of line segments for which the end points have the following coordinates:

   (i) $(-2, -3)$, $(2, 3)$ (ii) $(-4, 5)$, $(8, -3)$

   (iii) $(4, -5)$, $(-4, 3)$ (iv) $(0, 0)$, $(-3, 6)$

   (v) $(-\frac{1}{2}, 2)$, $(2\frac{1}{2}, -1)$ (vi) $(2, -3)$, $(2, 8)$

3. Interpret geometrically each of the following:

   (i) $\sqrt{x^2 + y^2}$ (ii) $\sqrt{(x-2)^2 + (y-1)^2}$

   (iii) $\dfrac{y+2}{x-1}$ (iv) $\left(\dfrac{x-1}{2}, \dfrac{y+3}{2}\right)$

   (v) $y = 3x + 5$ (vi) $\dfrac{y-3}{x-2} = \dfrac{2}{-7}$

4. Find the slope of the line:
   (i) on $A(2, -3)$ and $B(3, -4)$
   (ii) on the origin and $C(2, 2\sqrt{3})$
   (iii) perpendicular to the line $-\frac{1}{\sqrt{3}}$
   (iv) parallel to the line on $A(6, -2)$ and $B(5, -3)$
   (v) having equation $2y - 5 = 0$
   (vi) having equation $3x + 2 = 0$
   (vii) having equation $x + y + 5 = 0$
   (viii) parallel to the line with equation $\sqrt{3x} - y + 2 = 0$

(B)

5. $A(-3, -5)$, $B(5, 1)$, and $C(-2, 2)$ are vertices of a triangle. Prove that the triangle is right-angled by using the formula for (i) slope (ii) length of a line segment.
6. Calculate the lengths of the medians of the triangle with vertices $P(1, 4)$, $Q(-5, -2)$, and $R(5, 0)$.
7. Determine whether or not the points $A(-8, -1)$, $B(1, 1)$, and $C(6, 2)$ are collinear.
8. Find x so that the points $P(x, -2)$, $Q(0, 1)$, and $R(5, 2)$ are collinear.
9. Find the equation of the line on $A(-2, 3)$ and perpendicular to the line segment with end points $P(-6, -1)$ and $Q(5, 2)$.
10. Find the equation of the line which has x-intercept a and y-intercept b.
11. Find the centroid of the triangle with vertices $A(-2, 4)$, $B(6, -1)$, and $C(-1, 3)$. The centroid is a point of trisection of a median.
12. Prove that the lines joining the midpoints of the opposite sides of the quadrilateral with vertices $A(5, 7)$, $B(-3, 5)$, $C(1, -3)$, and $D(7,1)$ bisect each other.
13. Find the coordinates of a point on the y-axis which is equidistant from $A(-2, 6)$ and $B(4, -2)$.
14. A line is on $A(3, -2)$ and the midpoint of the line segment joining $B(-2, 4)$ and $C(6, -7)$. Find its equation.
15. By transforming the equations to the form $y = mx + b$, prove that the lines having equations $2x - 3y + 7 = 0$ and $6x + 4y - 5 = 0$ are perpendicular.

Chapter VI

SYSTEMS OF LINEAR EQUATIONS

***6•1* Solving a system of two linear equations in two variables.** The relations

$A = \{(x, y) \mid 2x - y = -2, x, y \in R\}$ and $B = \{(x, y) \mid x - 2y = 2, x, y \in R\}$ each determine a set of ordered pairs, the graph of which is a straight line (*Fig. 6-1*), page 115. The graphs of A and B intersect at the point whose coordinates are $(-2, -2)$. $(-2, -2)$ is the ordered pair which is common to relation A and relation B. Thus $\{(-2, -2)\}$ is the intersection set of A and B. This set is the set determined by the relation

$$C = \{(x, y) \mid 2x - y = -2 \text{ and } x - 2y = 2, x, y \in R\}$$

whose defining sentence involves the system of linear equations

$$\begin{cases} 2x - y = -2, & (1) \\ x - 2y = 2\ . & (2) \end{cases}$$

$\{(-2, -2)\}$ is the solution set of the system.

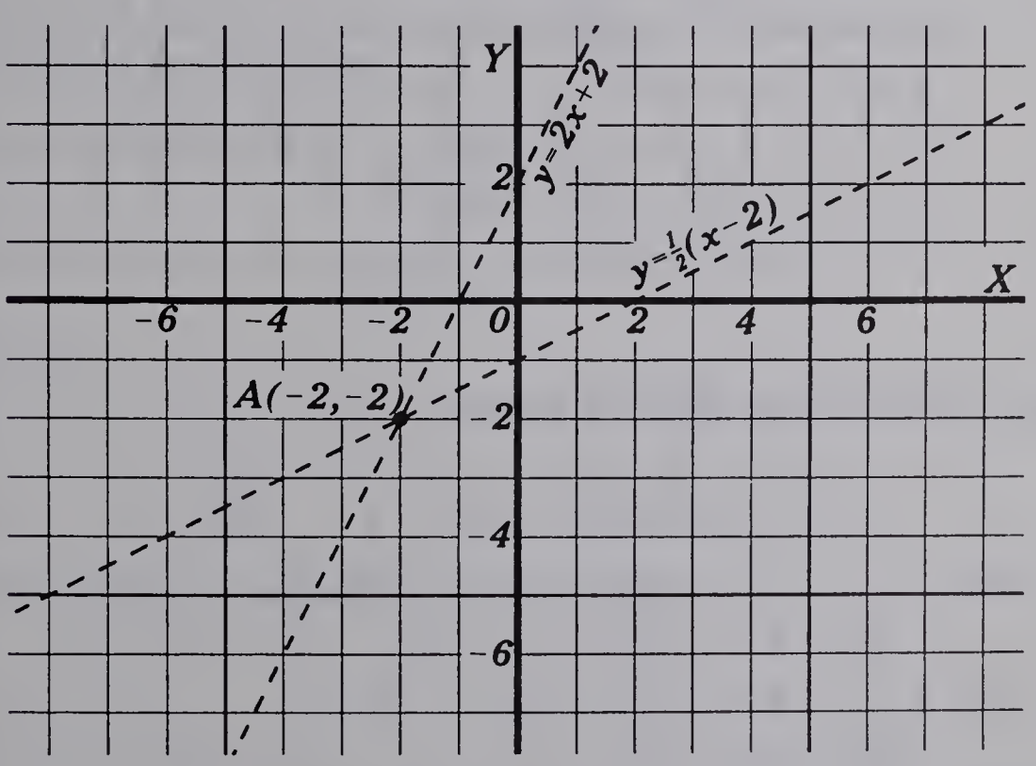


Fig. 6-1

Thus C determines

$$A \cap B = \{(x, y) \mid 2x - y = -2, x, y \in R\} \cap \{(x, y) \mid x - 2y = 2, x, y \in R\}.$$

The solution set of a system of equations in two variables x and y is the set of all ordered pairs (x, y) which are common to the solution sets defined by each of the given systems.

To solve a sytem of equations algebraically it is necessary to determine simpler equivalent systems until a system is obtained from which the solution set may be read. Since the systems are equivalent they will have the same solution set.

There are three commonly used methods of solving a system of two linear equations in two variables:

(a) elimination of a variable by comparison;

(b) elimination of a variable by substitution;

(c) elimination of a variable by addition or subtraction.

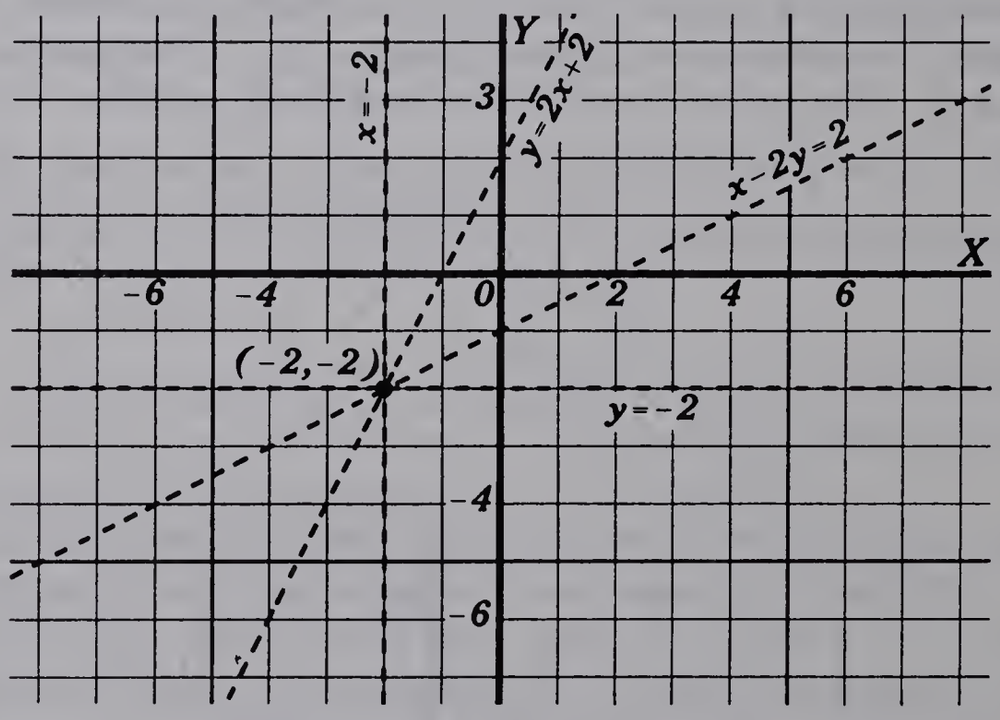


Fig. 6-2

Fig. 6-2 illustrates that the systems

$$\text{(i)} \begin{cases} 2x - y = -2 & (1) \\ x - 2y = 2 & (2) \end{cases}$$

$$\text{(ii)} \begin{cases} x = -2 \\ 2x - y = -2 \end{cases} \qquad \text{(iii)} \begin{cases} x = -2 \\ x - 2y = 2 \end{cases}$$

$$\text{(iv)} \begin{cases} y = -2 \\ 2x - y = -2 \end{cases} \qquad \text{(v)} \begin{cases} y = -2 \\ x - 2y = 2 \end{cases}$$

$$\text{(vi)} \begin{cases} x = -2 \\ y = -2 \end{cases}$$

are equivalent in the sense that they have the same solution set $\{(-2, -2)\}$.

a. Determining the solution set by comparison.

$$\begin{cases} 2x - y = -2 & (1) \\ x - 2y = 2 & (2) \end{cases}$$

From (1): $y = 2x + 2$ (3)

From (2): $y = \dfrac{x - 2}{2}$ (4)

By comparison, $2x + 2 = \dfrac{x - 2}{2}$

$$\leftrightarrow \quad 4x + 4 = x - 2$$

$$\leftrightarrow \quad x = -2. \qquad (5)$$

To obtain the solution set we solve either of the two systems

$$\begin{cases} x = -2 & (5) \\ 2x - y = -2 & (1) \end{cases} \qquad \text{or} \qquad \begin{cases} x = -2 & (5) \\ x - 2y = 2 & (2) \end{cases}$$

By substitution from (5) into (1),

$$-4 - y = -2$$

$$\leftrightarrow \quad y = -2. \qquad (6)$$

By substitution from (5) into (2),

$$-2 - 2y = 2$$

$$\leftrightarrow \quad y = -2. \qquad (6)$$

The system obtained is $\begin{cases} x = -2 & (5) \\ y = -2. & (6) \end{cases}$

The steps of the argument are reversible, and hence the systems indicated in the solution are equivalent.

Thus the solution set is

$$\{(-2, -2)\}.$$

Computation may be checked or verified as follows.

Verification.

L.S. (1) $= 2(-2) - (-2)$
$= -2.$

R.S. (1) $= -2.$

L.S. (2) $= -2 - 2(-2)$
$= 2.$

R.S. (2) $= 2.$

b. Determining the solution set by substitution.

$$\begin{cases} 2x - y = -2 & (1) \\ x - 2y = 2 & (2) \end{cases}$$

From (1): $y = 2x + 2$ (3)

Substitute in (2):
$$\begin{aligned} & x - 2(2x + 2) = 2 \\ \leftrightarrow\ & x - 4x - 4 = 2 \\ \leftrightarrow\ & -3x = 6 \\ \leftrightarrow\ & x = -2. \end{aligned}$$

Substitute in (1):
$$\begin{aligned} & -4 - y = -2 \\ \leftrightarrow\ & y = -2. \end{aligned}$$

The steps of the argument are reversible and therefore the solution set is
$$\{(-2, -2)\}$$

The equivalent systems used in the solution are:

(i) $\begin{cases} 2x - y = -2 \\ x - 2y = 2 \end{cases}$ (ii) $\begin{cases} y = 2x + 2 \\ x - 2y = 2 \end{cases}$

(iii) $\begin{cases} x = -2 \\ 2x - y = -2 \end{cases}$ (iv) $\begin{cases} x = -2 \\ y = -2 \end{cases}$

The student should always keep in mind that the algebraic solution depends on the development of simpler equivalent systems, and he should set down the equivalent systems at each stage until the procedure is clearly understood.

The determination of a solution by addition or subtraction is discussed in Section 6.2 .

Exercise 6-1

(B)

For each of the following relations:

(i) *express the defining equations in the form* $y = mx + b$;

(ii) *express the relations as the intersection of two relations;*

(iii) *state the slope and the x- and y-intercepts of the graph defined by each equation;*

(iv) *draw the graph and from it determine the ordered pairs of the relation;*

(v) *describe the graph of each relation;*

(vi) *use the method of comparison to determine algebraically the ordered pairs of the relation.*

1. $A = \{(x, y) \mid x + y = 7, x - y = 1, x, y \in R\}$

2. $B = \{(x, y) \mid x + y = 8, 3x - y = 0, x, y \in R\}$

3. $C = \{(x, y) \mid 2x - y + 4 = 0, x + y = 1, x, y \in R\}$

4. $D = \{(x, y) \mid 5x - 2y = 3, 2x + y = 3, x, y \in R\}$

Determine the coordinates of the points of intersection of the lines defined by each of the equations in the following systems. Use the method of substitution.

5. $\begin{cases} 3x - y = 5 \\ x + 2y = -3 \end{cases}$

6. $\begin{cases} 7x - 5y = 11 \\ 3x + y = 11 \end{cases}$

7. $\begin{cases} 6x + y = -12 \\ 10x + 7y = -4 \end{cases}$

8. $\begin{cases} \frac{x}{9} - \frac{y}{6} = \frac{1}{2} \\ 5x + 2y = -25 \end{cases}$

9. Determine the set of ordered pairs

$$\{(a, b) \mid 3a + 2b = 4, 5a - 3b = 13\ a, b \in R\}.$$

10. Determine the set of ordered pairs

$$\left\{(y, z) \;\middle|\; \frac{y}{2} + \frac{2z}{5} = -\frac{1}{5}, \frac{y}{3} - \frac{z}{6} = \frac{7}{6}, y, z \in R\right\}.$$

6•2 Solving a system of two linear equations in two variables by addition or subtraction.

Example 1. Determine the ordered pairs of

$$C = \{(x, y) \mid 2x + y = 4, x + y = 3, x, y \in R\}$$

by elimination of a variable by addition or subtraction.

Solution. C is the solution set of the system of equations:

$$\text{(i)} \begin{cases} 2x + y = 4 & (1) \\ x + y = 3 & (2) \end{cases}$$

The graph of this system is shown in *Fig. 6-3* and the solution set is $\{(1, 2)\}$. From the graph it may be seen that the systems

$$\text{(ii)} \begin{cases} x = 1 \\ 2x + y = 4 \end{cases} \quad \text{(iii)} \begin{cases} x = 1 \\ x + y = 3 \end{cases} \quad \text{(iv)} \begin{cases} y = 2 \\ 2x + y = 4 \end{cases} \quad \text{(v)} \begin{cases} y = 2 \\ x + y = 3 \end{cases}$$

$$\text{(vi)} \begin{cases} x = 1 \\ y = 2 \end{cases}$$

are equivalent in the sense that they have the same solution set $\{(1, 2)\}$.

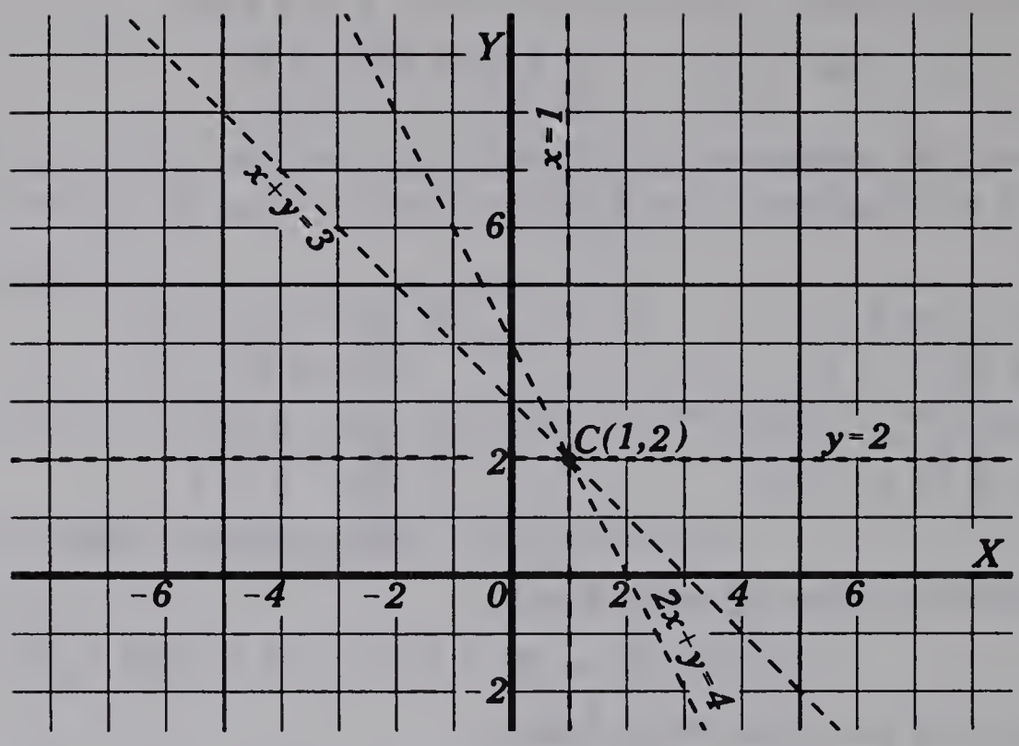


Fig. 6-3

The method of elimination of a variable by addition or subtraction is a convenient method for replacing the original system by a simpler equivalent system.

Subtracting (2) from (1) yields the equation $x = 1$, and taking either (1) or (2) with this equation produces the systems:

$$\begin{cases} x = 1 \\ 2x + y = 4 \end{cases} \quad \text{or} \quad \begin{cases} x = 1 \\ x + y = 3. \end{cases}$$

Multiplying (2) by 2 yields the equation $2x + 2y = 6$ which is equivalent to the equation $x + y = 3$, and taking this equation with equation (1) produces the equivalent system:

$$\begin{cases} 2x + 2y = 6 & (3) \\ 2x + y = 4 & (4) \end{cases}$$

Subtracting (4) from (3) yields the equation $y = 2$ and taking either (1) or (2) with this equation produces the systems:

$$\begin{cases} y = 2 \\ 2x + y = 4 \end{cases} \qquad \begin{cases} y = 2 \\ x + y = 3 \end{cases}$$

Any one of these simpler systems may be changed by the method of substitution to the still simpler system:

$$\begin{cases} x = 1 \\ y = 2 \end{cases}$$

which gives the solution set of the original system.

It is not immediately obvious that the systems used in the development are all algebraically equivalent systems. That they are will not be proved here although a graphical interpretation of the equivalence is given in Section 6·3 .

The algebraic solution is usually written in the following form:

$$\begin{cases} 2x + y = 4 & (1) \\ x + y = 3 & (2) \end{cases}$$

$2 \times (2)$:

$$\begin{cases} 2x + 2y = 6 & (3) \\ 2x + y = 4 & (1) \end{cases}$$

$(3) - (1)$:

$$\begin{cases} y = 2 & (5) \\ x + y = 3 & (2) \end{cases}$$

Substituting $y = 2$ in (2):

$$x + 2 = 3$$

$$\leftrightarrow \quad x = 1. \qquad (6)$$

$$\therefore \begin{cases} x = 1 & (6) \\ y = 2. & (5) \end{cases}$$

Verification.

L.S. (1) $= 2(1) + (2)$
$= 4.$
R.S. (1) $= 4.$

L.S. (2) $= (1) + (2)$
$= 3.$
R.S. (2) $= 3.$

$$\therefore C = \{(1, 2)\}.$$

Example 2. Solve and verify:

$$\begin{cases} \frac{1}{4}x + \frac{1}{2}y = 7 \\ \frac{1}{3}x + \frac{1}{4}y = 6 \end{cases}$$

by elimination of a variable by addition or subtraction.

Solution.

$$\begin{cases} \frac{1}{4}x + \frac{1}{2}y = 7 & (1) \\ \frac{1}{3}x + \frac{1}{4}y = 6 & (2) \end{cases}$$

When an equation of a system involves fractional coefficients, it is usually convenient to replace the equation by an equivalent equation cleared of fractions and then solve the resulting equivalent system as outlined in Example 1.

$4 \times (1)$: $\quad \begin{cases} x + 2y = 28 & (3) \end{cases}$

$12 \times (2)$: $\quad \begin{cases} 4x + 3y = 72 & (4) \end{cases}$

The solution may be completed as in Example 1.

Exercise 6-2

(A)

In each of the following, the systems of equations are equivalent; state how each of the systems has been derived from the first system:

1. $\begin{cases} x + y = 5 \\ x - y = 1 \end{cases}$ and $\begin{cases} 2x + 2y = 10 \\ 2x - 2y = 2 \end{cases}$ and $\begin{cases} 4x = 12 \\ x - y = 1 \end{cases}$

2. $\begin{cases} 2x + 3y = 5 \\ x - y = 0 \end{cases}$ and $\begin{cases} 2x + 3y = 5 \\ 2x - 2y = 0 \end{cases}$ and $\begin{cases} 2x + 3y = 5 \\ 5y = 5 \end{cases}$

3. $\begin{cases} 3x + 5y = 11 \\ 2x + 3y = 7 \end{cases}$ and $\begin{cases} 6x + 10y = 22 \\ 6x + 9y = 21 \end{cases}$ and $\begin{cases} y = 1 \\ 2x + 3y = 7 \end{cases}$

4. $\begin{cases} 6x + 5y = 27 \\ 5x + 6y = 28 \end{cases}$ and $\begin{cases} 11x + 11y = 55 \\ x - y = -1 \end{cases}$ and $\begin{cases} x + y = 5 \\ x - y = -1 \end{cases}$

and $\begin{cases} 2x = 4 \\ x - y = -1 \end{cases}$

5. $\begin{cases} 5x + 7y = 19 \\ 7x + 5y = 17 \end{cases}$ and $\begin{cases} 12x + 12y = 36 \\ 2x - 2y = -2 \end{cases}$ and $\begin{cases} x + y = 3 \\ x - y = -1 \end{cases}$

and $\begin{cases} 2x = 2 \\ x - y = -1 \end{cases}$

(B)

Determine the coordinates of the point of intersection of the lines defined by each of the equations of the following systems. Use the method of addition and subtraction. Verify.

6 $\begin{cases} x + y = 5 \\ x - y = 1 \end{cases}$

7. $\begin{cases} 5x + 4y = 14 \\ 3x + 2y = 8 \end{cases}$

8. Determine $\{(x, y) \mid 3x - 5y = 1,\ 2x + 7y = 11,\ x, y \in R\}$.

9. Determine $\left\{(x, y) \mid \frac{3x}{10} - \frac{y}{4} = \frac{2}{5},\ \frac{2x}{9} + \frac{y}{6} = 1,\ x, y \in R\right\}$.

Determine by any convenient algebraic method, the ordered pairs defined by each of the following systems.

10. $\begin{cases} 3x + 2y = 15 \\ 5x - y = 38 \end{cases}$

11. $\begin{cases} \frac{5}{2}p - q = 4 \\ \frac{3}{2}p - q = 2 \end{cases}$

12. $\begin{cases} 3x + 2y = 67 \\ 5x - 2y = 53 \end{cases}$

13. $\begin{cases} 1.5x + 1.7y = 4.7 \\ x + y = 3 \end{cases}$

14. Determine $\{(x, y) \mid y = 2x + 5, 5x + 7y + 22 = 0, x, y \in R\}$.

15. Determine $\{(x, y) \mid 3y - 5x = 3, 3x = y + 3, x, y \in R\}$.

(C)

16. Draw graphs of the following with reference to the same set of axes:

(i) $\{(x, y) \mid x + 2y - 5 = 0, x, y \in R\}$

(ii) $\{(x, y) \mid 2x + y - 3 = 0, x, y \in R\}$

(iii) $\{(x, y) \mid (x + 2y - 5) + (2x + y - 3) = 0, x, y \in R\}$

(iv) $\{(x, y) \mid (x + 2y - 5) - (2x + y - 3) = 0, x, y \in R\}$

(v) $\{(x, y) \mid (x + 2y - 5) - 2(2x + y - 3) = 0, x, y \in R\}$

(vi) $\{(x, y) \mid 2(x + 2y - 5) - (2x + y - 3) = 0, x, y \in R\}$

17. (i) What are the coordinates of the point which is the intersection set of the graphs in question 16?

(ii) Describe the graph of

$\{(x, y) \mid k_1(x + 2y - 5) - k_2(2x + y - 3) = 0, x, y \in R\}$

where $k_1, k_2 \in R$ and k_1, k_2 are not both zero.

6·3 The graphical interpretation of the elimination of one variable by addition or subtraction (supplementary).

Example 1. (a) (i) Draw graphs of $A = \{(x, y) \mid 2x - y - 4 = 0, x, y \in R\}$ and $B = \{(x, y) \mid x + 2y + 3 = 0, x, y \in R\}$.

(ii) Using the same set of axes, draw the graphs of:

$E = \{(x, y) \mid 3(x + 2y + 3) + (2x - y - 4) = 0, x, y \in R\}$;

$F = \{(x, y) \mid 4(x + 2y + 3) - 7(2x - y - 4) = 0, x, y \in R\}$;

$G = \{(x, y) \mid 2(x + 2y + 3) - (2x - y - 4) = 0, x, y \in R\}$;

$H = \{(x, y) \mid (x + 2y + 3) + 2(2x - y - 4) = 0, x, y \in R\}$.

(b) Determine the ordered pairs of

$C = \{(x, y) \mid x + 2y + 3 = 0, 2x - y - 4 = 0, x, y \in R\}$,

by elimination of each variable by addition or subtraction.

Solution. (a) (i) See *Fig. 6-4* .

(ii) $E = \{(x, y) \mid x + y + 1 = 0, x, y \in R\}$.

The graph of E is a line with x-intercept -1 and y-intercept -1 (*Fig. 6-4*).

$$F = \{(x, y) \mid 2x - 3y - 8 = 0, x, y \in R\}.$$

The graph of F is a line with x-intercept 4 and y-intercept $-\frac{8}{3}$ (*Fig. 6-4*).

$$G = \{(x, y) \mid y + 2 = 0, x, y \in R\}.$$

The graph of G is a line parallel to the x-axis with y-intercept -2 (*Fig. 6-4*).

$$H = \{(x, y) \mid x - 1 = 0, x, y \in R\}.$$

The graph of H is a line parallel to the y-axis with x-intercept 1 (*Fig. 6-4*).

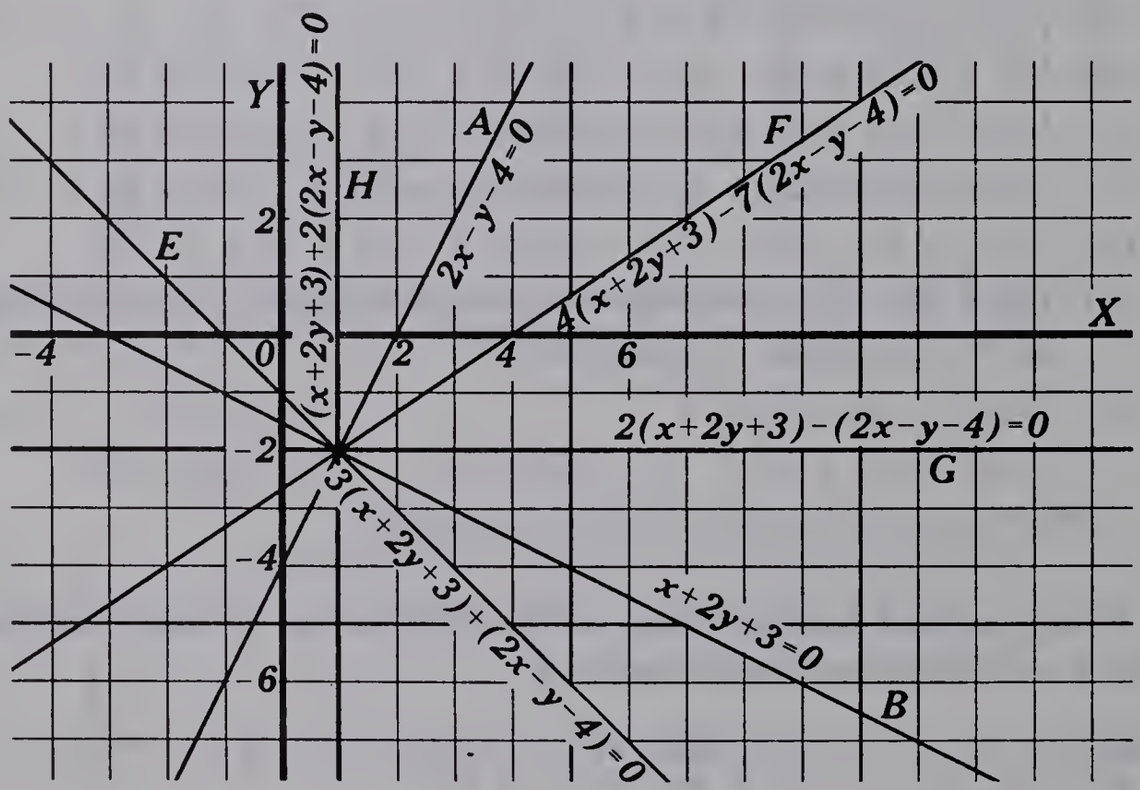


Fig. 6-4

Discussion.

The relations E, F, G, and H have defining equations which are formed by taking a multiple of the left side of the defining equation of A and a multiple of the left side of the defining equation of B, and setting the sum equal to zero. In each case the resulting relation has as one of its ordered pairs the intersection of A and B.

In general, the relation

$P = \{(x, y) \mid k_1(x + 2y + 3) + k_2(2x - y - 4) = 0,\ x, y \in R\}$

where $k_1, k_2 \in R$ and k_1, k_2 are not both zero,

has a graph which

(i) *is a line,*

(ii) *contains the point* $P\ (1, -2)$ *defined by* $A \cap B$.

That this is true may be proved as follows:

(i) The equation

$$k_1(x + 2y + 3) + k_2(2x - y - 4) = 0$$

may be written

$$(k_1 + 2k_2)x + (2k_1 - k_2)y + (3k_1 - 4k_2) = 0$$

which is a linear equation of the first degree in two variables and hence defines a straight line for all real k_1 and k_2 (not both zero). The equation represents a family of lines with parameters k_1 and k_2.

(ii) For the ordered pair $(1, -2)$ the expression

$$k_1(x + 2y + 3) + k_2(2x - y - 4)$$

becomes $k_1(1 - 4 + 3) + k_2(2 + 2 - 4)$,

or $k_1(0) + k_2(0)$,

which is zero for all real replacements of k_1 and k_2. Thus the equation is satisfied by the ordered pair $(1, -2)$ for all real replacements of k_1 and k_2, and therefore the graph of each member of the family lies on the point $P(1, -2)$ defined by $A \cap B$.

The set of all lines with defining equations produced by letting the parameters k_1 and k_2 vary, is called the family of lines on the point of intersection of the lines defined by A and B.

$$k_1(x + 2y + 3) + k_2(2x - y - 4) = 0$$

is called the defining equation of this family of lines.

Any pair of equations, chosen from the above set, form a system having as its solution set $A \cap B$. All such systems are equivalent to:

$$\begin{cases} x + 2y + 3 = 0 \\ 2x - y - 4 = 0, \end{cases}$$

The simplest equivalent system is:

$$\begin{cases} x = 1 \\ y = -2. \end{cases}$$

As is illustrated in (b), the process of eliminating variables by addition or subtraction is the algebraic method of finding this simple equivalent system.

(b)
$$\begin{cases} x + 2y + 3 = 0 & (1) \\ 2x - y - 4 = 0 & (2) \end{cases}$$
$$\leftrightarrow \begin{cases} (x + 2y + 3) + 2(2x - y - 4) = 0 & (3) \\ 2(x + 2y + 3) - (2x - y - 4) = 0 & (4) \end{cases}$$
$$\leftrightarrow \begin{cases} x + 2y + 3 + 4x - 2y - 8 = 0 & (3) \\ 2x + 4y + 6 - 2x + y + 4 = 0 & (4) \end{cases}$$
$$\leftrightarrow \begin{cases} 5x - 5 = 0 & (3) \\ 5y + 10 = 0 & (4) \end{cases}$$
$$\leftrightarrow \begin{cases} x = 1 & (5) \\ y = -2 & (6) \end{cases}$$

Verification.

L.S. (1) $= (1) + 2(-2) + 3$ L.S. (2) $= 2(1) - (-2) - 4$

$= 0.$ $= 0.$

R.S. (1) $= 0.$ R.S. (2) $= 0.$

$\therefore C = \{(1, -2)\}.$

Exercise 6-3

(B)

1. (i) State the equations of three relations which contain the intersection set of:

$$A = \{(x, y) \mid 3x + 2y - 18 = 0, x, y \in R\}$$
$$\text{and } B = \{(x, y) \mid x - y - 1 = 0, x, y \in R\}.$$

(ii) Draw graphs of the relations A and B, on the same set of axes.

(iii) Using the equations of A and B find the equations of the lines parallel to (a) the y-axis, (b) the x-axis, and containing $A \cap B$.

2. (i) Write the equation of the family of lines on the intersection of the lines with defining sentences $x + y - 3 = 0$ and $2x + 3y - 8 = 0$. Use two real parameters k_1 and k_2.

(ii) Draw the graphs of the members of this family of lines defined by the equations obtained if $k_1 = 1$ and k_2 is replaced by $+1$, $-\frac{2}{5}$, $-\frac{1}{2}$, $-\frac{4}{9}$, $-\frac{1}{3}$ respectively.

(iii) From the result of (ii), state the coordinates of the point of intersection of the lines defined in (i).

3. l_1 and l_2 are two lines defined by the equations

(1) $3x + y + 2 = 0$ and (2) $x + y - 7 = 0$

respectively.

(i) Using parameters k_1 and k_2 write the equation of the family of lines on the intersection point of l_1 and l_2.

(ii) For what replacements of k_1 and k_2 is the equation equivalent to (a) equation (2), (b) equation (1)?

(iii) Divide the equation by k_1. What restriction must be imposed if this is done?
This in a sense eliminates one parameter because $k_2 \div k_1$ is the real number k which is the quotient of k_2 divided by k_1. Rewrite the equation of the family using k as the only parameter. (This is also equivalent to making $k_1 = 1$.)

(iv) Using a single parameter, k as in (iii), eliminates one member of the family defined by the equation in (i). Which member is eliminated?

(v) Write the equation with parameter k in the form of the general linear equation $ax + by + c = 0$.

(vi) From the equation in (v) write down:
(a) the family slope;
(b) the family x-intercept;
(c) the family y-intercept.
Note that the parameter k is involved in each of these expressions.

(vii) Use (vi) to determine the equation of the member of the family with the particular characteristic:
(a) slope 2;
(b) x-intercept 3;
(c) y-intercept -4.

(viii) Determine the equation of the member of the family on the point $A(3, 2)$.

Find the equations of the lines described as follows, without obtaining the coordinates of the point of intersection of the given lines. (Hint: start by writing the equation of the family of lines on the intersection point of the given lines.)

4. On the point of intersection of the lines defined by
(1) $2a + b - 7 = 0$ and (2) $2a + 3b - 7 = 0$
and on the point $A(1, 3)$.

5. On the point of intersection of the lines defined by
(1) $3x - 2y + 1 = 0$ and (2) $2x - y + 3 = 0$
and on the point $B(3, -1)$.

6. Parallel to the line with equation $4x - 2y + 3 = 0$ and on the intersection of the lines with equations
(1) $4x + y = 7$ and (2) $3x - 2y + 3 = 0$.

7. On the point of intersection of the lines with equations
(1) $x - y + 3 = 0$ and (2) $3x + 2y = 14$
and (a) perpendicular to the line $2x + y + 3 = 0$,
(b) with x-intercept 4,
(c) with y-intercept 5.

(C)

8. Given the defining sentences of two intersecting straight lines,
(1) $a_1x + b_1y + c_1 = 0$ (2) $a_2x + b_2y + c_2 = 0$
prove that
$$k_1(a_1x + b_1y + c_1) + k_2(a_2x + b_2y + c_2) = 0,$$
where $k_1, k_2 \in R$ and k_1, k_2 are not both zero,
(i) defines a family of straight lines;
(ii) each member of the family lies on the point of intersection of the lines defined by (1) and (2).

6·4 The number of solutions of a system of two linear equations in two variables.

Example. Determine the ordered pairs of each of

(i) $A = \{(x, y) \mid 3x - 2y = 6, x + 2y = -2, x, y \in R\}$,
(ii) $B = \{(x, y) \mid 3x - 2y = 6, 6x - 4y = 12, x, y \in R\}$,
(iii) $C = \{(x, y) \mid 3x - 2y = 6, 6x - 4y = 3, x, y \in R\}$,
(a) graphically, (b) algebraically.

Solution. (a) *Graphical solution:*

(i) Let $A_1 = \{(x, y) \mid 3x - 2y = 6\}$ and $A_2 = \{(x, y) \mid x + 2y = -2\}$; then $A = A_1 \cap A_2$. The graphs are shown in *Fig. 6-5*.

(ii) Let $B_1 = \{(x, y) \mid 3x - 2y = 6\}$ and $B_2 = \{(x, y) \mid 6x - 4y = 12\}$; then $B = B_1 \cap B_2$. The graphs are shown in *Fig. 6-6*.

(iii) Let $C_1 = \{(x, y) \mid 3x - 2y = 6\}$ and $C_2 = \{(x, y) \mid 6x - 4y = 3\}$; then $C = C_1 \cap C_2$. The graphs are shown in *Fig. 6-7*.

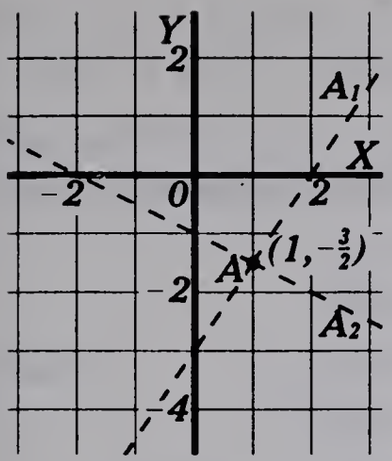


Fig. 6-5

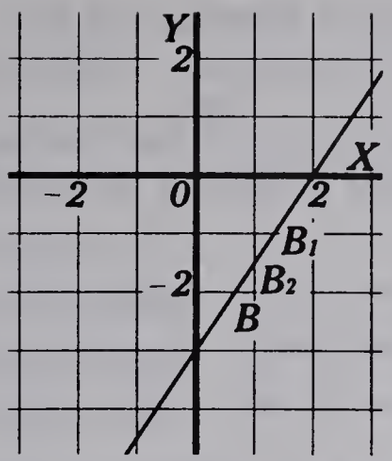


Fig. 6-6

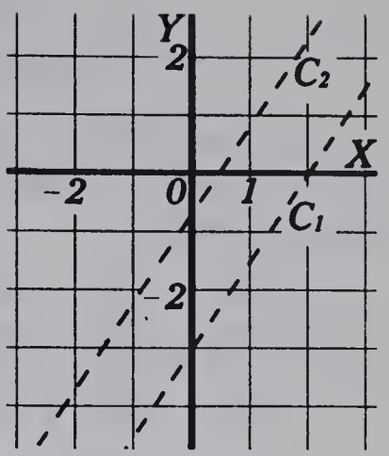


Fig. 6-7

The graph of A is the point of intersection of the graphs of A_1 and A_2.

$\therefore\ A = \{(1, -\frac{3}{2})\}$.

The graphs of B_1 and B_2 are coincident.
The graph of B is a straight line, the intersection of the graphs of B_1 and B_2.

$\therefore\ B =$
$\{(x, y) \mid 3x - 2y = 6\}$.

The graphs of C_1 and C_2 are parallel. The graph of C is the intersection of the graphs of C_1 and C_2.

$\therefore\ C = \emptyset$.

(b) *Algebraic solution:*

(i)
$$\begin{cases} 3x-2y=6 & (1) \\ x+2y=-2 & (2) \end{cases}$$

$$4x = 4$$

$$\begin{cases} x=1 & (3) \\ x+2y=-2 & (2) \end{cases}$$

Substituting $x = 1$ in (2):

$$1 + 2y = -2$$

$$\therefore\quad y = -\frac{3}{2} \quad (4)$$

$$x = 1. \quad (3)$$

Since the steps are reversible,

$\therefore\ A = \left\{\left(1, -\frac{3}{2}\right)\right\}$.

The system of equations has a unique solution.

(ii)
$$\begin{cases} 3x-2y=6 & (1) \\ 6x-4y=12 & (2) \end{cases}$$

Since equations (1) and (2) are equivalent, any ordered pair (x, y) which satisfies (1) also satisfies (2).

$\therefore\ B =$
$\{(x, y) \mid 3x - 2y = 6\}$.

The system of equations has an unlimited number of solutions.

(iii)
$$\begin{cases} 3x-2y=6 & (1) \\ 6x-4y=3 & (2) \end{cases}$$

$$\leftrightarrow \begin{cases} 3x-2y=\frac{3}{2} & (3) \\ 3x-2y=6 & (1) \end{cases}$$

(1) − (3):

$$0x+0y=\tfrac{9}{2} \quad (4)$$

There is no ordered pair which satisfies (4).

$\therefore\ C = \emptyset$.

The system of equations has no solution.

Exercise 6-4

(B)

Determine the ordered pairs of each of the following (i) *graphically, and* (ii) *algebraically:*

1. $A = \{(x, y) \mid 3x - 2y = 7,\ 2x - 5y = 12,\ x, y \in R\}$

2. $B = \{(x, y) \mid 2x - 3y = 8,\ 6x - 9y = 24,\ x, y \in R\}$

3. $C = \{(x, y) \mid 3x + 4y = 12,\ 6x + 8y = -2,\ x, y \in R\}$

4. $D = \left\{(x, y) \mid 3x - 2y = 7,\ \frac{3}{4}x - \frac{y}{2} = 3,\ x, y \in R\right\}$

5. $E = \{(x, y) \mid 3x - 2y = 7,\ 12x - 8y = 28,\ x, y \in R\}$

6·5 The solution of systems of two linear equations with literal coefficients. When literal coefficients occur, it is usually convenient to use elimination of a variable by addition or subtraction to form equivalent systems.

Example 1. If b and c represent real constants, solve the following system for a unique solution (if any); verify.

$$\begin{cases} bx + y = b^2 + c \\ x + cy = b + c^2 \end{cases}$$

Solution.

$$\begin{cases} bx + y = b^2 + c & (1) \\ x + cy = b + c^2 & (2) \end{cases}$$

$b \times (2)$:

$$\begin{cases} bx + bcy = b^2 + bc^2 & (3) \\ bx + y = b^2 + c & (1) \end{cases}$$

(3) − (1):

$$bcy - y = bc^2 - c \qquad (4)$$

$$\therefore\ y(bc - 1) = c(bc - 1) \qquad (5)$$

$$\therefore \begin{cases} y = c & (6) \quad (bc - 1 \neq 0) \\ x + cy = b + c^2 & (2) \end{cases}$$

Substituting $y = c$ in (2):

$$x + c^2 = b + c^2$$

$$\therefore\ x = b \qquad (7)$$

$$\therefore \begin{cases} x = b & (6) \\ y = c \end{cases}$$

Verification.

L.S. (1) $= b^2 + c$. L.S. (2) $= b + c^2$.

R.S. (1) $= b^2 + c$. R.S. (2) $= b + c^2$.

$$\therefore\ x = b,\ y = c, \text{ if } bc - 1 \neq 0.$$

If $bc - 1 = 0$, *then* (5) *does not define* y *uniquely, since* (5) *then gives* $y \times 0 = c \times 0$. *This possibility is discussed in Section* 6.6.

Example 2. If a, b, c, d represent real constants, solve the following system for any unique solution:

$$\begin{cases} ax + by = cd \\ cx - dy = ab \end{cases}$$

Solution.

$$\begin{cases} ax + by = cd & (1) \\ cx - dy = ab & (2) \end{cases}$$

$c \times (1)$:
$a \times (2)$:

$$\begin{cases} acx + bcy = c^2d & (3) \\ acx - ady = a^2b & (4) \end{cases}$$

(3) − (4):

$$bcy + ady = c^2d - a^2b$$

$$\therefore\ y(bc + ad) = c^2d - a^2b$$

$$\therefore\ y = \frac{c^2d - a^2b}{bc + ad}, \qquad (5)\ (bc + ad \neq 0).$$

Equation (5) together with either equation (1) or equation (2) forms a system which is equivalent to the original system and may be solved by substitution. Since the substitution is cumbersome, a better method is to eliminate y by addition or subtraction and thus replace the original system by a simpler equivalent system.

$d \times (1)$:
$b \times (2)$:

$$\begin{cases} adx + bdy = cd^2 & (6) \\ bcx - bdy = ab^2 & (7) \end{cases}$$

(6) + (7):

$$adx + bcx = cd^2 + ab^2$$

$$\therefore\ x(bc + ad) = cd^2 + ab^2$$

$$\therefore\ x = \frac{cd^2 + ab^2}{bc + ad} \qquad (8)\ (bc + ad \neq 0).$$

$$y = \frac{c^2d - a^2b}{bc + ad} \qquad (5)\ (bc + ad \neq 0).$$

$$\therefore\ x = \frac{cd^2 + ab^2}{bc + ad},\ y = \frac{c^2d - a^2b}{bc + ad}, \text{ if } bc + ad \neq 0.$$

Exercise 6-5

(A)

Solve for x *and* y:

1. $\begin{cases} x + y = a \\ x - y = b \end{cases}$

2. $\begin{cases} ax + y = r \\ ax - y = s \end{cases}$

Solve for x and y:

3. $\begin{cases} mx + ny = 2mn \\ mx - ny = 0 \end{cases}$

4. $\begin{cases} rx + sy = c \\ rx - sy = d \end{cases}$

(B)

Solve the following systems of equations for x and y:

5. $\begin{cases} rx - sy = r^2 - s^2 \\ x + y = r + s \end{cases}$

6. $\begin{cases} ax + by = b \\ bx + ay = a \end{cases}$

7. $\begin{cases} ax + by = a^2 + b^2 \\ x + y = 2a \end{cases}$

8. $\begin{cases} a^2x + aby = 2a \\ a^2x - b^2y = a - b \end{cases}$

9. $\begin{cases} hx = ky \\ 3hx - 2ky = hk \end{cases}$

10. $\begin{cases} px + qy = r \\ x + y = 0 \end{cases}$

11. $\begin{cases} 2x - ay = c \\ x + by = c \end{cases}$

12. $\begin{cases} ax + by = c \\ -bx + ay = c \end{cases}$

13. $\begin{cases} ax + by = 2ab \\ bx - ay = b^2 - a^2 \end{cases}$

14. $\begin{cases} \dfrac{x}{a} + \dfrac{y}{b} = 2 \\ ax - by = a^2 - b^2 \end{cases}$

(C)

15. (i) Find $\{(x, y) \mid a_1x + b_1y = c_1,\ a_2x + b_2y = c_2\}$
when (a) $a_1b_2 - b_1a_2 \neq 0$, (b) $a_1b_2 - b_1a_2 = 0$.
(ii) Describe the graphs which would result in (i)(a) and (i)(b).

6·6 The solution and classification of a system of two linear equations in two variables (supplementary). The general system of two linear equations in two variables may be represented by:

$$\begin{cases} a_1x + b_1y = c_1 & (1) \\ a_2x + b_2y = c_2 & (2) \end{cases}$$

where $a_1, b_1, c_1, a_2, b_2, c_2, x, y \in R$ with a_1, b_1 not both zero, and a_2, b_2 not both zero.

Eliminating x: $(a_1b_2 - a_2b_1)y = a_1c_2 - a_2c_1$ (3)

$$\therefore\ y = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}, \text{ if } a_1b_2 - a_2b_1 \neq 0.$$

Eliminating y: $(a_1b_2 - a_2b_1)x = b_2c_1 - b_1c_2$ (4)

$$\therefore\ x = \frac{b_2c_1 - b_1c_2}{a_1b_2 - a_2b_1}, \text{ if } a_1b_2 - a_2b_1 \neq 0.$$

The steps used to find x and y are reversible.

$$\therefore\ x = \frac{b_2c_1 - b_1c_2}{a_1b_2 - a_2b_1} \text{ and } y = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}.$$

Thus (i) if $a_1b_2 - a_2b_1 \neq 0$, the system has a *unique* solution.

If $a_1b_2 - a_2b_1 = 0$, then equation (3) becomes

$$0y = a_1c_2 - a_2c_1.$$

Two cases now arise:

(ii) If $a_1c_2 - a_2c_1 \neq 0$, then $0y \neq 0$, and no real value of y satisfies equation (3).

Further, if $b_2c_1 - b_1c_2 \neq 0$, no real value of x satisfies equation (4).

Thus if $a_1b_2 - a_2b_1 = 0$, and either $a_1c_2 - a_2c_1 \neq 0$ or $b_2c_1 - b_1c_2 \neq 0$, the system has *no solution.*

(iii) If $a_1c_2 - a_1c_2 = 0$, then equation (3) becomes $0y = 0$, which *any* real value of y will satisfy.

Further, if $b_2c_1 - b_1c_2 = 0$, any real value of x satisfies equation (4).

If $b_1 \neq 0$, the ordered pair $\left(x, \frac{c_1 - a_1x}{b_1}\right)$ satisfies equation (1). Substituting $x = x$ and $y = \frac{c_1 - a_1x}{b_1}$ in L.S. (2):

$$\begin{aligned}
\text{L.S. (2)} &= a_2x + \frac{b_2c_1 - a_1b_2x}{b_1} \\
&= \frac{b_2c_1 - (a_1b_2 - a_2b_1)x}{b_1} \\
&= \frac{b_2c_1}{b_1} && \text{since } a_1b_2 - a_2b_1 = 0 \\
&= \frac{b_1c_2}{b_1} && \text{since } b_1c_2 - b_2c_1 = 0 \\
&= c_2. \\
\text{R.S. (2)} &= c_2.
\end{aligned}$$

$\therefore \left(x, \frac{c_1 - a_1x}{b_1}\right)$ also satisfies equation (2), for any $x \in R$.

Similarly if $b_1 = 0$, then $a_1 \neq 0$ and it can be shown that $\left(\frac{c_1 - b_1y}{a_1}, y\right)$ or $\left(\frac{c_1}{a_1}, y\right)$ is a solution of the system for any $y \in R$.

Thus if $a_1b_2 - a_2b_1 = 0$, $a_1c_2 - a_2c_1 = 0$, and $b_2c_1 - b_1c_2 = 0$, the system has an unlimited number of solutions.

Summary: If a_1, b_1 are not both zero, and a_2, b_2 are not both zero, the system has:

(i) a unique solution	if	(i) $a_1b_2 - a_2b_1 \neq 0$
(ii) no solution	and	(ii) $a_1b_2 - a_2b_1 = 0$, and either $a_1c_2 - a_2c_1 \neq 0$, $b_2c_1 - b_1c_2 \neq 0$
(iii) an unlimited number of solutions	only if	(iii) $a_1b_2 - a_2b_1 = 0$, $a_1c_2 - a_2c_1 = 0$, $b_2c_1 - b_1c_2 = 0$.

Corollary. If none of a_2, b_2, c_2, are zero, then the system has

(i) a unique solution	if	$\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$
(ii) no solution	and only	$\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$
(iii) an unlimited number of solutions	if	$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$.

Example 1. Classify the following systems of equations according to the number of solutions. Describe the graph of the solution set.

(i) $\begin{cases} 3x + 4y = 7 \\ 2x - 5y = 9 \end{cases}$ (ii) $\begin{cases} 3x + 4y = 7 \\ 12x + 16y = 28 \end{cases}$ (iii) $\begin{cases} 3x + 4y = 7 \\ 6x + 8y = 10 \end{cases}$

Solution. (i) $a_1 = 3$, $b_1 = 4$, $c_1 = 7$,
$a_2 = 2$, $b_2 = -5$, $c_2 = 9$.

$a_1b_2 - a_2b_1$
$= 3(-5) - 2(4)$
$\neq 0$.

or

$\frac{a_1}{a_2} = \frac{3}{2}$, $\frac{b_1}{b_2} = \frac{4}{-5}$

$\therefore \frac{a_1}{a_2} \neq \frac{b_1}{b_2}$.

Therefore there is a unique solution. The graph of the solution set is a point.

(ii) $a_1 = 3$, $b_1 = 4$, $c_1 = 7$,
$a_2 = 12$, $b_2 = 16$, $c_2 = 28$.

$a_1b_2 - a_2b_1$ $a_1c_2 - a_2c_1$ $b_2c_1 - b_1c_2$
$= 48 - 48$ $= 84 - 84$ $= 112 - 112$
$= 0$, $= 0$, $= 0$,

or

$\frac{a_1}{a_2} = \frac{1}{4}$, $\frac{b_1}{b_2} = \frac{1}{4}$, $\frac{c_1}{c_2} = \frac{1}{4}$.

$\therefore \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$.

Therefore there is an unlimited number of solutions. The graph of the solution set is a straight line.

(iii) $a_1 = 3, \quad b_1 = 4, \quad c_1 = 7,$
$a_2 = 6, \quad b_2 = 8, \quad c_2 = 10.$

$a_1b_2 - a_2b_1 \quad a_1c_2 - a_2c_1 \quad b_2c_1 - b_1c_2$
$= 24 - 24 \quad = 30 - 42 \quad = 56 - 40$
$= 0, \quad \neq 0, \quad \neq 0,$

or

$$\frac{a_1}{a_2} = \frac{1}{2}, \qquad \frac{b_1}{b_2} = \frac{1}{2}, \qquad \frac{c_1}{c_2} = \frac{7}{10},$$

$$\therefore \frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}.$$

Therefore there is no solution. The solution set (null set) has no graph.

Example 2. Determine the number of ordered pairs in the following set and describe its graph:

$$\{(x, y) \mid 3x - y = 5,\ 6x - 2y = 9,\ x, y \in R\}.$$

Solution. For the system $\begin{cases} 3x - y = 5 & (1) \\ 6x - 2y = 9 & (2) \end{cases}$

$$\frac{a_1}{a_2} = \frac{1}{2}, \qquad \frac{b_1}{b_2} = \frac{1}{2}, \qquad \frac{c_1}{c_2} = \frac{5}{9}.$$

$$\therefore \frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}.$$

$\therefore$ the set has no ordered pairs.

$\therefore \{(x, y) \mid 3x - y = 5,\ 6x - 2y = 10,\ x, y \in R\} = \emptyset$.

There is no graph.

Exercise 6-6

(A)

Classify each pair of the following systems of equations as to the number of solutions: one, unlimited, none. State the kind of graph obtained in each case:

1. $\begin{cases} 2x + 3y = 6 \\ 3x - 2y = 7 \end{cases}$ **2.** $\begin{cases} 2x - 5y = 3 \\ 6x - 5y = 9 \end{cases}$ **3.** $\begin{cases} 2a - 4b + 5 = 0 \\ 6a - 12b + 7 = 0 \end{cases}$

4. $\begin{cases} 14m - 5n = 4 \\ 7m - \frac{5}{2}n = 7 \end{cases}$ **5.** $\begin{cases} 3x - 7y = 5 \\ 6x - 14y = 100 \end{cases}$ **6.** $\begin{cases} 5x - 2y - 3 = 0 \\ 10x - 4y = 6 \end{cases}$

(B)

Calculate the values of $\frac{a_1}{a_2}$, $\frac{b_1}{b_2}$, $\frac{c_1}{c_2}$ *for the systems of equations in each of the following. Classify each as to the number of ordered pairs in the relation: one, unlimited, none. State the kind of graph in each case. Draw the graphs of* 7, 8, *and* 9.

7. $R_1 = \{(x, y) \mid 2x - y = 4,\ x - \frac{1}{2}y = 1,\ x, y \in R\}$

8. $R_2 = \{(x, y) \mid 2x - y = 4,\ 3x - 2y = 5,\ x, y \in R\}$

9. $R_3 = \{(x, y) \mid x = 3 - \frac{1}{3}y,\ 3x + y = 9,\ x, y \in R\}$

10. $R_4 = \{(x, y) \mid y = 6 - 3x,\ x + \frac{1}{3}y = 1,\ x, y \in R\}$

11. $R_5 = \{(x, y) \mid y = 3x - 4,\ 6x - 2y = 8,\ x, y \in R\}$

12. $R_6 = \{(x, y) \mid 2x + 3y = 7,\ 4x + 6y = 14,\ x, y \in R\}$

13. $R_7 = \left\{(x, y) \mid \frac{x}{3} = \frac{y}{4},\ \frac{x - 4}{4} - \frac{y - 13}{3} = 1,\ x, y \in R\right\}$

(C)

State the number of ordered pairs in each of the following and where possible determine the ordered pairs of each:

14. $R_8 = \{(x, y) \mid \sqrt{2}x + y = 3,\ 3\sqrt{2}x - y = 5,\ x, y \in R\}$

15. $R_9 = \{(x, y) \mid \sqrt{3}x + \sqrt{5}y = 3 + \sqrt{5},\ (3\sqrt{3} - \sqrt{15})x + (3\sqrt{5} - 5)y = 4,\ x, y \in R\}$

6·7 Graphs defined by systems of linear inequalities in two variables (supplementary).

Example 1. Draw the graph defined by the system on $R \times R$,

$$\begin{cases} x \geqq 0 & (1) \\ y \geqq 0 & (2) \\ 3x + 4y \leqq 12. & (3) \end{cases}$$

The sentence $x \geqq 0$ defines the closed half-plane consisting of all the points on the line defined by $x = 0$ (y-axis) and all the points to the right of it (*Fig. 6-8*).

Similarly $y \geqq 0$ defines the closed half-plane consisting of the x-axis and the region above it (*Fig. 6-9*).

The sentence $3x + 4y \leqq 12$ defines the closed half-plane consisting of the line defined by $3x + 4y = 12$ and the region below the line (*Fig 6-10*).

The above regions are described as *closed* half-planes because the set includes the boundary line. An open half-plane does not include its boundary line.

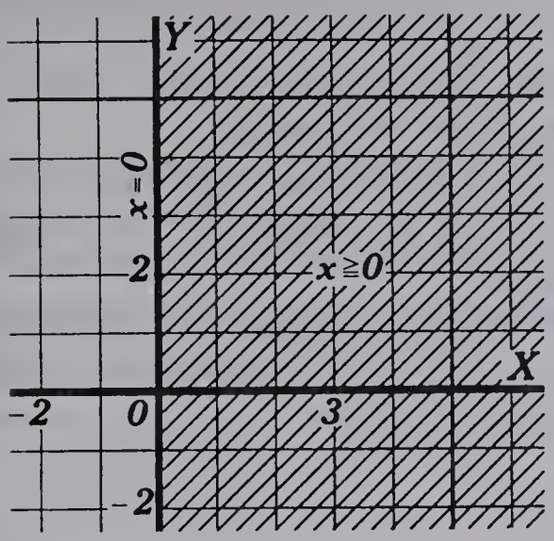


Fig. 6-8

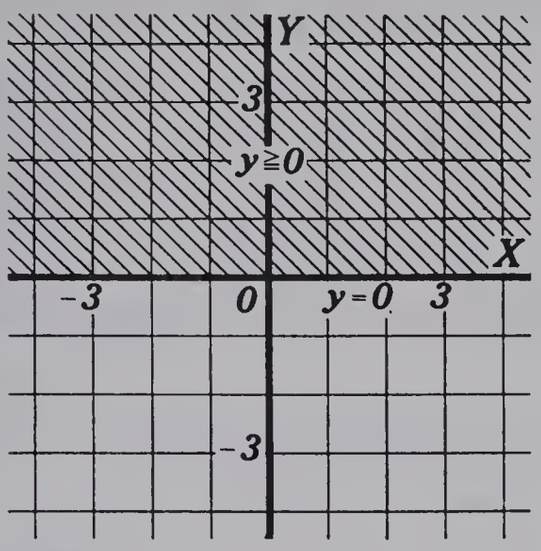


Fig. 6-9

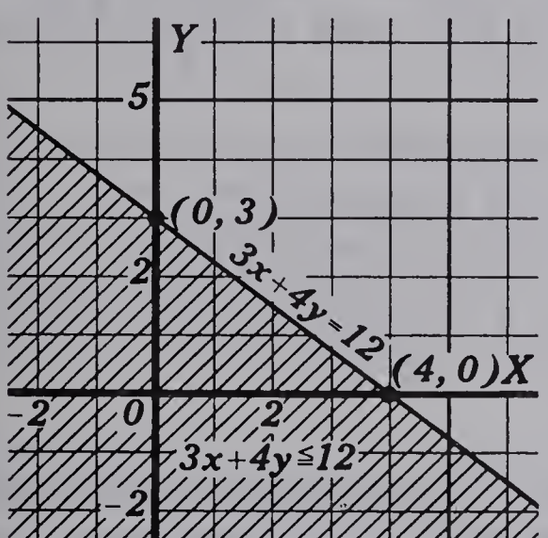


Fig. 6-10

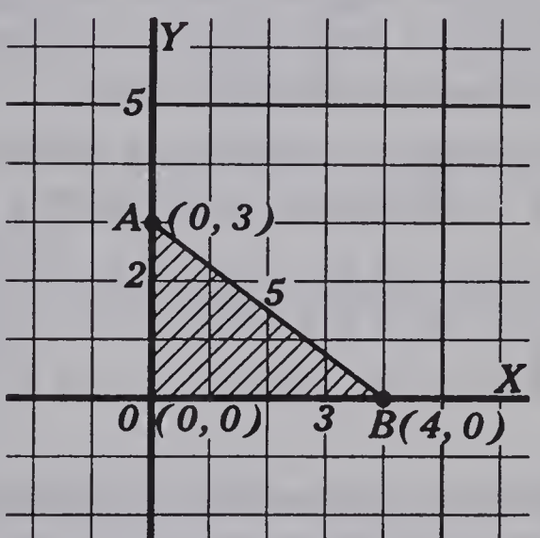


Fig. 6-11

The system of inequations defines the intersection set of these three regions: in this case a triangle and its interior (*Fig. 6-11*).

This region contains all the points which satisfy the system of inequalities. This set is called a *polygonal convex set.*

A set of points is said to be *convex* if whenever it contains two points it also contains the line segment determined by them.

All closed (or open) half-planes are convex sets. It can be shown that the intersection of a finite number of closed half-planes is a convex set. Any such set is called a polygonal convex set.

Write solutions to the following problems; compare your solutions with those on page 423.

1. (i) Draw the graph of the polygonal convex set

$$\{(x, y) \mid x \geqq 0, y \geqq 0, x + y \geqq 3, x, y \in R\}.$$

(ii) Name and give the coordinates of the vertices of the region.

2. (i) Draw the graph of the polygonal convex set defined by the system

$$\begin{cases} x + y \geqq 2 \\ x - 6y \leqq 2 \\ 4x - 3y \geqq -6 \\ 2x + 5y \leqq 36 \\ x \leqq 8 \end{cases}$$

(ii) Name the vertices and state their coordinates (from the graph).

(iii) Write on the graph the equations of the lines of which the edges of the polygon are segments.

(iv) List the systems of equations which must be solved to find the coordinates of the vertices algebraically.

Exercise 6-7

Graph the polygonal convex sets defined as follows; mark on the graph the coordinates of the vertices, and the equations of the lines of which the edges of the polyons are segments or rays.

1. $\{(x, y) \mid 3x + 2y \leqq 6, x \geqq 0, y \geqq 0, x, y \in R\}$

2. $\{(m,n) \mid m \geqq 0, n \geqq 0, m \leqq 3, m + 2n \leqq 4, m, n \in R\}$

3. $\{(p, q) \mid p \geqq -2, p \leqq 3, q \leqq 6, p - 2q \leqq -2, p, q \in R\}$

4. $\{(a, b) \mid b \leqq a + 2, b \geqq a - 2, -2 \leqq b \leqq 3, b, a \in R\}$

5. $\{(x, y) \mid 2x + 3y \leqq 12, y \leqq x, y \geqq 0, x, y \in R\}$

6·8 Linear programming (supplementary). Many problems in economics, business administration, and other fields require the determination, from the system of equations or inequations which define the problem, of *a particular solution or solutions* which *maximize* or *minimize* some factor, such as cost, involved in the problem.

In many instances, contemporary mathematics is not sufficiently advanced to contribute a neat solution, but in the particular case in which

(i) the equations or inequations of the system are linear, and

(ii) the expression to be maximized or minimized is a linear polynomial ($ax + by + c$), methods have been developed to solve the problem. These are known as *linear programming methods.*

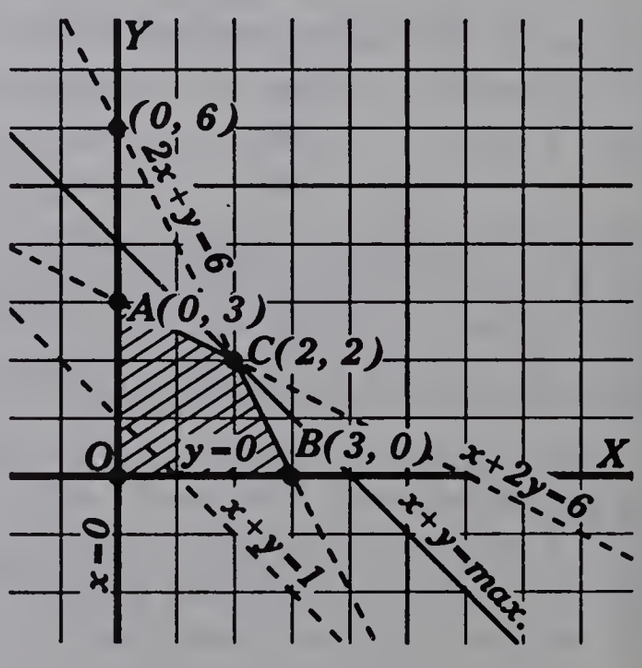


Fig. 6-12

In *Fig.6-12* the polygonal convex set defined by the system of inequations

$$\begin{cases} x \geqq 0 \\ y \geqq 0 \\ 2x + y \leqq 6 \\ x + 2y \leqq 6 \end{cases}$$

is illustrated. The coordinates of the vertices may be read from the graph or determined by solving the equations, arising from the inequalities, in pairs as indicated in the table.

SYSTEM OF INEQUATIONS	EQUATIONS	VERTICES OF THE POLYGON	
		SOLVE	OBTAIN
$x \geqq 0$	$x = 0$ (1)	(1) and (2)	(0, 0)
$y \geqq 0$	$y = 0$ (2)	(2) and (3)	(3, 0)
$2x + y \leqq 6$	$2x + y = 6$ (3)	(1) and (4)	(0, 3)
$x + 2y \leqq 6$	$x + 2y = 6$ (4)	(3) and (4)	(2, 2)

The polygonal convex set defined by the inequalities contains all the points whose coordinates satisfy the *constraints* or inequalities. These points are sometimes called *feasible points.*

Suppose now we wish to find from the set of feasible points the one or ones which make the linear polynomial

$$x + y$$

a maximum.

To discover what values of x and y make $x + y$ a maximum under the four constraints above, we must find the coordinates of the feasible point or points whose sum $x + y$ is a maximum.

Consider the line whose equation is $x + y = 1$, *Fig. 6-12.* The coordinates of all the points on the line segment cut off by the polygon satisfy all the conditions or constraints and also make the sum $x + y$ equal to 1. In a similar manner, lines drawn parallel to this line and intersecting the quadrilateral determine points whose coordinates satisfy all the constraints and make $x + y$ equal to sums other than 1. Of these lines, the line on $C(2, 2)$, being farthest to the right, has the equation with the maximum absolute term. For this line, $x + y$ is 4. Thus the values of x and y which maximize $x + y$ defined on this polygonal convex set are $x = 2, y = 2$.

Similarly $x = 0, y = 0$ minimize $x + y$ in the set of feasible points.

The coordinates of any other feasible point make the sum $x + y$ a number between 0 and 4.

The preceding is an example of a graphical approach to linear programming.

The linear polynomial $x + y$ defined on the convex set $AOBC$ takes on its maximum at the point (2, 2) and its minimum at the point (0, 0), each

of which are *vertices of the polygon.* It can be proved that the maximum or minimum will always occur at a vertex of the polygon. We express the fact as a theorem.

THEOREM. *If a linear polynomial $ax + by + c$ attains a maximum or a minimum value at a feasible point $P(x, y)$, then the value is attained at one of the vertices of the polygonal convex set determined by the constraints of the problem.*

Thus to solve such a problem it is necessary to:

(i) determine the coordinates of the vertices of the polygonal convex set defined by the constraints;

(ii) test each in the given linear polynomial to determine the maximum or minimum point.

Example. A certain problem led to the following constraints:

$$\begin{cases} x \geqq 0 \\ y \geqq 0 \\ 2x + y \geqq 8 \\ x + y \geqq 6 \\ 2x + 3y \geqq 14\,. \end{cases}$$

Find the points (values of x and y) which minimize the linear polynomial $4x + 6y$.

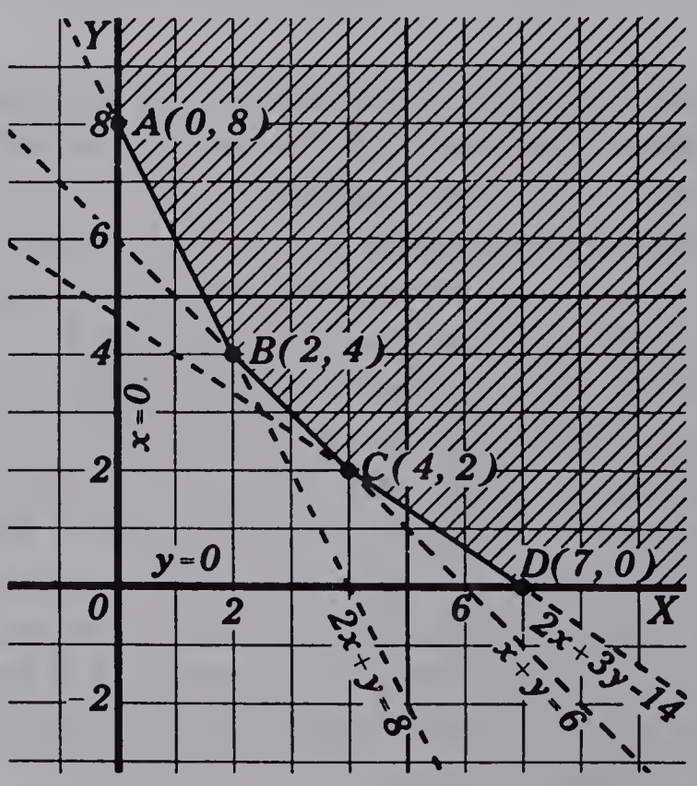


Fig. 6-13

Solution. A sketch, *Fig. 6-13*, may be drawn to help determine the equation pairs of the systems which must be solved to find the vertices of the

convex set. The solutions together with the corresponding value of the polynomial $4x + 6y$ are given in the table.

POINT	x	y	$4x + 6y$
A	0	8	48
B	2	4	32
C	4	2	28
D	7	0	28

The two points (4, 2) and (7, 0) both provide minimum solutions. Also any point on the line segment CD is a solution, since the line $4x + 6y = 28$ or $2x + 3y = 14$ is the line on C and D. This does not contradict our theorem. The minimal solution does occur at a vertex. Other minimal solutions also occur.

The polygonal set in this example is of the *unbounded* variety and has an infinite area. This example illustrates a case where a maximum value is not attained at all among the feasible points.

Exercise 6-8

(B)

1. Draw the graph of the polygonal convex set determined by the system

$$\begin{cases} 2x - y + 4 \geqq 0 \\ 5x + 2y - 10 \leqq 0 \\ x + 4y - 4 \geqq 0. \end{cases}$$

2. Draw the graph of the convex set defined by the system

$$\begin{cases} 2x - y + 4 \leqq 0 \\ 5x + 2y - 10 \geqq 0 \\ x + 4y - 4 \geqq 0. \end{cases}$$

3. The three lines whose equations are

$$2x - y + 4 = 0 \qquad (1)$$
$$5x + 2y - 10 = 0 \qquad (2)$$
$$x + 4y - 4 = 0 \qquad (3)$$

separate the plane into seven convex regions, two of which are described in questions 1 and 2.

(i) Draw the graphs of the three lines and mark the seven convex regions $A, B, C, \ldots G$.

(ii) For each of the seven regions of (i) write a system of three inequations which define the region.

(iii) There are eight ways of putting inequality signs into the three given equations. Write this eighth system and describe the locus it defines.

4. (i) Determine the coordinates of the vertices of the convex polygon defined by the system

$$\begin{cases} a \geqq 0 \\ b \geqq 0 \\ 3a + 2b \geqq 6 \\ 2a + 3b \geqq 6. \end{cases}$$

(ii) Determine the coordinates of the point or points of the polygonal region which minimize the expression $4a + b$ defined over the region.

5. (i) Determine the vertices of the convex polygon defined by the system

$$\begin{cases} x + y \leqq 4 \\ x + 4y \geqq 7 \\ -x + 2y \leqq 5. \end{cases}$$

(ii) Determine the point or points of the polygon which maximize or minimize the linear expression $3x + 2y - 5$.

6. (i) Find the vertices of the convex polygon defined by the system

$$\begin{cases} 2p + q + 9 \geqq 0 \\ p + 2q - 3 \leqq 0 \\ -p + 3q + 6 \geqq 0 \\ p + q \leqq 0. \end{cases}$$

(ii) Find the maximum and minimum of the expression

$$3p + 2q - 1$$

over the convex polygon given in part (i).

7. Find the coordinates of the vertices of the polygonal convex set determined by the constraints

$$y \geqq 2, \quad y \leqq \sqrt{3}x + 2 - 3\sqrt{3}, \quad y \leqq -\sqrt{3}x + 2 + 13\sqrt{3}.$$

Determine the point of the set for which $y - 2x$ is (i) a maximum, (ii) a minimum.

8. Two workmen, George and Jim, make rake and shovel handles. George can make 6 rake and 4 shovel handles per hour and Jim can make 10 rake and 4 shovel handles per hour. If George earns \$1.50 per hour and Jim \$2.00 per hour, for how many hours should each work to fill an order of 60 rake and 32 shovel handles at minimum labour cost?

Development of the constraint inequations.

Let the number of hours George works be x, $x \in R$,

and the number of hours Jim works be y, $y \in R$.

Then (i) $y \geqq 0$ (ii) $x \geqq 0$.

Together they produce $6x + 10y$ rake handles

and $4x + 4y$ shovel handles.

To fill the order: (iii) $6x + 10y \geqq 60$ or $3x + 5y \geqq 30$

and (iv) $4x + 4y \geqq 32$ or $x + y \geqq 8$.

The polynomial to be minimized is

$$1.5x + 2y.$$

9. A small cart manufacturer is to make two models I and II of carts for a steady market which will take all he can produce. Three machines are used in the process of manufacture and are required as follows:

	MACHINE HOURS		
MACHINE	1	2	3
Model I	1	2	$\frac{8}{5}$
Model II	2	1	$\frac{8}{5}$

No machine may be operated for more than 8 hours a day. The profit on model I is \$4.00 and on model II is \$3.00.

(i) Write the inequations defining the constraints in the problem;

(ii) draw the polygonal convex set for the problem;

(iii) determine the number of each model which should be produced per day to provide maximum profit.

10. Three stores P, Q, and R draw on two warehouses W_1 and W_2 for their supplies. Each warehouse has 12 cartons of a particular article in stock. The three stores each want 8 cartons. Shipping costs from warehouse to store are as follows:

	STORE		
WAREHOUSE	P	Q	R
W_1	\$2	\$2	\$3
W_2	\$4	\$3	\$4

(i) Write the constraint inequations;

(ii) draw the polygonal convex set;

(iii) write the cost equation;

(iv) determine the method of distribution for minimum cost.

6·9 The length of the perpendicular from a point to a line (supplementary).

Discovery Exercise 6-9

(B)

Write solutions for the following problems and compare them with those on page 424.

1. In *Fig. 6-14:*

(i) find the slope of AB;
(ii) state the slope of PM;
(iii) state the equation of PM;
(iv) find the coordinates of M;
(v) find the length of PM .

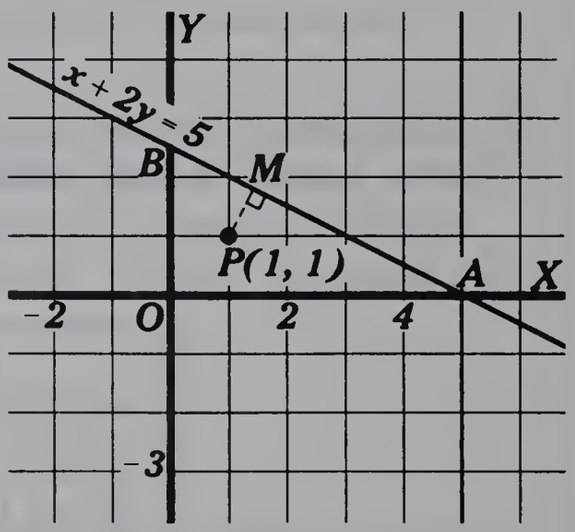


Fig. 6-14

2. In *Fig. 6-15:*

(i) find the slope of ED;
(ii) state the slope of P_1M;
(iii) state the equation of P_1M;
(iv) find the coordinates of M;
(v) find the length P_1M .

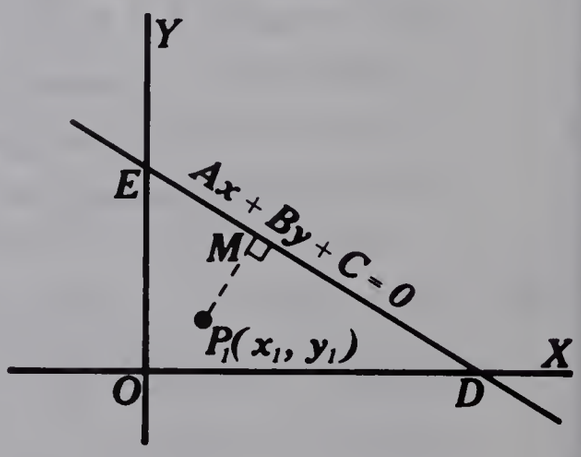


Fig. 6-15

The solution to problem 2 proves that:

The length of the perpendicular from $P_1(x_1, y_1)$ *to the line defined by* $Ax + By + C = 0$ *is*

$$\frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}} .$$

Example 1. Find the length of the perpendicular d from $R(-1, 4)$ to the line with equation $y = 2x + 1$.

Solution. The equation of the line may be written $2x - y + 1 = 0$.

$$d = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}$$

$$= \frac{|2(-1) - 1(4) + 1|}{\sqrt{2^2 + (-1)^2}}$$

$$= \frac{|-5|}{\sqrt{5}}$$

$$= \frac{5}{\sqrt{5}}$$

$$= \sqrt{5}.$$

$\therefore$ the length of the perpendicular is $\sqrt{5}$.

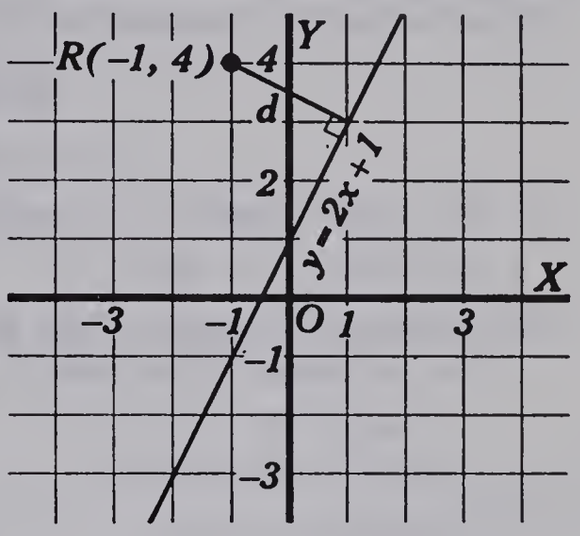


Fig. 6-16

Example 2. Find the length of the perpendicular from $A(2, -3)$ to the line with equation (i) $x = -2$ (ii) $2y - 3 = 0$.

Solution.

(i) For the equation $x + 2 = 0$,

$$A = 1, B = 0, C = 2.$$

$$\therefore d_1 = \frac{|1(2) + 0 + 2|}{\sqrt{1^2 + 0^2}} = \frac{|4|}{1} = 4.$$

$\therefore$ the length of the perpendicular is 4.

(ii) For the equation $2y - 3 = 0$,

$$A = 0, B = 2, C = -3.$$

$$\therefore d_2 = \frac{|0 + 2(-3) - 3|}{\sqrt{0 + 2^2}}$$

$$= \frac{|-9|}{2}$$

$$= 4\tfrac{1}{2}.$$

$\therefore$ the length of the perpendicular is $4\frac{1}{2}$.

It should be noted that these results could have been obtained from the diagram by inspection.

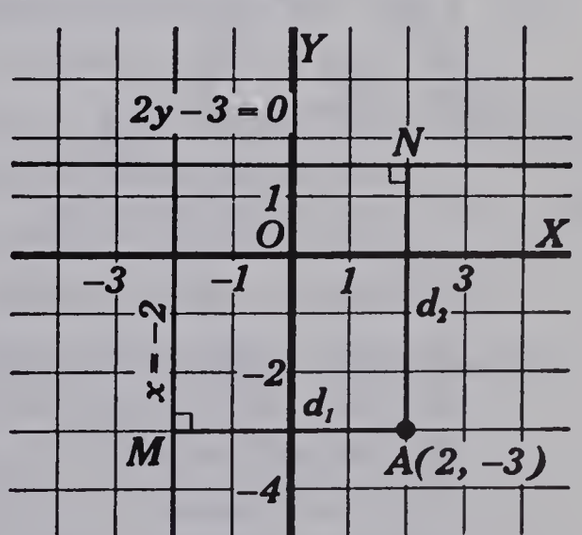


Fig. 11-48

Write solutions for the following problems and compare them with those on page 426.

1. A triangle has vertices $A(-1, 3)$, $B(0, -3)$, and $C(4, 0)$. Find the length of the altitude from A to BC.
2. Find the distance between the parallel lines having equations $x - 3y + 6 = 0$ and $x - 3y - 3 = 0$.

Exercise 6-10

(A)

1. Find the length of a perpendicular from $A(2, 1)$ to the line with equation $3x + 4y = 0$.
2. Find the distance from the origin to the line defined by each of the following equations:

(i) $6x + 8y - 25 = 0$ (ii) $5x - 12y + 13 = 0$
(iii) $x - y + 2 = 0$ (iv) $3x - 4y + 10 = 0$
(v) $3x + y = 0$ (vi) $y = 2x + 5$

(B)

3. Find the length of the perpendicular from the point to the line indicated in each of the following:

(i) $(-1, 3)$, $3x - 4y = 0$ (ii) $(0, 2)$, $5x - 12y - 2 = 0$
(iii) $(0, 0)$, $8x + 15y - 17 = 0$ (iv) $(2, 7)$, $3x - y + 1 = 0$
(v) $(2, 0)$, $6x - 8y - 2 = 0$ (vi) $(3, -1)$, $x + 2y + 4 = 0$
(vii) $(-3, 2)$, $x - 5 = 0$ (viii) $(0, -2)$, $y = -6$
(ix) $(0, 0)$, $y = mx + b$ (x) $(0, 0)$, $y - y_1 = m(x - x_1)$

4. Find b so that the length of the perpendicular from the origin to the line with equation $y = 2x + b$ is $\sqrt{5}$.
5. Find the distance between the parallel lines having equations $3x - 4y - 12 = 0$ and $6x - 8y - 9 = 0$.
6. For the triangle with vertices $A(1, 4)$, $B(0, -2)$, and $C(5, 0)$, find:
   (i) the length of the altitude from A;
   (ii) the length of the base BC;
   (iii) the area of the triangle.
7. Find the length of the perpendicular from the point of intersection of the lines having equations $2x - 3y - 7 = 0$ and $x + 2y - 7 = 0$ to the line with equation $5x - 12y - 13 = 0$ and interpret the result.
8. Find the abscissas of the points with ordinate 2 for which the distance to the line having equation $8x - 15y - 11 = 0$ is one unit.
9. Find the radius of a circle with centre $C(-4, 2)$ and touching the line with equation $3x + 4y - 6 = 0$.
10. Calculate the distance between the parallel lines having equations $5x + 12y = 0$ and $10x + 24y - 39 = 0$.

6·10 The area of a triangle expressed in terms of the coordinates of the vertices (supplementary).

In coordinate geometry, for convenience, we express an area simply as a positive real number and it is understood to be in terms of the unit of the system.

Discovery Exercise 6-11

(B)

Write solutions for the following problems and compare them with those on page 426.

1. In *Fig. 6-18*:

(i) find the equation of the line on P_2P_3;

(ii) find the length P_1D (the length of the perpendicular from P_1 to P_2P_3;

(iii) find the length P_2P_3;

(iv) find the area $\triangle P_1P_2P_3$.

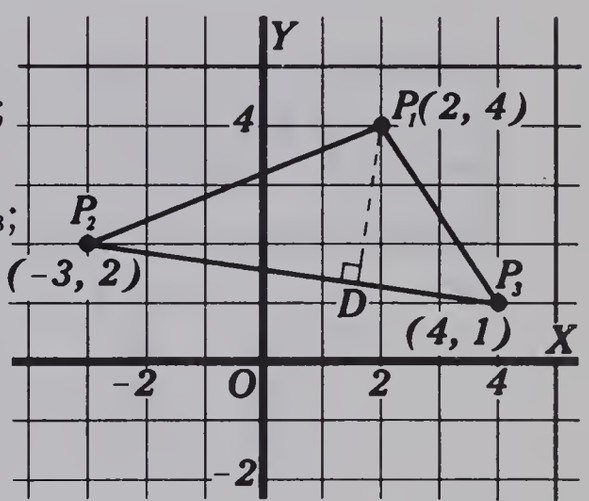


Fig. 6-18

2. In *Fig.* 6-19:

(i) find the equation of the line on P_2P_3;

(ii) find the length P_1D (the length of the perpendicular from P_1 to P_2P_3);

(iii) find the length P_2P_3;

(iv) find the area $\triangle P_1P_2P_3$.

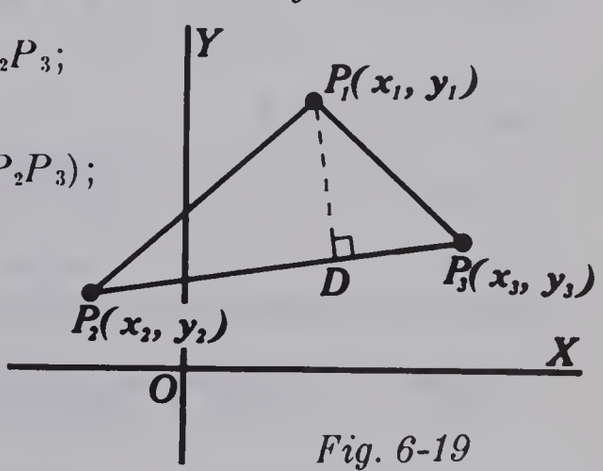


Fig. 6-19

The solution of problem 2 proves that for a triangle with vertices $P_1(x_1, y_1)$, $P_2(x_2, y_2)$, and $P_3(x_3, y_3)$

$$\text{area } \triangle P_1P_2P_3 = \tfrac{1}{2}|(x_1y_2 + x_2y_3 + x_3y_1) - (x_1y_3 + x_3y_2 + x_2y_1)| .$$

The array at the right may be used to facilitate computation, since: $(x_1y_2 + x_2y_3 + x_3y_1)$ is the sum of the "down-products" and $(x_1y_3 + x_3y_2 + x_2y_1)$ is the sum of the "up-products".

$$\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_1 & y_1 \end{vmatrix}$$

Example 1. Calculate the area of the triangle, *Fig. 6-20*, with vertices $A(-4, 3)$, $B(5, 1)$, and $C(-1, -2)$.

Solution: Using the array:

$$\begin{vmatrix} -4, & 3 \\ 5, & 1 \\ -1, & -2 \\ -4, & 3 \end{vmatrix}$$

$$\begin{aligned} \text{Area } \triangle ABC &= \tfrac{1}{2}\,|\,-4 - 10 - 3 - 8 + 1 - 15\,| \\ &= \tfrac{1}{2}\,|\,-39\,| \\ &= 19.5. \end{aligned}$$

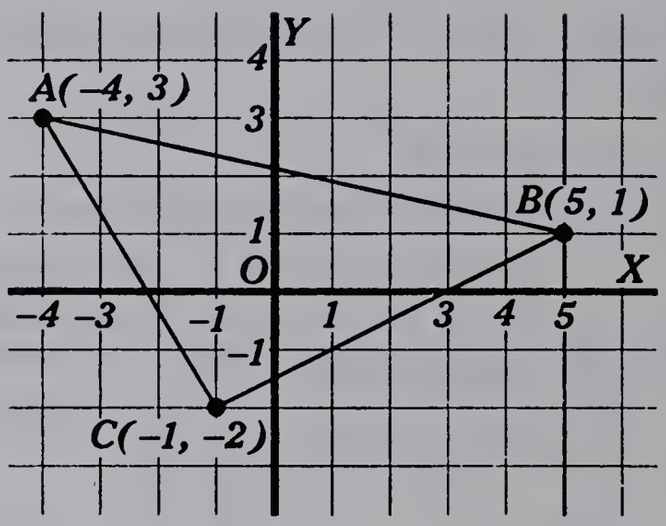


Fig. 6-20

Example 2. The triangle $A(x, -3)$, $B(4, 2)$, $C(3, -1)$ has area 7. Calculate x.

Solution. Using the array:

$$\begin{vmatrix} x, & -3 \\ 4, & 2 \\ 3, & -1 \\ x, & -3 \end{vmatrix}$$

$$\begin{aligned} \text{Area } \triangle ABC &= \tfrac{1}{2}\,|\,2x - 4 - 9 + x - 6 + 12\,| \\ &= \tfrac{1}{2}\,|\,3x - 7\,|\,. \end{aligned}$$

But the area is 7.

$$\begin{aligned} \text{Thus } \tfrac{1}{2}\,|\,3x - 7\,| &= 7 \\ \leftrightarrow \quad |\,3x - 7\,| &= 14 \\ \leftrightarrow 3x - 7 = 14 \quad &\text{or} \quad 3x - 7 = -14 \\ \text{or} \leftrightarrow \quad x = 7 \quad &\text{or} \quad x = -2\tfrac{1}{3}\,. \end{aligned}$$

Hence, A may have two positions: namely, $A_1(7, -3)$ and $A_2(-2\frac{1}{3}, -3)$.

***6·11* Collinearity of points.** The following are tests for collinear points. P_1, P_2, and P_3 are collinear:

(i) if slope P_1P_2 = slope P_2P_3 (slope test);

(ii) if $P_1P_2 + P_2P_3 = P_1P_3$ (length test);

(iii) if area $\triangle P_1P_2P_3 = 0$ (area test).

Example. Prove that the points $A(-7, -1)$, $B(1, 3)$, and $C(5, 5)$ are collinear using (i) the slope test (ii) the length test (iii) the area test.

Solution.

(i) Slope $AB = \dfrac{3-(-1)}{1-(-7)} = \dfrac{4}{8} = \dfrac{1}{2}$.

Slope $BC = \dfrac{5-3}{5-1} = \dfrac{2}{4} = \dfrac{1}{2}$.

$\because$ slope AB = slope BC,

$\therefore$ A, B, and C are collinear.

(ii) $AB = \sqrt{64+16} = \sqrt{80} = 4\sqrt{5}$.

$BC = \sqrt{16+4} = \sqrt{20} = 2\sqrt{5}$.

$AC = \sqrt{144+36} = \sqrt{180} = 6\sqrt{5}$.

$\because$ $AB + BC = AC$,

$\therefore$ A, B, and C are collinear.

(iii) Area $\triangle ABC = \frac{1}{2}|-21+5-5+35-15+1|$

$= \frac{1}{2}|0|$

$= 0$.

$\because$ area $\triangle ABC$ is 0,

$\therefore$ the points A, B, and C are collinear.

$$\begin{vmatrix} -7, & -1 \\ 1, & 3 \\ 5, & 5 \\ -7, & -1 \end{vmatrix}$$

Exercise 6-12

(B)

1. Find the area of the triangle with vertices $A(3, -4)$, $B(-2, 1)$, and $C(1, 6)$ by listing the coordinates of the points in (i) counter-clockwise order (ii) clockwise order.
2. Calculate the area of a triangle for which the coordinates of the vertices are:

   (i) $(7, -3)$, $(3, 5)$, $(-3, 2)$ (ii) $(9, 4)$, $(4, -3)$, $(-3, -2)$

   (iii) $(2, -2)$, $(-3, -1)$, $(-1, 3)$

   (iv) $(4, 7)$, $(1, 1)$, $(-3, -1)$ (v) $(-6, 4)$, $(0, 0)$, $(3, -2)$
3. Test for collinearity the points $A(3, -1)$, $B(0, 1)$, and $C(-6, 5)$.
4. Find x for which the points $A(x, 4)$, $B(-1, 0)$, and $C(2, -2)$ are collinear.
5. For the triangle with vertices $A(4, 0)$, $B(-1, 3)$, and $C(-4, -1)$, calculate (i) the area (ii) the altitude from A.
6. State algebraically that the triangle with vertices $P(x, y)$, $Q(-5, 2)$, and $R(2, 0)$ has area 5.

7. (i) Find the equation of the line on $A(3, -2)$ and $B(-1, 2)$ by stating algebraically that the area of the triangle formed by the points A, B, and $P(x, y)$ is zero.
   (ii) Verify by using the slope-point form of the equation of a line.
8. Two vertices of a triangle are $A(-1, 0)$ and $B(4, 1)$. Find the abscissa of the third vertex if its ordinate is 3 and the area of the triangle is 5.
9. Find the ordinate of the point with abscissa -2 which is collinear with the points $M(3, -5)$ and $N(0, 1)$.

Review Exercise 6-13

(B)

Determine, by any convenient algebraic method, the ordered pairs defined by each of the following systems:

1. $\begin{cases} \frac{1}{3}x + 2y = 1 \\ \frac{2}{3}x - 3y = -5 \end{cases}$
2. $\begin{cases} x = 3y \\ \frac{1}{2}x - y = 1 \end{cases}$
3. $\begin{cases} 7y = 5x + 21 \\ 20x - 9y = 68 \end{cases}$
4. $\begin{cases} 3 = 2m + b \\ -1 = 5m + b \end{cases}$

5. If 6 pens and 4 pencils cost \$13.26, and 4 pens and 6 pencils cost \$10.14, find the cost of each.
6. A number consisting of two digits is three less than five times the sum of its digits. If the digits are reversed, the number is increased by 18. Find the number.
7. Two persons 26 miles apart meet in 4 hours if they walk in opposite directions, and in 17 hours 20 minutes if they walk in the same direction. Find their rates of walking.
8. One train travels from station A to station B in 2 hr. 24 min. A second train travelling 10 m.p.h. faster makes the trip in 2 hr. What is the average speed of the faster train?
9. The cost of admission to a basketball game was 60¢ for adults and 15¢ for students. If the gate receipts from 724 paid admissions amounted to \$339.45, how many adults and how many students attended the game?
10. An airliner travelled with the wind 1,040 miles in 2 hr. and returned against the wind in 2 hr. 10 min. Find the airspeed of the airliner in still air and determine the wind velocity.
11. A marksman's score in an archery contest is 46 for 10 shots. If a bull's-eye counts 5 and an inner counts 4, find how many there were of each if he scored on every shot.

12. If A gave B \$100, A would have one-half as much as B; if B gave A \$100, B would have one-third as much as A. How much does each have?

13. Find the equation of the line having slope -1 and on the point of intersection of the lines having equations $2x - y - 5 = 0$ and $x + 3y + 1 = 0$.

Solve for x and y:

14. $\begin{cases} px + qy = p \\ pqx - y = -pq \end{cases}$

15. $\begin{cases} (a+b)x - (a-b)y = a^2 + b^2 \\ x - y = a - b \end{cases}$

Calculate the values of $\frac{a_1}{a_2}$, $\frac{b_1}{b_2}$, $\frac{c_1}{c_2}$ *for the systems of equations in each of the following. Classify each as to the number of ordered pairs in the relation: one, unlimited, none. State the kind of graph in each case. Draw the graphs of* 19, 20, *and* 21.

16. $A = \{(x, y) \mid 3x - y = 5,\ x - \tfrac{1}{3}y = 1,\ x, y \in R\}$

17. $B = \{(x, y) \mid 3x - y = 5,\ 2x + y = 4,\ x, y \in R\}$

18. $C = \{(x, y) \mid x = 2 - \frac{y}{2},\ 2x + y = 4,\ x, y \in R\}$

19. $D = \{(x, y) \mid 45x - 27y = 21,\ 9y = 15x - 7,\ x, y \in R\}$

20. $E = \{(x, y) \mid \frac{x}{3} = \frac{y}{4},\ \frac{x-4}{4} - \frac{y-13}{3} = 1,\ x, y \in R\}$

21. $F = \{(x, y) \mid \frac{2x}{3} - 7y = 1,\ \frac{21}{2}y = x - 6,\ x, y \in R\}$

22. Graph the polygonal convex set defined as follows:
$\{(x, y) \mid y < 2x - 2,\ y < -x + 5,\ y > -1,\ x, y \in R\}$

23. A plant makes use of three machines, I, II, and III. These machines are used to manufacture two models of TV sets. The standard model yields a profit of \$20 per set, and the deluxe model yields a profit of \$30 per set.

The standard model requires:

39 hours of work on machine I;
$\frac{20}{3}$ hours of work on machine II;
1 hour of work on machine III.

The deluxe model requires:

1 hour of work on machine I;
$\frac{20}{3}$ hours of work on machine II;
$\frac{19}{2}$ hours of work on machine III.

No machine may work more than 40 hours per week. How many sets of each model should be manufactured per week to produce a maximum profit?

24. Find the coordinates of the foot of the perpendicular from $A(1, 5)$ to the line segment with end points $B(1, -1)$ and $C(3,0)$.

25. Find the length of the altitude from A for a triangle with vertices $A(3, -2)$, $B(-5, -4)$, and $C(0, -8)$.

26. Find the equations which express the condition that the length of a perpendicular from $P(x, y)$ to the line with equation $5x - 12y + 6 = 0$ is 2.

27. Find the distance between the parallel lines having equations $3x - 4y - 6 = 0$ and $6x - 8y + k = 0$.

28. Find y so that the area of the triangle with vertices $A(-3, y)$, $B(2, -4)$, and $C(-1, 5)$ is 12.

Chapter VII

EXPONENTS
THE EXPONENTIAL FUNCTION

7·1 Introduction. Previous experience, in Chapter 2 and elsewhere, leads to the conclusion that some real numbers may be expressed in exponential form; for example: $32 = 2^5$.

The question arises:

Can any positive real number, M, be expressed in exponential form to a given base a where $a \in R$?

In this Chapter we search out an answer to this question. The discussion is based on:

(i) *the definition of* a^m, $a \in R$, $n \in {}^+I$:

$$\begin{cases} a^1 = a, a \in R, \\ a^{n+1} = a \cdot a^n, n \in {}^+I; \end{cases}$$

(ii) *the exponential laws for integral exponents* developed inductively in Chapter 2.

If $n, m \in I$, and $a \neq 0$, $b \neq 0$, then

(i) Law of a product $a^n \times a^m = a^{n+m}$ (ii) Zero exponent $a^0 = 1$

(iii) Negative exponent $a^{-n} = \dfrac{1}{a^n}$ (iv) Law of a quotient $a^n \div a^m = a^{n-m}$

(v) Power law $(a^n)^m = a^{nm}$ (vi) Power of a product $(a \cdot b)^n = a^n \cdot b^n$

(vii) Power of a quotient $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$.

The exponential laws for integral exponents which are consequents of:

(i) the definition of a^n, $a \in R$, $n \in {}^+I$ (Section 4.2);

(ii) the exponential law of a product for positive integral exponents: If $a \in R$, $n, m \in {}^+I$, then $a^n \times a^m = a^{n+m}$;

(iii) the definition $a^0 = 1$, $a \neq 0$;

(iv) the definition $a^{-n} = \dfrac{1}{a^n}$, $a \neq 0$, $n \in {}^+I$

will not be proved here but particular cases of the exponential laws for integral exponents are dealt with in Exercise 7-1.

In the following development letter symbols represent real numbers unless otherwise specified.

Write solutions for the following problems and compare your solutions with those on page 427.

1. State equivalent expressions for each of the following.

(i) $(-3)^{-2}$ (ii) $(ab)^{-2}$ (iii) $\left(\frac{a}{b}\right)^{-3}$ (iv) $(m^3n^2p^4)^{-3}$

(v) $\frac{9x^4y^5}{3x^5y^6}$ (vi) 8 as a power of 2 (vii) 9^4 as a power of 3

2. Simplify:

(i) $3^{-4} \cdot 3^2$ (ii) $(-2)^4(-2)^{-3}$ (iii) $(-3)^{-2} \cdot (-3)^0$

(iv) $\left(\frac{2}{3}\right)^2 \times \left(\frac{3}{2}\right)^2$ (v) $\frac{2^3 \times 3^3 \times 2^4 \times 3^5}{2^5 \times 3^4}$

3. Simplify; (exponents are integers):

(i) $x^{27} \div x^{15}$ (ii) $3^4 \div 3^{-2}$ (iii) $(-2)^{-6} \div (-2)^3$

(iv) $a^{3x} \div a^{-4x}$, $a \neq 0$ (v) $\frac{4^{3a} \times 8^{2a}}{16^a}$ (vi) $\frac{5^{4b} \times 25^{2b+2}}{125^{b-1}}$

4. Solve: (i) $2^{3x} = 2^{2x+3}$, $x \in I$ (ii) $3^{4y} = 9^{y-3}$, $y \in I$

***7·2* Scientific (standard) notation.** Numerals used to represent large quantities such as the diameter of the sun, which is approximately 4,567,000,000 feet, and to represent small quantities such as the radius of the hydrogen atom, which is approximately 0.000000000174 feet, contain many zeros. It is useful to find a more convenient way to represent such numerals. One of the most useful is to use powers as follows:

4,567,000,000 may be expressed 4.567×10^9

and 0.000000000174 may be expressed 1.74×10^{-10}.

These are two examples of a method of expression which is considered a *standard form* for writing numerals. It is usually called *scientific notation.*

Scientific notation consists of writing a numeral as the indicated product of a decimal between 1 *and* 10 *and the appropriate power of* 10.

This notation will be made use of in some of the work which follows.

Exercise 7-1

(*The letter symbols represent numbers for which the expressions are defined.*)

(A)

1. State the definition of a^x, $x \in {}^+I$.

2. State the definition of a^0.

3. State the definition of a^x, $x \in {}^-I$, $a \neq 0$.
4. State the law of a product for integral exponents.
5. State the law of a quotient for integral exponents.
6. State the power law for integral exponents.
7. State the law of the power of a product for integral exponents.
8. State the law of the power of a quotient for integral exponents.

Using the exponential laws state simplified or equivalent forms for each of the following:

9. $2^2 \times 2^5$
10. $7^{15} \div 7^8$
11. $(2^2)^3$
12. $(2 \times 5)^2$
13. $(11 \times 13 \times 17)^7$
14. $\left(\frac{5}{7}\right)^8$
15. $0^5 \times 0^7$
16. $2^2 \times 3^3$
17. $\frac{3^3 \times 5^3}{2^3 \times 7^3}$
18. 1^{527}
19. $(-1)^{365}$
20. $\frac{29^5}{17^5}$

State equivalent expressions with positive exponents for each of the following:

21. x^{-1}
22. $\frac{a^{-5}}{a^2}$
23. $a^{-2}b^{-3}c^{-4}$
24. $\left(\frac{x}{y}\right)^{-3}$
25. $\frac{2a^{-1}}{b^{-4}}$
26. $\frac{(-3)^0(-3)^4}{(-3)^{-2}}$

Express each of the following with denominator 1:

27. $\frac{3a}{b^2}$
28. $\frac{4}{n^2(a+b)^{-2}}$
29. $\frac{45a^0b^{-2}}{9ab^{-4}}$

Evaluate:

30. $(-5)^0$
31. 4^{-2}
32. $3^2 \times 2^0$
33. $3^{-4} \times 3^4$
34. $\frac{1}{(-3)^{-2}}$
35. $(-7)^{-2}$

(B)

36. Show that the *law of a product* for integral exponents holds for each of the following cases:
    (i) $a^3 \times a^0$ (ii) $b^0 \times b^{-3}$ (iii) $c^2 \times c^{-4}$ (iv) $d^{-3} \times d^{-2}$
37. Show that the *law of a quotient* for integral exponents holds for each of the following cases:
    (i) $a^2 \div a^0$ (ii) $b^0 \div b^3$ (iii) $c^2 \div c^{-4}$ (iv) $d^{-3} \div d^{-2}$

38. Show that the *power law* for integral exponents holds for each of the following cases:

(i) $(a^2)^0$ (ii) $(b^0)^3$ (iii) $(c^2)^{-3}$ (iv) $(d^{-3})^{-2}$

39. Show that the *power of a product* law for integral exponents holds for each of the following cases:

(i) $(a \cdot b)^0$ (ii) $(c \cdot d)^{-3}$ (iii) $e^0 f^0$ (iv) $g^{-4}h^{-4}$

40. Show that the *power of a quotient* law for integral exponents holds for each of the following cases:

(i) $\left(\frac{a}{b}\right)^0$ (ii) $\left(\frac{c}{d}\right)^{-3}$ (iii) $\frac{e^0}{f^0}$ (iv) $\frac{g^{-2}}{h^{-2}}$

41. Show that if $a = 0$, then a^n, $n \in {}^+I$ is zero.

42. Show that if $a = 1$, then a^n, $n \in I$ is unity.

43. Show that if $a = -1$, then a^n, $n \in I$ is ± 1 depending on whether n is even or odd.

44. Write four examples which illustrate each of the following statements:

(i) $a^n > 0$ for all $a \in R$, $a \neq 0$, if n is an even integer;

(ii) $a^n > 0$ for all $a \in R$, $a > 0$, if n is an odd integer;

(iii) $a^n < 0$ for all $a \in R$, $a < 0$, if n is an odd integer.

Simplify, or by a direct application of the laws of exponents for integral exponents, write in an equivalent form:

45. $x^4 \times x^5$ **46.** $\frac{a^4 \cdot a^2 \cdot b^3}{a^3 b^2}$ **47.** $(a^2b^3c^4)^4$

48. $\frac{3^a 3^b 3^c}{3^{2b}}$ **49.** $\frac{2c^2d \times 3c^3d^4}{3cd^2 \times 4c^2d^3}$ **50.** $\left(\frac{3}{5}\right)^2 \div \left(\frac{3}{5}\right)^3$

51. $\frac{4^x \times 2^{x+3}}{8^x}$ **52.** $\left(\frac{r^2}{s^2}\right)^{-2}$ **53.** $\left(\frac{a}{b}\right)^{-5} \times \left(\frac{b}{a}\right)^2$

54. Express 4^5 as a power of 2 and 9^4 as a power of 3.

55. Express 8^n as a power of 2 and 27^v as a power of 3.

56. Simplify: $\frac{2^n \times 2^{2n-3} \times 2^3}{4^{n+2}}$.

57. Simplify: $\frac{9^{3n} \times 27^{3n-1} \times 3^n}{81^{4n}}$.

Solve each of the following exponential equations:

58. $2^{3x+2} = 2^{2x+4}$ **59.** $3^{5m-4} = 3^{6m-7}$

60. $4^{2x} = 2^{6x+8}$ **61.** $9^{4x-3} = 27^{2x+8}$

Express each of the following numerals in scientific notation:

62. 280 **63.** 0.720

64. 349000 **65.** 0.000003572

66. 49,600,000,000 **67.** 0.00000000485

68. 1 **69.** 0.1

***7•3* The principal *n*th root of a real number.** The following examples illustrate the meaning of a root of a real number.

$+5$ is a square root of 25	because	$(+5)^2 = 25.$
-5 is a square root of 25	because	$(-5)^2 = 25.$
2 is a cube root of 8	because	$2^3 = 8.$
-2 is a cube root of -8	because	$(-2)^3 = -8.$
3 is a cube root of 27	because	$3^3 = 27.$
-3 is a cube root of -27	because	$(-3)^3 = -27.$
2 is a fourth root of 16	because	$2^4 = 16.$
-2 is a fourth root of 16	because	$(-2)^4 = 16.$
2 is a fifth root of 32	because	$2^5 = 32.$
-2 is a fifth root of -32	because	$(-2)^5 = -32.$
2 is a sixth root of 64	because	$2^6 = 64.$
-2 is a sixth root of 64	because	$(-2)^6 = 64.$
0 is an *n*th root of 0	because	$0^n = 0.$

In general,

if $a, b \in R$ and $n \in {}^+I$, then

b is an nth root of a if and only if $b^n = a$.

This definition implies the following:

(i) positive real numbers have two real *n*th roots if n is even;

(ii) positive real numbers have one positive real *n*th root if n is odd;

(iii) negative real numbers have one negative real *n*th root if n is odd;

(iv) zero is the only *n*th root of zero;

Note that b^n, where $n \in \{2, 4, 6, 8, \ldots\}$, is always a *positive* real number or zero.

Thus a negative real number, such as -4, has no real *n*th root if $n \in \{2, 4, 6, 8, \ldots\}$.

Since some real numbers have more than one real *n*th root it is customary to speak of the *principal nth root* of a real number and define it as follows:

DEFINITION: *Principal nth root of a real number.*

(i) *If $a > 0$, the principal nth root of a is the positive real number, b, such that $b^n = a$, $n \in {}^+I$;*

(ii) *if* $a < 0$, *the principal nth root of a is the negative real number*, b, *such that* $b^n = a$, $n \in {}^+I$,

(*Note:* the principal nth root of $a < 0$, for example $\sqrt[n]{-27}$, is defined only when $n \in \{1, 3, 5, 7, \ldots\}$);

(iii) *if* $a = 0$, *the principal* (*and only*) *nth root of a is* 0 .

7·4 Definition of $a^{\frac{1}{n}}$, $n \in {}^+I$. Symbols such as

$$5^{\frac{1}{2}},\ 8^{\frac{1}{3}},\ 9^{\frac{1}{7}},\ \ldots,\ a^{\frac{1}{n}},\ n \in {}^+I$$

have no meaning with reference to the definition of an integral exponent, since the fractional index cannot mean an indicated product in which the number of 5's is $\frac{1}{2}$ or the number of 8's is $\frac{1}{3}$ or the number of a's is $\frac{1}{n}$.

Consider the symbol $5^{\frac{1}{2}}$. If a meaning is to be given to the symbol $5^{\frac{1}{2}}$ consistent with the exponential laws developed for an integral exponent, the following must be considered.

(i) If the multiplication law is to hold, then

$$5^{\frac{1}{2}} \times 5^{\frac{1}{2}} = 5^{\frac{1}{2}+\frac{1}{2}} = 5^1 = 5.$$

For this to be so, $5^{\frac{1}{2}}$ must represent a square root of 5, since 5 is the product of two equal factors, $5^{\frac{1}{2}}$ and $5^{\frac{1}{2}}$.

(ii) If the power law is to hold, then

$$\left(5^{\frac{1}{2}}\right)^2 = 5^{\frac{2}{2}} = 5^1 = 5.$$

For this to be true $5^{\frac{1}{2}}$ must represent a square root of 5.

Since any positive real number has two real square roots and it is essential to assign a unique meaning to $5^{\frac{1}{2}}$, we are led to define $5^{\frac{1}{2}}$ to be the *principal square root* of 5.

In a similar manner we define

$8^{\frac{1}{3}}$ to represent the principal cube root of 8,

$9^{\frac{1}{7}}$ to represent the principal 7th root of 9,

$a^{\frac{1}{n}}$, $n \in {}^+I$ to represent the principal nth root of a.

The definition,

$$a^{\frac{1}{n}} = \sqrt[n]{a},\quad n \in {}^+I,$$

must be accompanied by the restrictions:

(i) $a^{\frac{1}{n}}$ is defined for all $a \geqq 0$, $n \in {}^+I$;

and (ii) $a^{\frac{1}{n}}$ is defined for all $a < 0$, $n \in \{1, 3, 5, 7, \ldots\}$.

Example 1. State:

(i) the principal square root of 625;
(ii) the principal cube root of -64;
(iii) the principal fifth root of 32;
(iv) the principal sixth root of 729.

Solution. (i) $\sqrt{625} = 25$ (ii) $\sqrt[3]{-64} = -4$
(iii) $\sqrt[5]{32} = 2$ (iv) $\sqrt[6]{729} = 3$.

Example 2. State the meaning of:

(i) $a^{\frac{1}{3}}$ (ii) $z = 2^{\frac{1}{4}}$ (iii) $p^5 = 3, p > 0$ (iv) $(-3)^{\frac{1}{2}}$.

Solution. (i) $a^{\frac{1}{3}}$ means the principal cube root of a, if $a \in R$.
(ii) $z = 2^{\frac{1}{4}}$ means z is the principal 4th root of 2.
(iii) $p^5 = 3$, $p > 0$ means p is a fifth root of 3.
(iv) $(-3)^{\frac{1}{2}}$ is undefined.

Example 3. Evaluate:

(i) $49^{\frac{1}{2}}$ (ii) $125^{\frac{1}{3}}$ (iii) $(-343)^{\frac{1}{3}}$ (iv) $-625^{\frac{1}{4}}$ (v) $(-81)^{\frac{1}{4}}$.

Solution. (i) $49^{\frac{1}{2}} = \sqrt{49} = 7.$
(ii) $125^{\frac{1}{3}} = \sqrt[3]{125} = 5.$
(iii) $(-343)^{\frac{1}{3}} = \sqrt[3]{-343} = -7.$
(iv) $-625^{\frac{1}{4}} = -\sqrt[4]{625} = -5.$
(v) $(-81)^{\frac{1}{4}} = \sqrt[4]{-81}$ is undefined.

Example 4. Simplify: (i) $\sqrt{x^2}$ (ii) $\sqrt{x^2 + 2x + 1}$.

Solution. (i) $\sqrt{x^2} = |x|$. (ii) $\sqrt{x^2 + 2x + 1} = |x + 1|$.

Example 5. Evaluate: (i) $4^{\frac{1}{2}} \times 4^{\frac{1}{2}}$ (ii) $27^{\frac{1}{3}} \div 8^{\frac{1}{3}}$.
Show that the same result occurs if the appropriate law of exponents as defined for integral exponents is applied.

Solution. (i) $4^{\frac{1}{2}} \times 4^{\frac{1}{2}} = \sqrt{4} \times \sqrt{4}$
$= 2 \times 2$
$= 4.$

(ii) $27^{\frac{1}{3}} \div 8^{\frac{1}{3}} = \dfrac{\sqrt[3]{27}}{\sqrt[3]{8}}$
$= \dfrac{3}{2}.$

By the law of multiplication,

$$4^{\frac{1}{2}} \times 4^{\frac{1}{2}} = 4^{\frac{1}{2}+\frac{1}{2}}$$
$$= 4^1$$
$$= 4.$$

By the law of the power of a quotient,

$$\frac{27^{\frac{1}{3}}}{8^{\frac{1}{3}}} = \left(\frac{27}{8}\right)^{\frac{1}{3}}$$
$$= \sqrt[3]{\frac{27}{8}}$$
$$= \frac{3}{2}.$$

Exercise 7-2

(The variables represent real numbers for which the expressions are defined.)

(A)

State the meaning of:

1. a^0 **2.** $b^{\frac{0}{1}}$ **3.** $x^{\frac{1}{2}}$

4. $y^{\frac{1}{4}}$ **5.** $(xy)^{\frac{1}{5}}$ **6.** $\dfrac{1}{x^{\frac{1}{7}}}$

7. $a^{\frac{1}{4}}b^{\frac{1}{3}}$ **8.** $(-9)^{\frac{1}{2}}$ **9.** $(-16)^{\frac{1}{4}}$

10. $z = 49^{\frac{1}{2}}$ **11.** $p = 3^{\frac{1}{7}}$ **12.** $a^4 = 5,\ a > 0$

13. $b^3 = -27,\ b < 0$ **14.** $c = (-8)^{\frac{1}{4}}$ **15.** $x^{\frac{1}{4}}y^{\frac{1}{5}}z^{\frac{1}{6}}$

Evaluate:

16. $32^{\frac{1}{5}}$ **17.** $16^{\frac{1}{2}}$ **18.** $(-32)^{\frac{1}{5}}$

19. $-32^{\frac{1}{5}}$ **20.** $27^{\frac{1}{3}}$ **21.** $-27^{\frac{1}{3}}$

22. $(-27)^{\frac{1}{3}}$ **23.** $(-27)^{\frac{1}{4}}$ **24.** $125^{\frac{2}{3}}$

25. $625^{\frac{1}{4}}$ **26.** $-625^{\frac{1}{4}}$ **27.** $(-625)^{\frac{1}{4}}$

State an equivalent exponential expression for each of the following:

28. $\sqrt{5}$ **29.** $\sqrt[3]{-27}$ **30.** $\sqrt[5]{125}$

31. $\sqrt[7]{82}$ **32.** $\dfrac{1}{\sqrt[4]{7.5}}$ **33.** $\dfrac{2}{\sqrt[16]{74^{\frac{1}{2}}}}$

34. $\dfrac{\sqrt[4]{a}}{\sqrt[6]{b}}$ **35.** $\sqrt[9]{\dfrac{a}{b}}$

Assuming that all the exponential laws as defined for integral exponents hold for exponents of the form $\frac{1}{n}$, $n \in {}^+I$, *state the equivalent form for each of the following and name the exponential law used to obtain this form:*

36. $(ab)^{\frac{1}{2}}$ **37.** $(xyz)^{\frac{1}{3}}$ **38.** $(ab)^0$

39. $\left(\frac{x}{y}\right)^{\frac{1}{4}}$ **40.** $\left(\frac{m}{n}\right)^{\frac{1}{p}}$ **41.** $\left(\frac{x+y}{m+n}\right)^{\frac{1}{5}}$

42. $a^{\frac{1}{2}}b^{\frac{1}{2}}$ **43.** $x^{\frac{1}{3}}y^{\frac{1}{3}}$ **44.** $\frac{a^{\frac{1}{5}}b^{\frac{1}{5}}}{x^{\frac{1}{4}}y^{\frac{1}{4}}}$

45. $\frac{x^{\frac{1}{6}}}{y^{\frac{1}{6}}}$ **46.** $a^{\frac{1}{7}}b^{\frac{1}{7}}c^{\frac{1}{7}}$ **47.** $b^{\frac{1}{9}}c^{\frac{1}{9}}d^0$

48. $a^{\frac{1}{8}}b^{\frac{1}{8}}$ **49.** $b^{\frac{1}{3}}b^{\frac{1}{3}}b^{\frac{1}{3}}$ **50.** $c^{\frac{1}{4}}c^{\frac{1}{4}}c^{\frac{1}{4}}c^{\frac{1}{4}}$

51. $a^{\frac{1}{3}} \div a^{\frac{1}{3}}$ **52.** $x^{\frac{1}{4}} \div x^{\frac{1}{4}}$ **53.** $y^{\frac{1}{6}} \div y^{\frac{1}{6}}$

54. $(ab)^{\frac{1}{3}} \div (ab)^{\frac{1}{3}}$ **55.** $(a^{\frac{1}{4}}b^{\frac{1}{5}}) \div (a^{\frac{1}{4}}b^{\frac{1}{5}})$ **56.** $\left(\frac{a}{b}\right)^{\frac{1}{7}} \div \left(\frac{a}{b}\right)^{\frac{1}{7}}, b \neq 0$

(B)

Evaluate each of the following. Show that the same results occur if the appropriate exponential law as defined for integral exponents is applied. (See Example 5 of Section 7·4.)

57. $9^{\frac{1}{2}} \times 9^{\frac{1}{2}}$ **58.** $16^{\frac{1}{4}} \div 16^{\frac{1}{4}}$ **59.** $8^{\frac{1}{3}} \times 8^{\frac{1}{3}} \times 8^{\frac{1}{3}}$

60. $4^{\frac{1}{2}} \times 9^{\frac{1}{2}}$ **61.** $32^{\frac{1}{5}} \div 243^{\frac{1}{5}}$ **62.** $16^{\frac{1}{2}} \div 16^{\frac{1}{2}}$

63. $27^{\frac{1}{3}} \div 64^{\frac{1}{3}}$ **64.** $25^{\frac{1}{2}} \times 49^{\frac{1}{2}}$

***7·5* Definition of $a^{\frac{x}{n}}$, x, $n \in {}^+I$.** Since exponents of the form $\frac{1}{n}$, $n \in {}^+I$ were defined assuming the power law as defined for integral exponents, then

$$(5^{\frac{1}{7}})^6 = 5^{\frac{1}{7} \times 6} = 5^{\frac{6}{7}},$$

$$[(-27)^{\frac{1}{3}}]^7 = (-27)^{\frac{7}{3}},$$

$$(a^{\frac{1}{n}})^x = a^{\frac{x}{n}}, \; x, n \in {}^+I.$$

Since $5^{\frac{1}{7}} = \sqrt[7]{5}$, we define $5^{\frac{6}{7}}$ to be $(\sqrt[7]{5})^6$.

Also, we define $(-27)^{\frac{7}{3}}$ to be $(\sqrt[3]{-27})^7$.

In general, define

$$a^{\frac{x}{n}} \text{ to be } (\sqrt[n]{a})^x \text{ for all } x,\ n \in {}^+I.$$

It should be noted that this definition is meaningful only for those real numbers a for which $a^{\frac{1}{n}}$ is defined. If $a < 0$, then $n \in \{1, 3, 5, 7, \ldots\}$.

This definition is consistent with the basic idea that the principal nth root of a real number raised to the exponent n is the number itself. Thus

$$(5^{\frac{1}{5}})^5 = 5^{\frac{5}{5}} = 5, \quad (a^{\frac{1}{n}})^n = a^{\frac{n}{n}} = a.$$

Example 1. Evaluate: (1) $4^{\frac{3}{2}}$ (ii) $-4^{\frac{3}{2}}$ (iii) $(-4)^{\frac{3}{2}}$ (iv) $(-125)^{\frac{4}{3}}$.

Solution. (i) $4^{\frac{3}{2}} = (\sqrt[2]{4})^3$
$= 2^3$
$= 8.$

(ii) $-4^{\frac{3}{2}} = -(\sqrt[2]{4})^3$
$= -(2)^3$
$= -8.$

(iii) $(-4)^{\frac{3}{2}} = (\sqrt[2]{-4})^3.$

Since -4 does not have a real square root, this symbol does not represent a real number.

(iv) $(-125)^{\frac{4}{3}} = (\sqrt[3]{-125})^4$
$= (-5)^4$
$= 625.$

The fact that $4^{\frac{3}{2}} = (\sqrt[2]{4})^3 = 8$
and $\sqrt[2]{4^3} = \sqrt[2]{64} = 8$
and hence $(\sqrt[2]{4})^3 = \sqrt[2]{4^3}$
suggests that in general

$$(\sqrt[n]{a})^x = \sqrt[n]{a^x}$$

or $$(a^{\frac{1}{n}})^x = (a^x)^{\frac{1}{n}}.$$

If the power law is to hold in such cases, then the above must be true for all cases in which both $a^{\frac{1}{n}}$ and a^x are defined. It can be shown that in all such cases

$$a^{\frac{x}{n}} = (a^{\frac{1}{n}})^x = (a^x)^{\frac{1}{n}}.$$

If a positive real number M is equivalent to a^x in exponential form, then the principal nth root of M is

$$M^{\frac{1}{n}} = (a^x)^{\frac{1}{n}} = a^{\frac{x}{n}}.$$

Example 2. Evaluate each of the following in two ways:

(i) $16^{\frac{3}{4}}$ (ii) $32^{\frac{4}{5}}$ (iii) $(-27)^{\frac{4}{3}}$.

Solution.

(i) $16^{\frac{3}{4}} = (\sqrt[4]{16})^3$
$= 2^3$
$= 8.$

or

$16^{\frac{3}{4}} = \sqrt[4]{16^3}$
$= \sqrt[4]{16 \times 16 \times 16}$
$= \sqrt[4]{2^4 \times 2^4 \times 2^4}$
$= \sqrt[4]{(2 \times 2 \times 2)^4}$
$= 2 \times 2 \times 2$
$= 8.$

(ii) $32^{\frac{4}{5}} = (\sqrt[5]{32})^4$
$= 2^4$
$= 16.$

or

$32^{\frac{4}{5}} = \sqrt[5]{32^4}$
$= \sqrt[5]{32 \times 32 \times 32 \times 32}$
$= \sqrt[5]{2^5 \times 2^5 \times 2^5 \times 2^5}$
$= \sqrt[5]{(2 \times 2 \times 2 \times 2)^5}$
$= 2 \times 2 \times 2 \times 2$
$= 16.$

(iii) $(-27)^{\frac{4}{3}} = (\sqrt[3]{-27})^4$
$= (-3)^4$
$= 81.$

or

$(-27)^{\frac{4}{3}} = \sqrt[3]{(-27)^4}$
$= \sqrt[3]{(-27)(-27)(-27)(-27)}$
$= \sqrt[3]{(-3)^3(-3)^3(-3)^3(-3)^3}$
$= \sqrt[3]{(-3)^{12}}$
$= (-3)^4$
$= 81.$

7·6 Definition of $a^{-\frac{x}{n}}$, $x, n \in {}^+I$. To be consistent with the definition of a negative integral exponent define

$$6^{-\frac{2}{5}} \text{ to be } \frac{1}{6^{\frac{2}{5}}},$$

$$9^{-\frac{7}{8}} \text{ to be } \frac{1}{9^{\frac{7}{8}}},$$

and *in general,*

$$a^{-\frac{x}{n}} \text{ to be } \frac{1}{a^{\frac{x}{n}}},\ x,\ n \in {}^+I,\ a \neq 0.$$

Defining an exponent of the form $\frac{x}{n}$, $x, n \in {}^+I$ and a negative exponent of the form $-\frac{x}{n}$, $x, n \in {}^+I$ extends the definition of an exponent to any rational exponent, because any fraction may be expressed in the form

$$\frac{x}{n} \text{ or } -\frac{x}{n} \text{ where } x, n \in {}^+I.$$

For example: $\frac{-3}{-2} = \frac{3}{2}$; $\frac{-7}{2} = -\frac{7}{2}$; $\frac{6}{-7} = -\frac{6}{7}$.

It is essential to appreciate this fact because by definition the denominator of a rational exponent always indicates a principal root and hence must be a positive integer.

For example:

$8^{\frac{2}{-3}}$ should be written $8^{-\frac{2}{3}}$ to clearly indicate its meaning.

Expressed in this form $8^{-\frac{2}{3}}$ means $\frac{1}{(\sqrt[3]{8})^2}$.

We could consider the form $8^{\frac{-2}{3}}$, which means $(\sqrt[3]{8})^{-2}$ and is equal to $\frac{1}{(\sqrt[3]{8})^2}$. The form $8^{-\frac{2}{3}}$ is most explicit.

7•7 The laws of exponents applied to rational exponents. The development of the definition for $a^{\frac{x}{n}}$, $\frac{x}{n} \in Q$ was based on the power law. It is still a question whether or not exponentials with rational exponents obey all the exponential laws as defined for integral exponents. The following applications of the laws to particular examples illustrate that the exponential laws do apply for rational exponents. It will be assumed that the laws hold in general.

a. *The law of a product.*

Show that $16^{\frac{3}{2}} \times 16^{-\frac{5}{4}} = 16^{\frac{3}{2}+(-\frac{5}{4})}$.

Solution.

$$16^{\frac{3}{2}} \times 16^{-\frac{5}{4}} = 64 \div 32 = 2. \quad \Big| \quad 16^{\frac{3}{2}+(-\frac{5}{4})} = 16^{\frac{1}{4}} = 2.$$

$$\therefore\ 16^{\frac{3}{2}} \times 16^{-\frac{5}{4}} = 16^{\frac{3}{2}+(-\frac{5}{4})}.$$

In general,

if $\frac{p}{q}, \frac{m}{n} \in Q$, *then* $a^{\frac{p}{q}} \times a^{\frac{m}{n}} = a^{\frac{p}{q}+\frac{m}{n}}$, $a \neq 0$.

b. *The law of a quotient.*

Show that $64^{\frac{4}{3}} \div 64^{\frac{5}{6}} = 64^{\frac{4}{3}-\frac{5}{6}}$.

Solution.

$$64^{\frac{4}{3}} \div 64^{\frac{5}{6}} = 256 \div 32 = 8. \quad \Big| \quad 64^{\frac{4}{3}-\frac{5}{6}} = 64^{\frac{1}{2}} = 8.$$

$$\therefore\ 64^{\frac{4}{3}} \div 64^{\frac{5}{6}} = 64^{\frac{4}{3}-\frac{5}{6}}.$$

In general,

if $\frac{p}{q}, \frac{m}{n} \in Q$, *then* $a^{\frac{p}{q}} \div a^{\frac{m}{n}} = a^{\frac{p}{q}-\frac{m}{n}}$, $a \neq 0$.

c. *The power of a quotient.*

Show that $\left(\frac{8}{27}\right)^{\frac{2}{3}} = \frac{8^{\frac{2}{3}}}{27^{\frac{2}{3}}}$.

Solution.

$$\left(\frac{8}{27}\right)^{\frac{2}{3}} = \left(\sqrt[3]{\frac{8}{27}}\right)^2 \qquad \frac{8^{\frac{2}{3}}}{27^{\frac{2}{3}}} = \frac{2^2}{3^2}$$

$$= \left(\frac{2}{3}\right)^2 \qquad = \frac{4}{9}.$$

$$= \frac{4}{9}.$$

$$\therefore \left(\frac{8}{27}\right)^{\frac{2}{3}} = \frac{8^{\frac{2}{3}}}{27^{\frac{2}{3}}}.$$

In general,

$$\text{if } \tfrac{p}{q} \in Q, \text{ then } \left(\frac{a}{b}\right)^{\frac{p}{q}} = \frac{a^{\frac{p}{q}}}{b^{\frac{p}{q}}},\ a \neq 0,\ b \neq 0.$$

d. *Power law.*

Show that $\left(8^{\frac{2}{3}}\right)^{-\frac{5}{2}} = 8^{(\frac{2}{3})(-\frac{5}{2})}$.

Solution.

$$\left(8^{\frac{2}{3}}\right)^{-\frac{5}{2}} = 4^{-\frac{5}{2}} \qquad 8^{(\frac{2}{3})(-\frac{5}{2})} = 8^{-\frac{5}{3}}$$

$$= \frac{1}{32}. \qquad = \frac{1}{32}.$$

$$\therefore \left(8^{\frac{2}{3}}\right)^{-\frac{5}{2}} = 8^{(\frac{2}{3})(-\frac{5}{2})}.$$

In general,

$$\text{if } \tfrac{p}{q},\ \tfrac{m}{n} \in Q, \text{ then } \left(a^{\frac{p}{q}}\right)^{\frac{m}{n}} = a^{\frac{pm}{qn}},\ a \neq 0.$$

Attention is again drawn to the fact that rational exponents are written with the denominator positive.

Exercise 7-3

(The variables represent numbers for which the expressions are defined.)

(A)

State the meaning of each of the following in two ways:

1. $3^{\frac{7}{2}}$ **2.** $4^{\frac{2}{3}}$ **3.** $5^{\frac{3}{5}}$

4. $8^{\frac{4}{7}}$ 5. $a^{\frac{3}{4}}$ 6. $b^{\frac{4}{3}}$

7. $x^{\frac{5}{6}}$ 8. $y^{\frac{p}{q}}$

Using the laws of exponents state equivalent expressions for each of the following:

9. $a^{\frac{1}{2}} \times a^{\frac{3}{4}}$ 10. $b^{\frac{2}{3}} \times b^{\frac{3}{5}}$ 11. $c^{\frac{1}{7}} \times c^{\frac{3}{4}} \times c^{\frac{2}{5}}$

12. $a^{\frac{1}{2}} \div a^{\frac{3}{4}}$ 13. $b^{\frac{2}{3}} \div b^{\frac{3}{5}}$ 14. $c^{\frac{1}{7}} \times c^{\frac{3}{4}} \div c^{\frac{2}{5}}$

15. $a^{\frac{3}{4}} \times b^{\frac{3}{4}}$ 16. $b^{\frac{4}{5}} \times c^{\frac{4}{5}}$ 17. $a^{\frac{2}{3}}b^{\frac{2}{3}}c^{\frac{2}{3}}$

18. $(xy)^{\frac{3}{2}}$ 19. $(xyz)^{\frac{3}{4}}$ 20. $(abcd)^{\frac{5}{6}}$

21. $\dfrac{a^{\frac{2}{3}}}{b^{\frac{2}{3}}}$ 22. $\dfrac{x^{\frac{3}{4}}}{y^{\frac{3}{4}}}$ 23. $\dfrac{3a^{\frac{2}{7}}}{4b^{\frac{2}{7}}}$

24. $\left(\dfrac{a}{b}\right)^{\frac{9}{8}}$ 25. $\left(\dfrac{c}{d}\right)^{\frac{5}{9}}$ 26. $\left(\dfrac{ab}{cd}\right)^{\frac{p}{q}}$

27. $a^{-\frac{3}{2}}$ 28. $(ab)^{-\frac{4}{5}}$ 29. $\dfrac{1}{x^{-\frac{3}{7}}}$

Evaluate the following:

30. $8^{\frac{2}{3}}$ 31. $16^{\frac{3}{4}}$ 32. $32^{\frac{3}{5}}$

33. $100^{\frac{3}{2}}$ 34. $27^{\frac{4}{3}}$ 35. $\left(\dfrac{8}{27}\right)^{\frac{2}{3}}$

36. $\left(\dfrac{4}{9}\right)^{\frac{3}{2}}$ 37. $\left(\dfrac{8}{343}\right)^{\frac{1}{3}}$ 38. $\left(\dfrac{1}{16}\right)^{\frac{0}{3}}$

Solve the following equations:

39. $x^{\frac{1}{2}} = 2$ 40. $x^{\frac{1}{3}} = 3$ 41. $y^{\frac{1}{4}} = 2$

42. $x^{\frac{2}{3}} = 4$ 43. $m^{\frac{3}{2}} = 8$ 44. $z^{\frac{3}{4}} = 27$

(B)

Evaluate or simplify the following:

45. $4^{-\frac{3}{2}}$ 46. $\dfrac{1}{16^{-\frac{3}{4}}}$ 47. $1000^{-\frac{2}{3}}$

48. $\left(\dfrac{8}{27}\right)^{-1}$ 49. $\left(\dfrac{8}{27}\right)^{-\frac{2}{3}}$ 50. $49^{-\frac{3}{2}}$

51. $2^{\frac{1}{3}} \times 2^{\frac{2}{3}}$ 52. $a^{\frac{3}{2}} \times a^{\frac{3}{4}}$ 53. $b^{\frac{1}{5}} \times b^{\frac{3}{10}}$

54. $x^{\frac{1}{2}} \times x^{\frac{3}{4}} \times x^{\frac{2}{3}}$ **55.** $y^m y^n y^p$ **56.** $a^{\frac{4}{7}} \div a^{\frac{3}{14}}$

57. $b^{\frac{5}{6}} \div b^{\frac{2}{3}}$ **58.** $3x^{\frac{2}{3}} \times 4x^{\frac{3}{5}}$ **59.** $\dfrac{a^{\frac{1}{2}} \cdot a^{\frac{5}{6}}}{a^{\frac{2}{3}}}$

60. $\dfrac{3^{\frac{2}{3}} \times 3^{\frac{3}{4}}}{3^{\frac{5}{12}}}$ **61.** $\dfrac{a^{\frac{1}{5}} \times a^{\frac{2}{3}}}{a^{\frac{8}{15}}}$ **62.** $\dfrac{x^{\frac{1}{m}} \cdot x^{\frac{1}{n}}}{x^{\frac{1}{p}}}$ **63.** $\dfrac{a^{\frac{2}{3}} b^{\frac{2}{3}}}{c^{\frac{2}{3}}}$

Using the laws of exponents show that the following statements are true:

64. $3 \times 5^{\frac{1}{2}} = 9^{\frac{1}{2}} \times 5^{\frac{1}{2}} = 45^{\frac{1}{2}}$ **65.** $2 \times 3^{\frac{1}{3}} = 8^{\frac{1}{3}} \times 3^{\frac{1}{3}} = 24^{\frac{1}{3}}$

66. $27 \times 2^{\frac{3}{2}} = 9^{\frac{3}{2}} \times 2^{\frac{3}{2}} = 18^{\frac{3}{2}}$ **67.** $54^{\frac{1}{3}} = 3 \times 2^{\frac{1}{3}}$

68. $72^{\frac{1}{2}} = 6 \times 2^{\frac{1}{2}}$ **69.** $48^{\frac{1}{4}} = 2 \times 3^{\frac{1}{4}}$

Solve the following equations:

70. $x = 9^{\frac{1}{2}}$ **71.** $x^{\frac{1}{2}} = 4$ **72.** $y^{\frac{1}{3}} = 3$

73. $x^{\frac{2}{3}} = 16$ **74.** $z^{\frac{3}{4}} = 27$ **75.** $m^{\frac{1}{3}} = \dfrac{3}{2}$

7·8 Definition of a^x for irrational exponents. Any irrational number may be approximated as closely as desired by some rational number. For example:

$$1 < \sqrt{2} < 2$$
$$1.4 < \sqrt{2} < 1.5$$
$$1.41 < \sqrt{2} < 1.42$$
$$1\,414 < \sqrt{2} < 1.415$$
and so on.

The sequence of rational numbers, 1, 1.4, 1.41, 1.414, . . . on the left as well as the sequence 2, 1.5, 1.42, 1.415, . . . on the right converges to $\sqrt{2}$ and approximates $\sqrt{2}$ as closely as desired.

Similarly, if a is a positive real number, the sequences

a^1	a^2
$a^{1.4}$	$a^{1.5}$
$a^{1.41}$	$a^{1.42}$
$a^{1.414}$	$a^{1.415}$

and so on

also converge to a unique real number. It seems reasonable to define this unique real number to be $a^{\sqrt{2}}$.

In general: if $x_1, x_2, x_3, \ldots, x_n, \ldots$ is any sequence of rational numbers which approaches as closely as desired to some irrational number, p, then $a^{x_1}, a^{x_2}, a^{x_3}, \ldots, a^{x_n}, \ldots$ also approaches some unique real number, M, as long as $a > 0$. Therefore define

$$a^p \text{ to be } M,\ a > 0,\ p \text{ irrational.}$$

This conclusion will be accepted without proof. With this understanding we have assigned a meaning to a^x, $a \in {}^+R$, $x \in R$.

We will also accept without proof that the basic laws of exponents hold for all real exponents. The laws hold for rational exponents which can be made to approximate irrational exponents as closely as desired.

7·9 The exponential function. It was concluded in Section 7·8 that if a is a positive real number and x is a real number, then a^x is a unique positive real number y.

$$\text{If } a \in {}^+R \text{ and } x \in R, \text{ then } a^x = y \text{ where } y \in {}^+R.$$

For example,

$$\text{if } a = 2 \text{ and } x = 1.41, \text{ then } y = 2^{1.41}.$$

The set of all real numbers represented by $y = 2^x$, $x \in R$ is the range of the function

$$E = \{(x, y) \mid y = 2^x, x \in R\}.$$

This function, the graph of which is shown in *Fig. 7-1*, is called an *exponential function.*

x	y
−2	$\frac{1}{4}$
−1	$\frac{1}{2}$
0	1
1	2
2	4
3	8
4	16
5	32

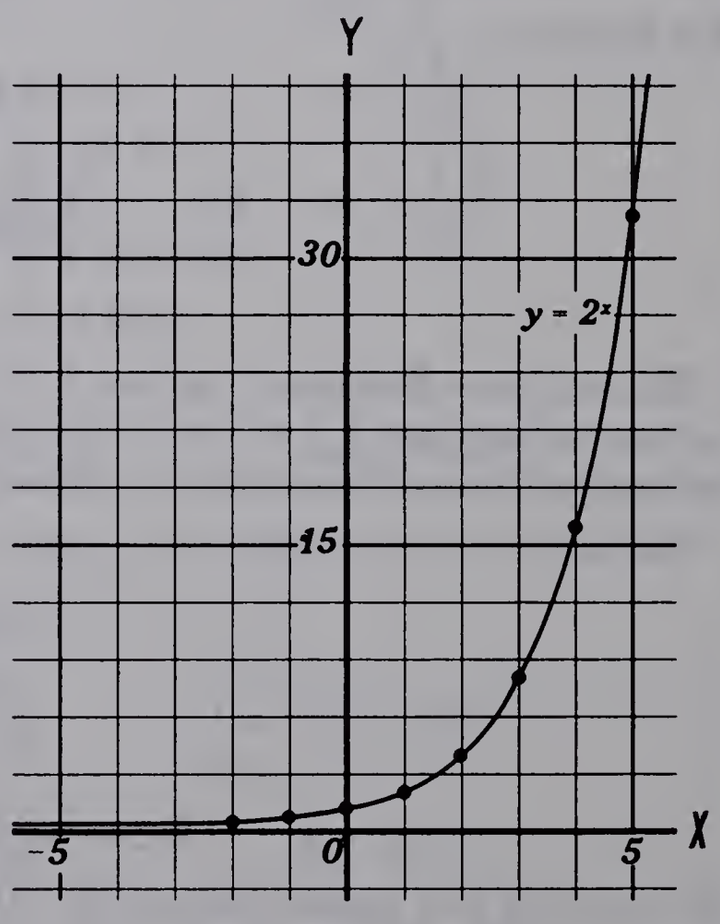


Fig. 7-1

A study of the graph leads (intuitively) to the following conclusions:

(i) the range of the function is the set of all positive real numbers, $\{y \mid y \in {}^{+}R\}$;

(ii) for each real number y of the range there corresponds one and only one (a unique) real number x in the domain, that is for each $y \in {}^{+}R$ there is a unique $x \in R$ such that $y = 2^x$.

A study of other exponential functions such as those defined by

$$y = 3^x,\ x \in R, \text{ and } y = 10^x,\ x \in R,$$

leads to the same conclusion. (The case where $a = 1$ is the only exception; $1^x = 1$ for all $x \in R$, therefore $a = 1$ is not used.)

Thus we will assume that any positive real number M can be expressed uniquely in exponential form to a given base a, where $a \in {}^{+}R$, and exponent x, where $x \in R$; thus

if $M \in {}^{+}R$, *then* $M = a^x$ *where* $a \in {}^{+}R$, $a \neq 1$, *and* $x \in R$.

Write solutions to the following problems and compare them with those on page 428.

1. With reference to the same set of axes, sketch the graphs of the following functions:

(i) $A = \{(x, y) \mid y = 3x,\ x \in R\}$ (ii) $B = \{(x, y) \mid y = 4^x,\ x \in R\}$

(iii) $C = \{(x, y) \mid y = 3^x,\ x \in R\}$ (iv) $D = \{(x, y) \mid y = 10^x,\ x \in R\}$.

2. Find the rate of change $\dfrac{\triangle y}{\triangle x}$ for each function of question 1 for the intervals: (i) $x = 1$ to $x = 2$ (ii) $x = 2$ to $x = 3$ (these increments may be read from the graphs).

3. Using the information from question 2, discuss the *rate of exponential growth* for a base $a > 1$, $a \in R$:

(i) if the base a is increasing, say from 3 to 4 to 10 and so on;

(ii) for equal increments in x, say from $x = 1$ to $x = 2$, $x = 2$ to $x = 3$ and so on, for any given base.

7•10 **Multiplication, division, extraction of roots.** The following examples illustrate how the graphs of exponential functions may be used to find approximately the products, quotients, and roots of positive real numbers.

Example 1. Use the graph, *Fig.7-2* , of the exponential function defined by $y = 2^x$, $x \in R$, to find the following indicated products:

(i) 2.1×3.1 (ii) 1.5×2.7 (iii) 7.3×4.7.

Solution. (i) From the graph

$2.1 \doteq 2^{1.1}$. 2.1 is read on the y-axis, exponent 1.1 on the x-axis.

$3.1 \doteq 2^{1.6}$. 3.1 is read on the y-axis, exponent 1.6 on the x-axis.

$$2.1 \times 3.1 \doteq 2^{1.1} \times 2^{1.6}$$
$$\doteq 2^{1.1+1.6}$$
$$\doteq 2^{2.7}$$
$$\doteq 6.5.$$

6.5 (nearest tenth) is read from the y-axis corresponding to the exponent 2.7 read on the x-axis.

(ii)
$$1.5 \doteq 2^{0.6}$$
$$2.7 \doteq 2^{1.4}.$$
$$1.5 \times 2.7 \doteq 2^{0.6} \times 2^{1.4}$$
$$\doteq 2^{2.0}$$
$$\doteq 4.0 \text{ (nearest tenth).}$$

(iii)
$$7.3 \doteq 2^{2.85}.$$
$$4.7 \doteq 2^{2.22}.$$
$$7.3 \times 4.7 \doteq 2^{2.85} \times 2^{2.22}$$
$$\doteq 2^{5.07}$$
$$\doteq 34.3 \text{ (nearest tenth).}$$

Example 2. Use the graph, *Fig. 7-2*, and the exponential law of division to find the following indicated quotients:

(i) $32 \div 8$ (ii) $28.7 \div 4.5$.

Solution. (i)
$$32 = 2^5.$$
$$8 = 2^3.$$
$$32 \div 8 = 2^{5-3}$$
$$= 2^2$$
$$= 4.$$

(ii)
$$28.7 \doteq 2^{4.87}.$$
$$4.5 \doteq 2^{2.18}.$$
$$28.7 \div 4.5 \doteq 2^{4.87-2.18}$$
$$\doteq 2^{2.69}$$
$$\doteq 6.4 \text{ (nearest tenth).}$$

Example 3. Use the graph, *Fig. 7-2*, and the definition of exponents for roots and the power law to determine the following:

(i) $\sqrt{25.1}$ (ii) $\sqrt[3]{30}$ (iii) $\sqrt[6]{24}$.

Solution.

(i)
$$25.1 \doteq 2^{4.68}.$$
$$\sqrt{25.1} \doteq (2^{4.68})^{\frac{1}{2}}$$
$$\doteq 2^{2.34}$$
$$\doteq 5.0$$
(nearest tenth).

(ii)
$$30 \doteq 2^{4.9}.$$
$$\sqrt[3]{30} \doteq (2^{4.9})^{\frac{1}{3}}$$
$$\doteq 2^{1.63}$$
$$\doteq 3.1$$
(nearest tenth).

(iii)
$$24 \doteq 2^{4.65}.$$
$$\sqrt[6]{24} \doteq (2^{4.65})^{\frac{1}{6}}$$
$$\doteq 2^{0.76}$$
$$\doteq 1.7$$
(nearest tenth).

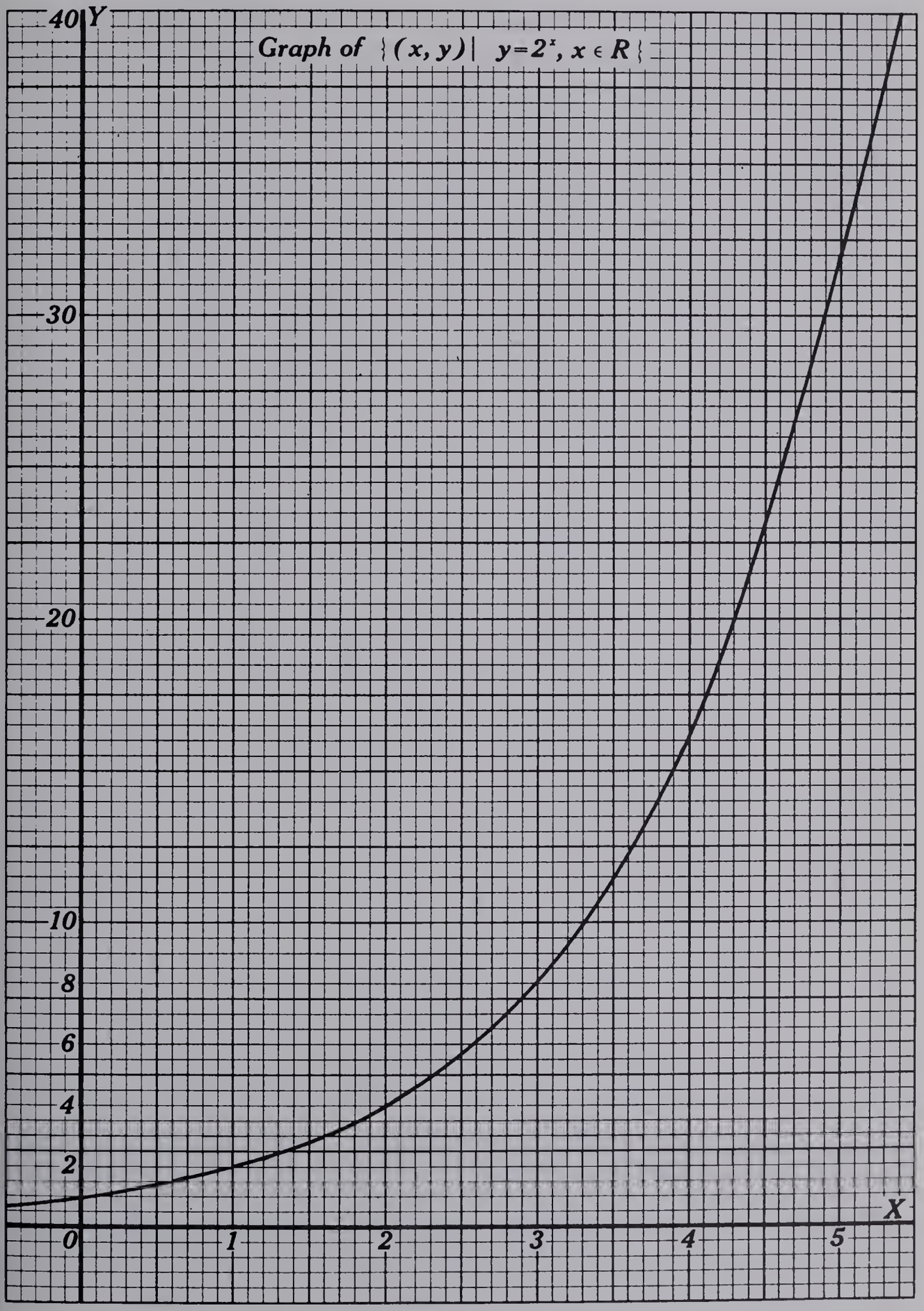


Fig. 7-2

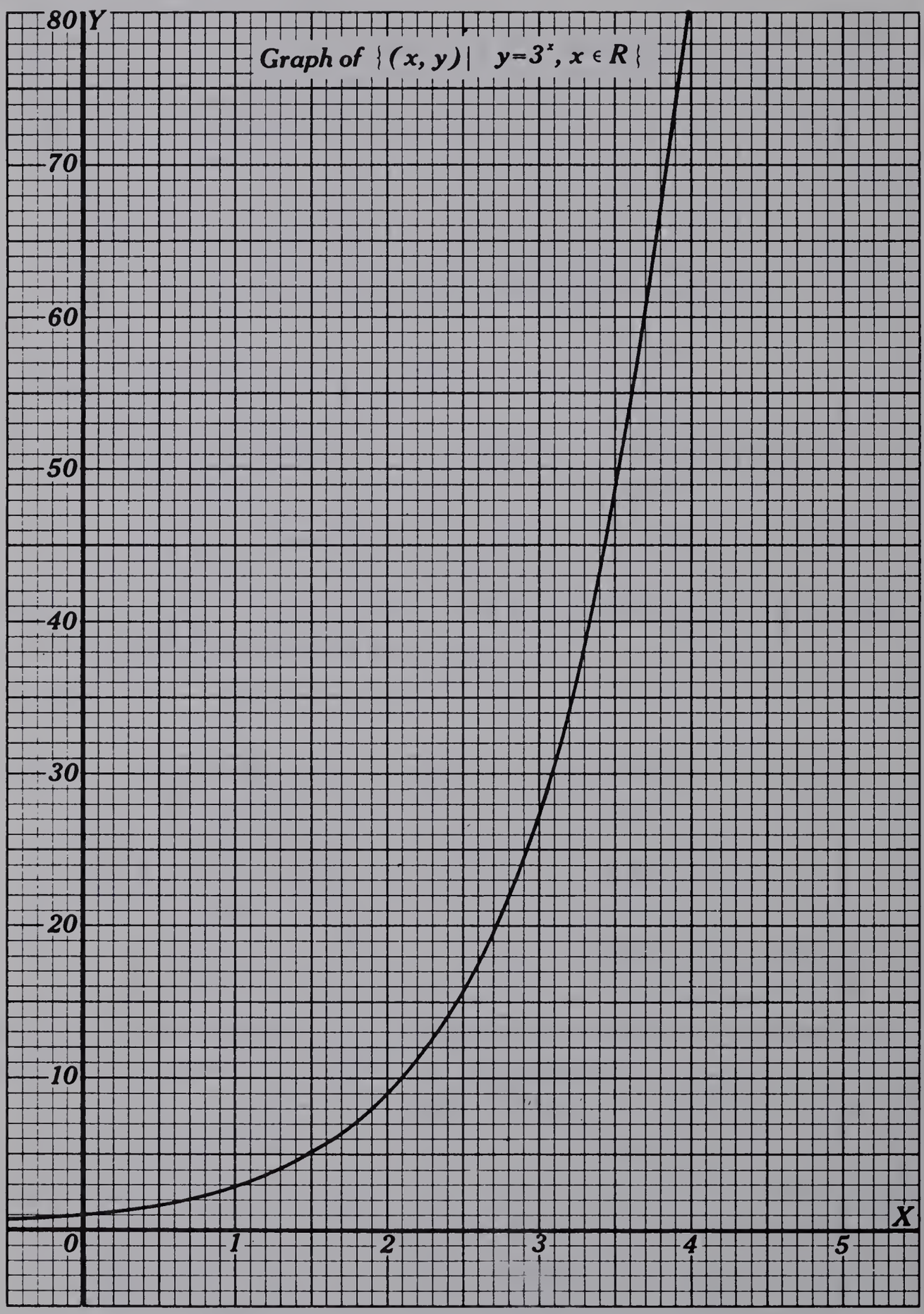


Fig 7-3

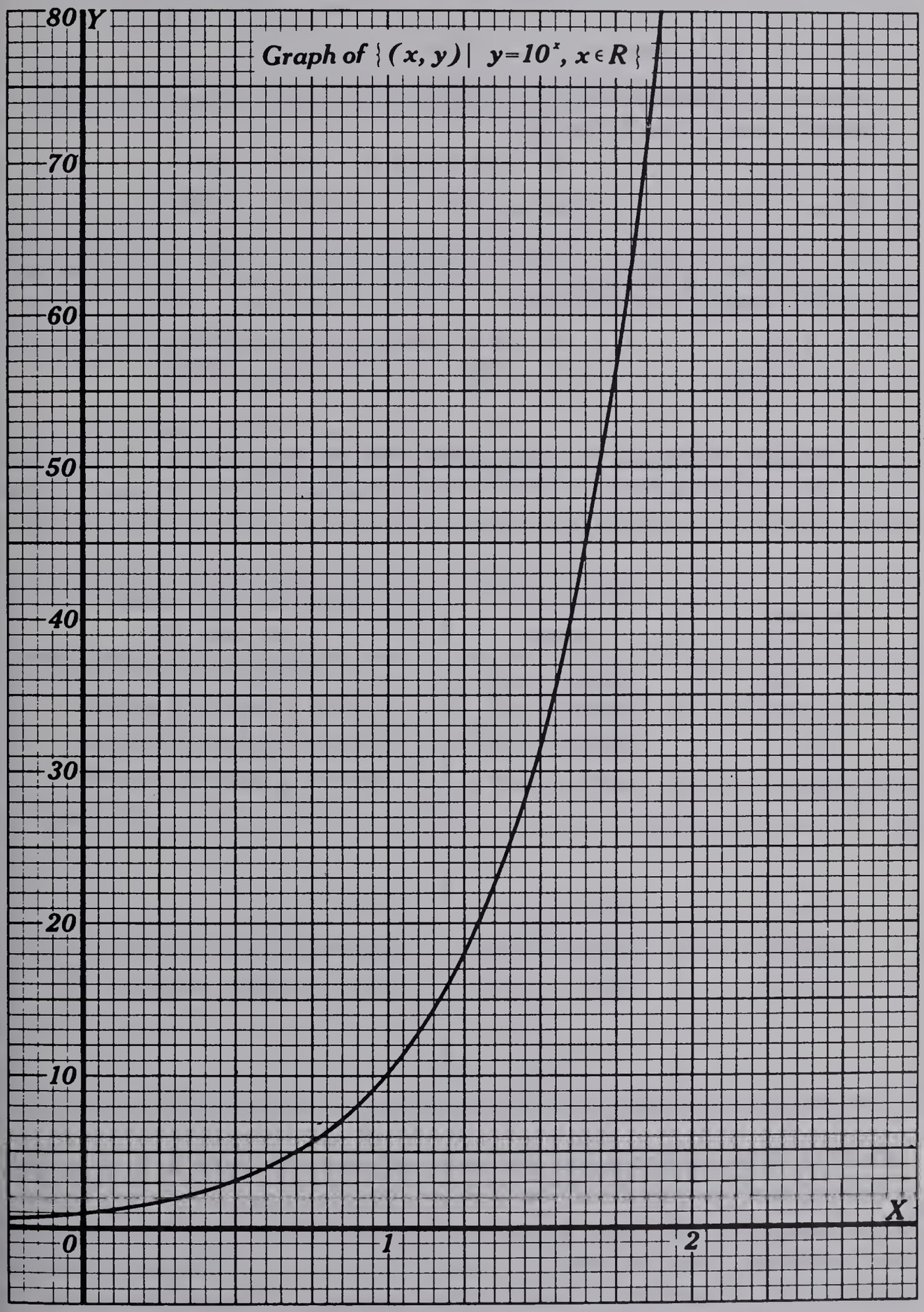


Fig. 7-4

Exercise 7-4

(B)

Use the laws of exponents and the graph defined by $y = 3^x$, $x \in R$, *Fig. 7-3, to determine each of the following:*

1. 15×3.5 **2.** 12.3×5.6 **3.** $75 \div 5$

4. $72.6 \div 7.7$ **5.** $\sqrt{72}$ **6.** $\sqrt[3]{56}$

Use the laws of exponents and the graph defined by $y = 10^x$, $x \in R$, *Fig. 7-4, to determine each of the following:*

7. 6.6×4.3 **8.** 18.4×4.8 **9.** $52.6 \div 12.4$

10. $73.7 \div 24.2$ **11.** $\sqrt{78}$ **12.** $\sqrt[4]{68}$

***7·11* Numbers expressed in exponential form to the base 10.** It may be shown that $2.541 \doteq 10^{0.405}$. The following table illustrates some of the consequents of this and the application of the law of a product for exponents.

NUMERAL	STANDARD FORM	EXPONENTIAL FORM TO BASE 10
2.541	2.541×10^0	$10^{0.405} \times 10^0 = 10^{0+0.405}$
25.41	2.541×10^1	$10^{0.405} \times 10^1 = 10^{1+0.405}$
254.1	2.541×10^2	$10^{0.405} \times 10^2 = 10^{2+0.405}$
2541.	2.541×10^3	$10^{0.405} \times 10^3 = 10^{3+0.405}$
•	•	•
•	•	•
•	•	•
0.2541	2.541×10^{-1}	$10^{0.405} \times 10^{-1} = 10^{-1+0.405}$
0.02541	2.541×10^{-2}	$10^{0.405} \times 10^{-2} = 10^{-2+0.405}$
0.002541	2.541×10^{-3}	$10^{0.405} \times 10^{-3} = 10^{-3+0.405}$
•	•	•
•	•	•
•	•	•

Each of these numerals has the same sequence of digits 2, 5, 4, 1, but each number represented is different due to the placing of the decimal point. The exponents of the base 10 in the exponential form corresponding to these numbers have the form $x + 0.405$, where $x \in I$. The part 0.405 of the exponent is common to each of the exponents. This is the case only when decimal numerals are expressed in exponential form to base 10. It may be seen from the graph of $y = 3^x$, $x \in R$, *Fig 7-3*, that

$$27 = 3^3 \text{ but } 2.7 \doteq 3^{0.9}.$$

The table on page 392 provides the sequence of digits to four figures corresponding to each power of ten from $10^{0.000}$ to $10^{0.999}$. Because of the property of base 10, mentioned in the previous paragraph, this table is sufficient to provide any power of 10 correct to four figures.

This is illustrated by the following example.

From the tables $10^{0.426} \doteq 2.667$.

To obtain this approximation, find the exponent .42 in the first column of the table, then look along the corresponding row to the column headed 6. The sequence of digits in this column and row, 2667, correspond to the exponent 0.426. Although the decimal point is not included in the table, the table is constructed so that

$$10^{0.426} \doteq 2.667.$$

Also since $10^{2+0.426} = 10^{0.426} \times 10^{2}$.

$$\therefore \quad 10^{2+0.426} \doteq 2.667 \times 10^{2}$$

$$\doteq 266.7 \qquad \text{(four figures)}.$$

Similarly $10^{-1+0.426} \doteq 2.667 \times 10^{-1}$

$$\doteq 0.2667.$$

Thus the table provides

(i) the exponents, x, to the base 10 of numbers, y, between 1 and 10; these are found in the first column and row;

(ii) the numbers y (to four digits) in the remaining rows and columns.

The table may be used to determine rational approximations for all ordered pairs of the exponential function

$$T = \{(x, y) \mid y = 10^{x}, x \in R\}$$

with x rounded off to the nearest thousandth and y to four digits.

One ordered pair is $(0.426, 2.667)$,

a second is $(2 + 0.426, 266.7)$,

a third is $(-1 + 0.426, 0.2667)$.

It is customary, to assist in using the tables, to keep the integral part of the exponent separate from the decimal part and to express the exponent as a sum.

Example 1. Use the table to find the number y or 10^{x} corresponding to the exponent:

(i) $x = .346$ (ii) $x = 2 + .648$ (iii) $x = -1 + .572$.

Solution.

(i) From the table, if $x = .346$, $y = 10^{x} \doteq 2.218$.

(ii) From the table, if $x = .648$, $10^x \doteq 4.446$.

$\therefore$ if $x = 2 + .648$, $y = 10^x \doteq 4.446 \times 10^2$

$\doteq 444.6$.

(iii) From the table, if $x = .572$, $10^x \doteq 3.733$.

$\therefore$ if $x = -1 + .572$, $y = 10^x \doteq 3.733 \times 10^{-1}$

$\doteq 0.3733$.

Example 2. Use the tables to find the power of 10 equivalent to the given number:

(i) 2.917 (ii) 660.7 (iii) 0.001239 (iv) 24.7.

Solution.

(i) From the table, $2.917 \doteq 10^{0.465}$.

(ii) From the table, $6.607 \doteq 10^{0.820}$.

$\therefore \quad 660.7 \doteq 10^{0.820} \times 10^2$

$\doteq 10^{2+0.820}$.

(iii) From the table, $1.239 \doteq 10^{0.093}$.

$\therefore \quad 0.001239 \doteq 10^{0.093} \times 10^{-3}$

$\doteq 10^{-3+0.093}$.

(iv) From the table, $2.47 \doteq 10^{0.393}$ $(2.472 \doteq 10^{0.393})$.

$\therefore \quad 24.7 \doteq 10^{0.393} \times 10^1$

$\doteq 10^{1+0.393}$.

Write solutions to the following problems, and compare them with those on page 428.

1. Using the table of the exponential function with base 10, complete the following ordered pairs:

(i) (0.437,) (ii) (1+0.632,) (iii) (3+0.216,)

(iv) (−1+0.524,) (v) (−2+0.320,) (vi) (−4+0.861,)

2. Using the tables, complete the following ordered pairs:

(i) (, 3.141) (ii) (, 98.40) (iii) (, 636.8)

(iv) (, 0 1303) (v) (, 0.02234) (vi) (, 0.0007603)

The following examples illustrate the use of the tables to determine the product and quotient of two numbers and a root of a positive real number.

Example. Using the tables find:

(i) 32.6×5.72 (ii) $103.2 \div 26.7$ (iii) $\sqrt[4]{452}$.

Solution.

(i) From the tables:

$$32.6 \doteq 10^{1+0.513} \qquad [3.258\ (\doteq 3.26) \doteq 10^{0.513}]$$
$$5.72 \doteq 10^{0.757} \qquad [5.715\ (\doteq 5.72) \doteq 10^{0.757}]$$
$$32.6 \times 5.72 \doteq 10^{1+1.270}$$
$$\doteq 10^{2+0.270}$$
$$\doteq 186.2 \qquad [1.862 \doteq 10^{0.270}]$$

(ii) From the tables:

$$103.2 \doteq 10^{2+0.014} \qquad \left[\begin{matrix} 10^{1+1.014} \\ 10^{1+0.426} \\ 10^{0.588} \end{matrix}\right.$$
$$26.7 \doteq 10^{1+0.426}$$
$$103.2 \div 26.7 \doteq 10^{0.588}$$
$$\doteq 3.873$$

(iii) $$452 \doteq 10^{2+0.655}$$
$$\sqrt[4]{452} \doteq 10^{\frac{1}{4}(2+0.655)}$$
$$\doteq 10^{0.664} \qquad \left[\frac{2.655}{4} \doteq 0.664\right]$$
$$\doteq 4.613$$

Exercise 7-5

(B)

(Refer to the table of the exponential function with base ten.)

1. Complete the following ordered pairs:

(i) (0.562,) (ii) (2 + 0.378,)

(iii) (− 1 + 0.732,) (iv) (− 4 + 0.816,)

(v) (, 24.66) (vi) (, 1.734)

(vii) (, 0.01820) (viii) (, 3622)

Find, using the tables:

2. 48.7×13.6 **3.** $342.6 \div 54.2$ **4.** $\sqrt[6]{7824}$

5. 356.2×24.5 **6.** $127.6 \div 82.1$ **7.** $\sqrt[8]{1625}$

8. 1.23×7.24 **9.** $92.3 \div 8.75$ **10.** $\sqrt[3]{7.63}$

11. 32.67×0.00726 **12.** $47.86 \div 1.622$ **13.** $\sqrt[4]{98.17}$

7·12 Exponential functions defined by $y = a^x$, $0 < a < 1$, $a, x \in R$ (supplementary).

Example. Discuss and draw the graph of

$$F = \{(x, y) \mid y = \left(\frac{1}{3}\right)^x, x \in R\}.$$

Solution.

x-intercepts. If $y = 0$, then $\left(\frac{1}{3}\right)^x = 0$.

Since no real x satisfies this equation there is no x-intercept.

y-intercepts. If $x = 0$, then $y = \left(\frac{1}{3}\right)^0$.

$\therefore\ y = 1.$

$\therefore$ the y-intercept is 1.

Domain. $y = \left(\frac{1}{3}\right)^x$.

$\therefore\ y \in R \leftrightarrow \left(\frac{1}{3}\right)^x \in R.$

But $\left(\frac{1}{3}\right)^x \in R$ for all $x \in R$,

$\therefore$ the domain of the function is R.

Range. $y = \left(\frac{1}{3}\right)^x$ or $y = \frac{1}{3^x}$, $x \in R$.

Consider (i) $x > 0$:

$\frac{1}{3^x}$ represents a positive real number for all $x \in {}^+R$.

The greater x is, the greater 3^x is and the lesser $\left(\frac{1}{3}\right)^x$ is.

As x is replaced by greater and greater numbers, the closer to zero $\left(\frac{1}{3}\right)^x$ is. $\left(\frac{1}{3}\right)^x$ is never zero but is as close to zero as desired.

(ii) $x = 0$.

If $x = 0$, then $y = 1$.

(iii) $x < 0$.

If $x < 0$, then $\left(\frac{1}{3}\right)^x = \frac{1}{3^x} = 3^{-x} = 3^n$ where $n \in {}^+R$.

3^n is as great as desired for sufficiently large n.

Although this discussion does not prove it, we intuitively assume that the range of the function is

$$\{y \mid y > 0, y \in R\}.$$

Table of values and graph.

x	y
-3	27
-2	9
-1	3
0	1
1	$\frac{1}{3}$
2	$\frac{1}{9}$

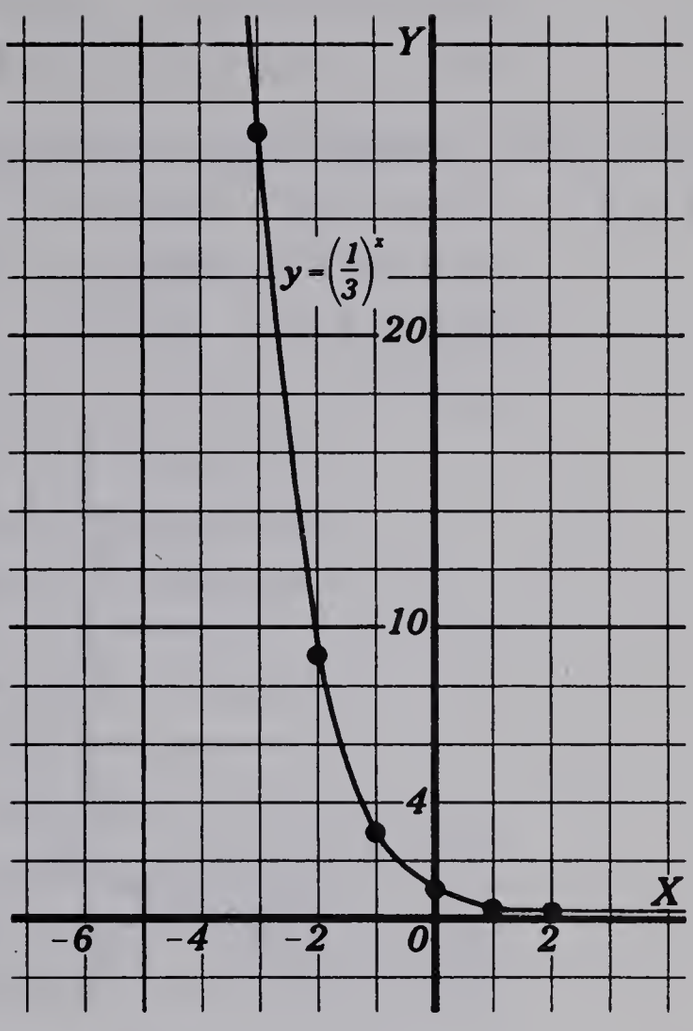


Fig. 7-5

Exercise 7-6

(B)

1. Discuss and draw the graph of the exponential function

$$G = \left\{(x, y) \mid y = \left(\frac{1}{2}\right)^x, x \in R\right\}.$$

***7·13* Order relations in exponential forms (supplementary).**

Example 1. Consider $y = 2^x$, $x \in R$, when (i) $x < 0$, (ii) $x = 0$, (iii) $x > 0$.

x	-3	-2	-1	0	1	2	3
y	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8

The table and graph, *Fig. 7-6*, suggest:

(i) for all $x < 0$, $\quad 0 < 2^x < 1$;

(ii) for $x = 0$, $\quad 2^x = 1$;

(iii) for all $x > 0$, $\quad 2^x > 1$;

(iv) if $x_1 < x_2$, $\quad 2^{x_1} < 2^{x_2}$.

This and other examples suggest the general conclusion:

if $a > 1$, (i) and $x > 0$, then $a^x > 1$;

(ii) and $x < 0$, then $0 < a^x < 1$;

(iii) and $x_1 < x_2$, then $a^{x_1} < a^{x_2}$.

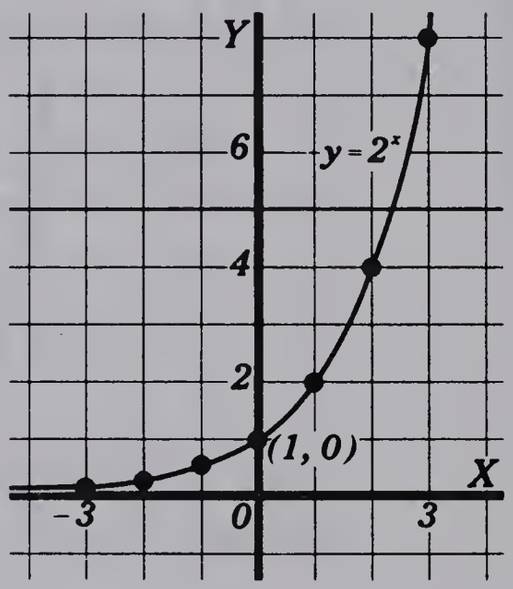


Fig. 7-6

Example 2.

Consider $y = \left(\frac{1}{2}\right)^x$, $x \in R$,

when (i) $x < 0$, (ii) $x = 0$,
(iii) $x > 0$.

x	-3	-2	-1	0	1	2	3
y	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$

Note:

Since $\left(\frac{1}{2}\right)^x = \frac{1^x}{2^x}$ and $1^x = 1$

for all $x \in R$,

$\therefore \left(\frac{1}{2}\right)^x = \frac{1}{2^x}$.

The table and graph, *Fig. 7-7*, suggest:

(i) for all $x < 0$, $\left(\frac{1}{2}\right)^x > 1$;

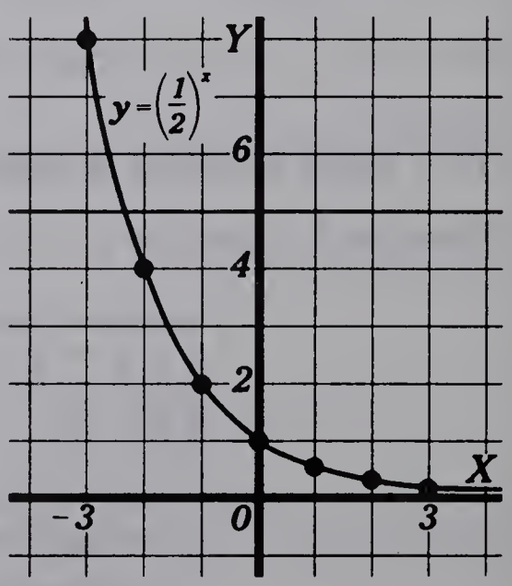


Fig. 7-7

(ii) for $x = 0$, $\left(\frac{1}{2}\right)^x = 1$;

(iii) for all $x > 0$, $0 < \left(\frac{1}{2}\right)^x < 1$;

(iv) if $x_1 < x_2$, $\left(\frac{1}{2}\right)^{x_1} > \left(\frac{1}{2}\right)^{x_2}$.

This and other examples suggest the general conclusion:

if $0 < a < 1$, (i) and $x < 0$, then $a^x > 1$;

(ii) and $x > 0$, then $0 < a^x < 1$;

(iii) and $x_1 < x_2$, then $a^{x_1} > a^{x_2}$.

Example 3. Consider $y = 2^x$ and $y = 3^x$, $x \in R$, when (i) $x < 0$, (ii) $x = 0$, (iii) $x > 0$.

x	-3	-2	-1	0	1	2	3
2^x	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8
3^x	$\frac{1}{27}$	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9	27

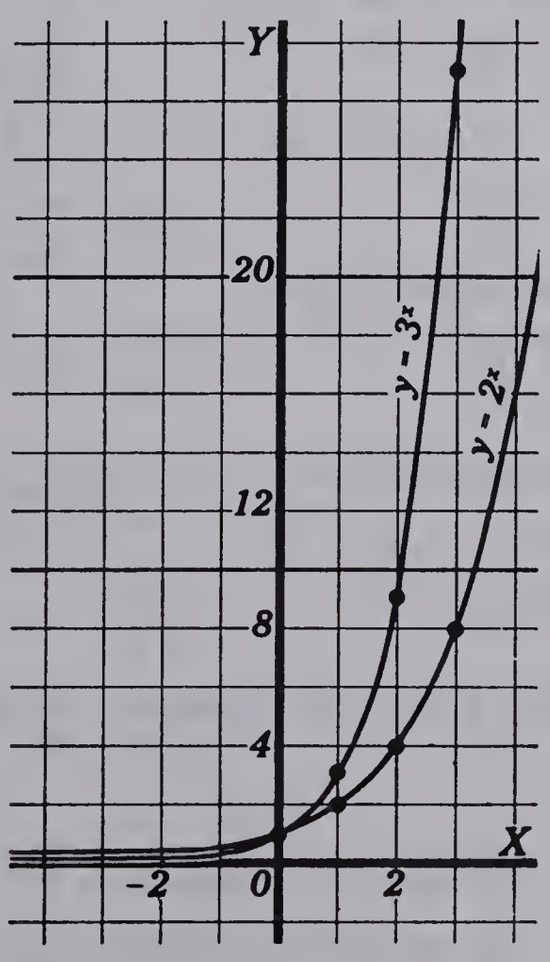


Fig. 7-8

The table and graph, *Fig.7-8* , suggest:

(i) if $x < 0$, $3^x < 2^x$;

(ii) if $x = 0$, $3^x = 2^x = 1$;

(iii) if $x > 0$, $3^x > 2^x$.

This and similar examples suggest the general conclusion:

if $a, b > 1$ and $a < b$, (i) and $x < 0$, then $a^x > b^x$;

(ii) and $x = 0$, then $a^x = b^x = 1$;

(iii) and $x > 0$, then $a^x < b^x$.

Example 4. Consider $y = (\frac{1}{2})^x$ and $y = (\frac{1}{3})^x$, $x \in R$, when (i) $x < 0$, (ii) $x = 0$, (iii) $x > 0$.

x	-3	-2	-1	0	1	2	3
$(\frac{1}{2})^x$	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$
$(\frac{1}{3})^x$	27	9	3	1	$\frac{1}{3}$	$\frac{1}{9}$	$\frac{1}{27}$

The table and graph, *Fig. 7-9*, suggest:

(i) if $x < 0$, $(\frac{1}{3})^x > (\frac{1}{2})^x$;

(ii) if $x = 0$, $(\frac{1}{3})^x = (\frac{1}{2})^x = 1$;

(iii) if $x > 0$, $(\frac{1}{3})^x < (\frac{1}{2})^x$.

This and similar examples suggest the general conclusion:

if $0 < a < b < 1$,

(i) and $x < 0$, then $a^x > b^x$;

(ii) and $x = 0$, then $a^x = b^x = 1$;

(iii) and $x > 0$, then $a^x < b^x$.

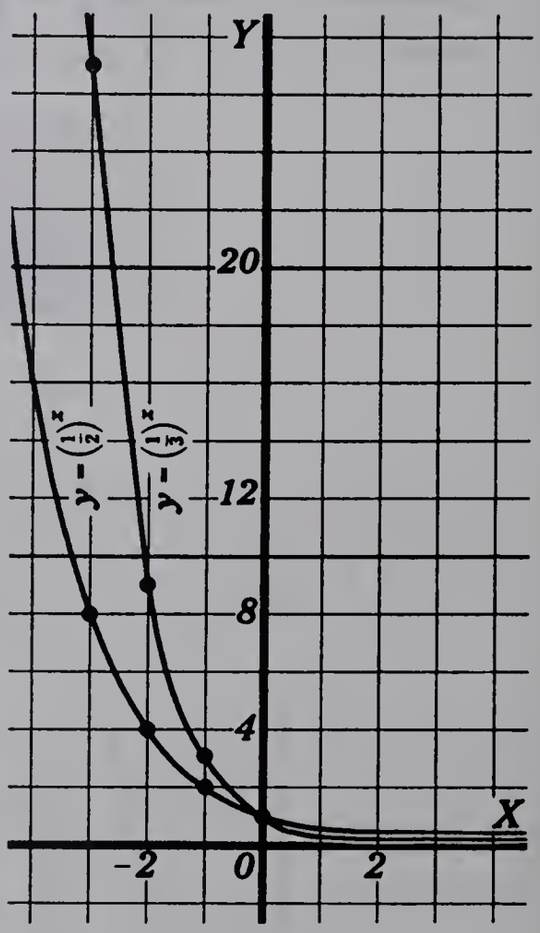


Fig. 7-9

Combining Examples 3 and 4 the following more general conclusion may be drawn:

If $0 < a < b$,

(i) and $x < 0$, then $a^x > b^x$;

(ii) and $x = 0$, then $a^x = b^x = 1$;

(iii) and $x > 0$, then $a^x < b^x$.

Practice Exercise 7-7

(The letter symbols represent numbers for which the expressions are defined.)

(B)

Using the exponential laws state simplified or equivalent forms of each of the following:

1. $5^5 \times 5^{10}$ **2.** $3^4 \div 3^5$ **3.** $2^5 \div 2^3$

4. $(5^4)^2$ **5.** $(7 \times 5 \times 3)^3$ **6.** $(\frac{3}{5})^7$

7. $(3^2 \times 5^6)^3$ **8.** $7^6 \div 7^6$ **9.** $(-1)^{35}$

10. $3^4 \times 3^5 \times 3^7 \times 3$ **11.** $\dfrac{3^2 \times 3^3 \times 3^5}{3^2 \times 3^4}$ **12.** $\left(\dfrac{2^2 \times 3^3}{5^3 \times 7^2}\right)^4$

13. $\dfrac{3^3}{7^3}$ **14.** $\dfrac{(abc)^2}{(abc)^2}$ **15.** $\dfrac{38^4}{13^4}$

State equivalent expressions with positive exponents for each of the following:

16. a^{-3} **17.** $\dfrac{1}{m^0}$ **18.** $a^{-3} \times a^0$

19. a^2b^{-3} **20.** $\dfrac{ab^{-2}}{mn^{-3}}$ **21.** $\dfrac{3x^{-5}}{2y^{-3}}$

22. $(-3)^{-1}$ **23.** $(-5)(-5)^2(-5)^{-3}$ **24.** $\left(\dfrac{3a}{2b}\right)^{-4}$

Express each of the following with denominator 1:

25. $\dfrac{7b^4}{14b^{-2}}$ **26.** $\dfrac{3a^{-2}}{b^2c^{-3}}$ **27.** $\dfrac{17x^2y^3}{m^{-4}b^5}$

28. $\dfrac{16b^7c^8}{48b^2c^{-2}}$ **29.** $\dfrac{256(ab)^{17}}{8a^5b^{-11}}$ **30.** $\dfrac{x^4y^{17}}{726x^{-4}y^{17}}$

Simplify, or by a direct application of the exponential laws, write each of the following in an equivalent form:

31. $\dfrac{a^5 \cdot a^6}{a^3}$ **32.** $\dfrac{9x^4y^3}{3x^2y^4}$ **33.** $\left(\dfrac{a^2}{b^3}\right)^3$

34. $\dfrac{a^pb^q}{a^rb^s}$ **35.** $\left(\dfrac{mp^3q^2}{r^2s}\right)^4$ **36.** $\dfrac{(ab)^3 \times (cd)^2}{(abc)^3 \times (ad)^4}$

37. $\left(\dfrac{4}{3}\right)^2 \times \left(\dfrac{3}{4}\right)^3$ **38.** $\dfrac{5^v \times 25^{v-2}}{125^{v-1}}$ **39.** $\dfrac{15a^3b^{-4}}{5a^{-2}b^{-2}}$

40. $\dfrac{a^x \cdot a^{-2x} \cdot a^{3x}}{a^{2x-3}}$ **41.** $\left(\dfrac{m}{n}\right)^6 \div \left(\dfrac{m}{n}\right)^{-3}$ **42.** $m^{a+b} \div m^{a-b}$

43. Express 125^4 as a power of 5 and 343^5 as a power of 7.

44. Express 125^n as a power of 5 and 343^n as a power of 7.

45. Simplify: $\dfrac{5^{3n} \times 25^{4n+6} \times 125^{-2n+3}}{125^{3n+4}}$.

46. Simplify: $\dfrac{7^{4n} \times 49^{n-7} \times 343^{-2n+1}}{343^{3n-3}}$.

Express each of the following in scientific notation:

47. 32.67

48. 0.0054

49. 324675.172

50. 0.0000726

51. 364,500,000,000

52. 0.000000000017

Practice Exercise 7-8

(The letter symbols represent numbers for which the expressions are defined.)

(B)

Using the exponential laws, state simplified or equivalent forms of each of the following:

1. $3^4 \times 3^7$

2. $9^5 \div 9^4$

3. $7^2 \div 7^{15}$

4. $(2^3)^4$

5. $(3 \times 7)^5$

6. $\left(\dfrac{2}{3}\right)^4$

7. $\left(\dfrac{7}{9}\right)^{13}$

8. $5^2 \div 5^2$

9. 1^{276}

10. $(-1)^{18}$

11. $2^2 \times 2^3 \times 2^4$

12. $\dfrac{2^4 \times 2^6}{2^3}$

13. $\dfrac{x^2y^3z^4}{xy^2z^3}$

14. $7^4 \times 5^4$

15. $\dfrac{2^5 \cdot 3^5 \cdot 5^5}{7^5 \cdot 9^5 \cdot 11^5}$

State equivalent expressions with positive exponents for each of the following:

16. $\dfrac{1}{b^{-2}}$

17. $\dfrac{p^{-2}}{q^{-2}}$

18. $a^{-3} \times a^0$

19. $2x^{-3}$

20. $\dfrac{3^{-2}a^2}{4^{-3}b^{-2}}$

21. $\dfrac{(-2)^2(-2)^3}{(-2)^{-4}}$

22. $\dfrac{(-7)^{-5}}{(-7)^{-5}}$

23. $\dfrac{3a^{-2}}{5b^{-2}}$

24. $\left(\dfrac{3x}{5y}\right)^{-7}$

Express each of the following with denominator 1:

25. $\dfrac{2b^2}{c^{-3}}$ **26.** $\dfrac{14ab^{-3}}{7a^{-3}b^4}$ **27.** $\dfrac{5x^{-2}}{y^{-2}z^{-3}}$

28. $\dfrac{36x^4y^{-5}}{12x^{-4}y^5}$ **29.** $\dfrac{45x^4}{15(a-b)^2}$ **30.** $\dfrac{72(xy)^6}{9x^{-2}y^4}$

Simplify, or by a direct application of the exponential laws, write each of the following in an equivalent form:

31. $\dfrac{a^2b^2}{a^3b^3}$ **32.** $\dfrac{(-2)^x \times (-2)^y}{(-2)^z}$ **33.** $\left(\dfrac{a^2b^4}{ab^2}\right)^4$

34. $\dfrac{3^x \times 3^{x-2} \times 3^{2x}}{3^{3x}}$ **35.** $\dfrac{3x^2y \times 2xy^2}{5xy^4 \times 7x^2y}$ **36.** $\left(\dfrac{a}{b}\right)^3 \times \left(\dfrac{2a}{3b}\right)^2 \times \left(\dfrac{3a^2}{5b^3}\right)^2$

37. $\left(\dfrac{2}{3}\right)^x \times \left(\dfrac{2}{3}\right)^y$ **38.** $\dfrac{3^x \times 3^{x-3} \times 3^{2x+1}}{9^{3x}}$ **39.** $\dfrac{6m^2n^{-3}p^4}{3m^{-2}n^3p^{-2}}$

40. $\dfrac{275m^{-3}q^{-2}}{25m^2q^{-4}}$ **41.** $\left(\dfrac{x}{y}\right)^p \div \left(\dfrac{x}{y}\right)^{-q}$ **42.** $(a+b)^{-3} \times \dfrac{1}{(a+b)^{-4}}$

43. Express 25^5 as a power of 5 and 49^3 as a power of 7 .

44. Express 25^n as a power of 5 and 49^n as a power of 7 .

45. Simplify: $\dfrac{5^{2n} \times 5^{3n-3} \times 5^{4n}}{25^{3n-6}}$.

46. Simplify: $\dfrac{7^{5n} \times 7^{4n+6} \times 7^{-3n+3}}{49^{5n+1}}$.

Express each of the following in scientific notation:

47. 34.2 **48.** 0.003425

49. 34672.5 **50.** 0.0000074362

51. 74562841.34 **52.** 0.46278215

Practice Exercise 7-9

(B)

State the meaning of:

1. $3^{\frac{1}{2}}$ **2.** $(-3)^{\frac{1}{5}}$ **3.** $(-7)^{\frac{1}{n}}$, $n \in {}^+I$

4. $3^{\frac{3}{5}}$ **5.** $(a)^{\frac{4}{7}}$, $a < 0$ **6.** $(x^3)^{\frac{5}{9}}$

Simplify:

7. $(x^2)^{\frac{1}{2}}$ **8.** $(x^2+8x+16)^{\frac{1}{2}}$ **9.** $(4a^2-20ab+25b^2)^{\frac{1}{2}}$

10. $(x^3)^{\frac{1}{3}}$ **11.** $[(x+y)^5]^{\frac{1}{5}}$ **12.** $[(a-b)^4]^{\frac{1}{4}}$

Simplify or evaluate:

13. $8^{-\frac{4}{3}}$ **14.** $32^{-\frac{3}{5}}$ **15.** $\left(\frac{x^4y^6}{a^8b^{10}}\right)^{\frac{3}{2}}$

16. $\left(\frac{x^{10}y^{15}}{m^{20}p^5}\right)^{\frac{3}{5}}$ **17.** $\frac{4^{\frac{2}{3}} \times 4^{\frac{5}{6}}}{4^{\frac{5}{12}}}$ **18.** $\frac{a^{\frac{3}{5}} \times a^{\frac{3}{10}}}{a^{\frac{4}{15}}}$

Solve the following equations:

19. $a^{\frac{1}{4}} = 2$ **20.** $b^{\frac{1}{5}} = 3$ **21.** $c^{\frac{1}{3}} = 5$

22. $a^{\frac{2}{3}} = 16$ **23.** $b^{\frac{5}{4}} = 32$ **24.** $c^{\frac{5}{7}} = 3125$

Chapter VIII

LOGARITHMIC FUNCTIONS

8·1 The logarithmic function. In *Fig. 8-1* are shown

(i) the graph of the exponential function

$$e = \{(x, y) \mid y = 2^x,\ x \in R\},\ \text{range}\ \{y \mid y \in {}^+R\};$$

(ii) the graph of the function

$$l = \{(x, y) \mid x = 2^y,\ x \in {}^+R\},\ \text{range}\ \{y \mid y \in R\}.$$

A comparison of the two functions indicates that l is related to e in that the variables x and y have been interchanged in the defining equation, and the domain and range have been interchanged.

(iii) the graph of the function

$$g = \{(x, y) \mid y = x,\ x \in R\}.$$

It may be seen from *Fig. 8-1*:

(i) that for any point A (p, q) of the graph of e there is a corresponding point B (q, p) of the graph of l;

(ii) the line segment AB is perpendicular to the line which is the graph of g and is bisected by the line;

(iii) the graphs of the two functions e and l are *reflections* of each other in the line graph of g .

Each of the functions e and l is the *inverse* function of the other. The inverse function of an exponential function is called a *logarithmic function*.

The *exponential form* of the defining sentence of the logarithmic function l is

$$x = 2^y,\ x \in {}^+R;$$

For e

For l

x	-4	-3	-2	-1	0	1	2	3	4
y	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8	16
x	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8	16
y	-4	-3	-2	-1	0	1	2	3	4

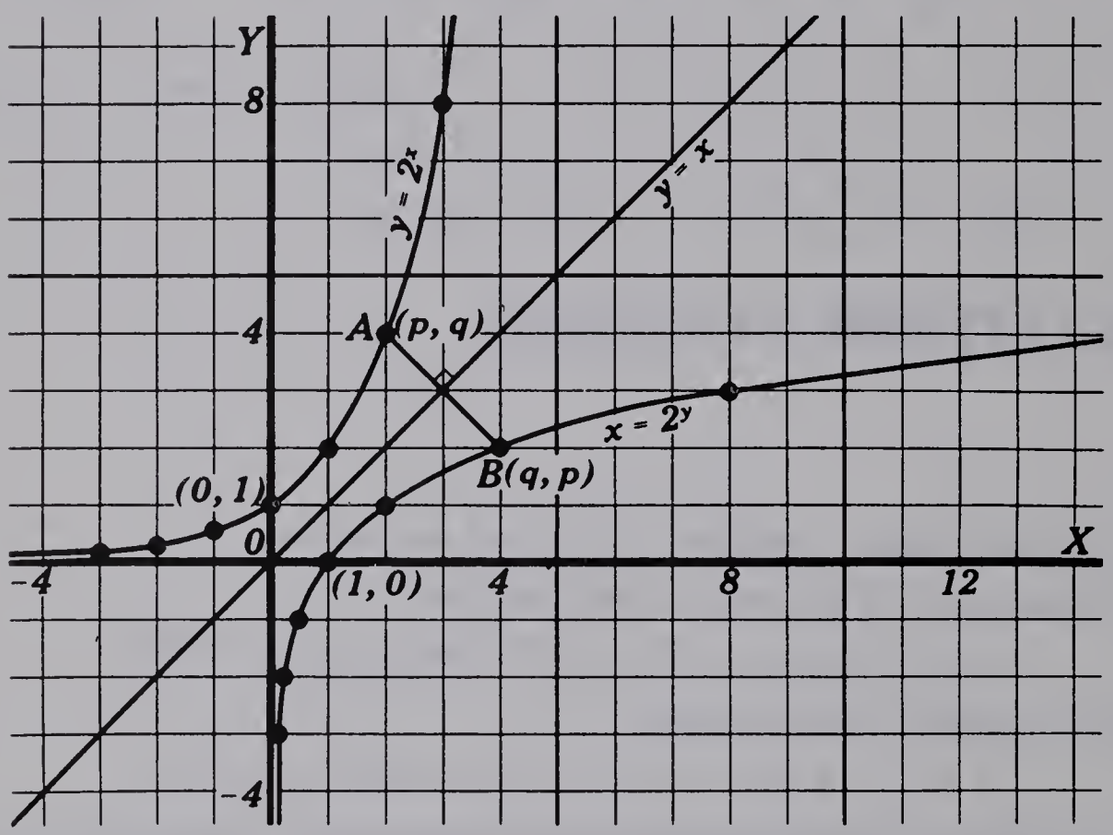


Fig. 8-1

the *logarithmic form* of the defining sentence is

$$y = \log_2 x,\ x \in {}^+R.$$

The equation is read "y equals the logarithm of x to base 2".

Thus $\qquad x = 2^y,\ x \in {}^+R \leftrightarrow y = \log_2 x,\ x \in {}^+R.$

DEFINING EQUATION OF EXPONENTIAL FUNCTION (DOMAIN: R)	DEFINING EQUATION OF THE INVERSE OR LOGARITHMIC FUNCTION (DOMAIN: ${}^+R$)	
	EXPONENTIAL FORM	LOGARITHMIC FORM
$y = 3^x$	$x = 3^y$	$y = \log_3 x$
$y = 4^x$	$x = 4^y$	$y = \log_4 x$
$y = 10^x$	$x = 10^y$	$y = \log_{10} x$
$y = 50^x$	$x = 50^y$	$y = \log_{50} x$
$y = a^x,\ a \in {}^+R,\ a \neq 1$	$x = a^y$	$y = \log_a x$

The general logarithmic function is

$$l = \{(x, y) \mid y = \log_a x,\ x \in {}^+R\} \text{ where } a \in {}^+R,\ a \neq 1.$$

This function may also be expressed

$$l = \{(x, y) \mid x = a^y,\ x \in {}^+R\}.$$

It follows that the *logarithm* of a number x to a given base a, where $a \in {}^+R$, $a \neq 1$, is the exponent y to which the base must be raised to equal the number.

Write solutions for each of the following; compare your solutions with those on page 430.

1. Express in logarithmic form:

(i) $x = 5^3$ (ii) $8 = 2^3$ (iii) $1000 = 10^3$ (iv) $p = q^r$.

2. Express in exponential form:

(i) $\log_7 x = 4$ (ii) $\log_5 125 = 3$ (iii) $\log_{10} 10^5 = 5$ (iv) $\log_l B = m$.

Exercise 8-1

(A)

1. Express in logarithmic form:

(i) $2^5 = 32$ (ii) $3^3 = 27$ (iii) $10^4 = 10000$

(iv) $5^0 = 1$ (v) $16^{\frac{1}{4}} = 2$ (vi) $81^{\frac{1}{4}} = 3$

(vii) $x^4 = b$ (viii) $x^a = b$ (ix) $a^p = 45$

2. Express in exponential form:

(i) $\log_2 64 = 6$ (ii) $\log_3 27 = 3$ (iii) $\log_8 8 = 1$

(iv) $\log_a n = b$ (v) $\log_4 8 = \frac{3}{2}$ (vi) $\log_9 27 = \frac{3}{2}$

3. Find the value of x in each of the following:

(i) $\log_x 27 = 3$ (ii) $\log_x 8 = 3$ (iii) $\log_x 10000 = 4$

(iv) $\log_x 128 = 7$ (v) $\log_x 125 = 3$ (vi) $\log_x 0.01 = 2$

(B)

4. By expressing the statement in exponential form, find x in each of the following:

(i) $\log_{10} 100 = x$ (ii) $\log_5 125 = x$ (iii) $\log_3 81 = x$

5. Find:

(i) $\log_2 32$ (ii) $\log_{10} 1000$ (iii) $\log_4 64$

(iv) $\log_{10} 10000$ (v) $\log_3 27$ (vi) $\log_5 625$

(vii) $\log_2 8$ (viii) $\log_3 \sqrt[3]{27}$ (ix) $\log_{10} \sqrt[3]{1000}$

6. Make a table of values and sketch the graph of the exponential function defined by $y = 5^x$, $x \in R$, and the graph of the logarithmic (inverse) function defined by $y = \log_5 x$, $x \in {}^+R$, with respect to the same axes.

7. Express in logarithmic form:

(i) $5^0 = 1$ (ii) $\left(\frac{1}{8}\right)^{-\frac{4}{3}} = 16$ (iii) $2^{-3} = \frac{1}{8}$

(iv) $10^{0.3010} = 2$ (v) $\left(\frac{1}{81}\right)^{-\frac{3}{4}} = 27$ (vi) $a^b = c$

8. Express in exponential form:

(i) $\log_9 27 = \frac{3}{2}$ (ii) $\log_4 x = \frac{3}{4}$ (iii) $\log_b M = a$

(iv) $\log_x \frac{1}{16} = -\frac{4}{5}$ (v) $\log_8 16 = x$ (vi) $\log_4 \frac{1}{16} = -2$

9. Find y in each of the following:

(i) $\log_4 y = \frac{3}{2}$ (ii) $\log_{10} y = 0$ (iii) $\log_9 y = -\frac{1}{2}$

(iv) $\log_8 y = \frac{4}{3}$ (v) $\log_7 y = -2$ (vi) $\log_{\frac{1}{8}} y = -\frac{4}{3}$

(C)

10. Find x in each of the following:

(i) $\log_x \frac{1}{16} = -\frac{4}{5}$ (ii) $\log_x \frac{9}{4} = -\frac{2}{3}$ (iii) $\log_x \frac{1}{49} = -2$

(iv) $\log_x \frac{125}{27} = -3$ (v) $\log_x 16 = -\frac{4}{3}$ (vi) $\log_x 625 = -4$

11. Find A if $A = \log_3 27 + \log_{10} 0.01 + \log_2 \frac{1}{16}$.

12. Find z if $z = \log_6 36 - \log_4 8 - \log_3 \frac{1}{81}$.

13. Find w if

$$w = \frac{\log_4 64 + \log_3 81 - \log_4 \frac{1}{16} + \log_{\frac{1}{3}} 81}{\log_{\frac{1}{9}} 27 - \log_8 32 - \log_4 \frac{1}{64}}.$$

8•2 Basic properties of logarithms. The basic properties of the function defined by $y = \log_a x$, $x \in {}^+R$ are illustrated in the following examples.

Example 1. Show that $\log_2 (8 \times 4) = \log_2 8 + \log_2 4$.

Solution.

$$\begin{aligned} \log_2 (8 \times 4) &= \log_2 32 \\ &= 5. \qquad (\because 32 = 2^5) \end{aligned} \quad \Bigg| \quad \begin{aligned} \log_2 8 + \log_2 4 &= 3 + 2 \\ &= 5. \end{aligned}$$

$$\therefore \log_2 (8 \times 4) = \log_2 8 + \log_2 4.$$

Example 2. Show that $\log_2 \frac{32}{8} = \log_2 32 - \log_2 8$.

Solution.

$$\begin{aligned} \log_2 \frac{32}{8} &= \log_2 4 \\ &= 2. \end{aligned} \quad \Bigg| \quad \begin{aligned} \log_2 32 - \log_2 8 &= 5 - 3 \\ &= 2. \end{aligned}$$

$$\therefore \log_2 \frac{32}{8} = \log_2 32 - \log_2 8.$$

Example 3. Show that $\log_{10} 100^3 = 3 \log_{10} 100$.

Solution.

$$\begin{aligned} \log_{10} 100^3 &= \log_{10} (10^2)^3 \\ &= \log_{10} 10^6 \\ &= 6. \end{aligned} \quad \Bigg| \quad \begin{aligned} 3 \log_{10} 100 &= 3 \times 2 \\ &= 6. \end{aligned}$$

$$\therefore \log_{10} 100^3 = 3 \log_{10} 100.$$

Example 4. Show that $\log_3 \sqrt[3]{27} = \frac{1}{3} \log_3 27$.

Solution.

$$\begin{aligned} \log_3 \sqrt[3]{27} &= \log_3 (27)^{\frac{1}{3}} \\ &= \log_3 3 \\ &= 1. \end{aligned} \quad \Bigg| \quad \begin{aligned} \frac{1}{3} \log_3 27 &= \frac{1}{3} \log_3 (3^3) \\ &= \frac{1}{3} (3) \\ &= 1. \end{aligned}$$

$$\therefore \log_3 \sqrt[3]{27} = \frac{1}{3} \log_3 27.$$

Exercise 8-2

(B)

Show that each of the following is true:

1. $\log_7 (343 \times 49) = \log_7 343 + \log_7 49$

2. $\log_3 \dfrac{243}{27} = \log_3 243 - \log_3 27$

3. $\log_5 (125)^3 = 3 \log_5 125$

4. $\log_5 \sqrt[3]{5^6} = \dfrac{1}{3} \log_5 5^6$

5. $\log_7 (\sqrt[3]{343})^2 = \dfrac{2}{3} \log_7 343$

6. $\log_3 \dfrac{27 \times 81}{9} = \log_3 27 + \log_3 81 - \log_3 9$

7. $\log_5 \dfrac{5^4}{25 \times 625} = \log_5 5^4 - (\log_5 25 + \log_5 625)$

From the results of the previous problems, suggest an alternate expression for each of the following:

8. $\log_a (37 \times 129)$ **9.** $\log_b \dfrac{537}{29}$ **10.** $\log_c 31^5$ **11.** $\log_w \sqrt[k]{47}$

12. $\log_x (a \times b)$ **13.** $\log_v \left(\dfrac{l}{f}\right)$ **14.** $\log_z K^t$ **15.** $\log_p \sqrt[s]{q^r}$

8·3 Proofs of basic properties of logarithms. Since considerable use is made of the basic properties of logarithms, it is important to establish each in a formal manner.

THEOREM 1. *The logarithm of a product.*

If $M, N, a \in {}^+R, a \neq 1$, *then*

$$\log_a MN = \log_a M + \log_a N.$$

Proof.

Let $\log_a M = x, x \in R$; then $M = a^x$.		(definition)
Let $\log_a N = y, y \in R$; then $N = a^y$.		(definition)
	$MN = a^x \times a^y$	
$\leftrightarrow$	$MN = a^{x+y}$	(exponential law)
$\leftrightarrow$	$\log_a MN = x + y$	(definition)
$\leftrightarrow$	$\log_a MN = \log_a M + \log_a N.$	(substitution)

Corollary. $\log_a MNP \ldots Q = \log_a M + \log_a N + \log_a P + \ldots + \log_a Q.$

THEOREM 2. *The logarithm of a quotient.*

If $M, N, a \in {}^+R, a \neq 1$, *then*

$$\log_a \frac{M}{N} = \log_a M - \log_a N.$$

Proof. Let $\log_a M = x,\ x \in R$; then $M = a^x$. (definition)
Let $\log_a N = y,\ y \in R$; then $N = a^y$. (definition)

$$\frac{M}{N} = \frac{a^x}{a^y}$$

$$\leftrightarrow \quad \frac{M}{N} = a^{x-y} \qquad \text{(exponential law)}$$

$$\leftrightarrow \log_a \frac{M}{N} = x - y \qquad \text{(definition)}$$

$$\leftrightarrow \log_a \frac{M}{N} = \log_a M - \log_a N. \qquad \text{(substitution)}$$

THEOREM 3. *The logarithm of a power (and of a root).*
If $M, a \in {}^+R, a \neq 1$ *and* $p \in R$, *then*

$$\log_a M^p = p \log_a M..$$

Proof. Let $\log_a M = x,\ x \in R$; then $M = a^x$. (definition)

$$M^p = (a^x)^p$$

$$\leftrightarrow \quad M^p = a^{xp} \qquad \text{(exponential law)}$$

$$\leftrightarrow \log_a M^p = xp \qquad \text{(definition)}$$

$$\leftrightarrow \log_a M^p = p \log_a M. \qquad \text{(substitution)}$$

Corollaries. (i) Since $\sqrt[q]{M} = M^{\frac{1}{q}},\ q \in {}^+I,$

$$\text{then } \log_a M^{\frac{1}{q}} = \frac{1}{q} \log_a M.$$

$$\therefore \log_a \sqrt[q]{M} = \frac{1}{q} \log_a M.$$

(ii) Since $(\sqrt[q]{M})^p = M^{\frac{p}{q}},\ p \in I,\ q \in {}^+I,$

$$\text{then } \log_a M^{\frac{p}{q}} = \frac{p}{q} \log_a M.$$

$$\therefore \log\ (\sqrt[q]{M})^p = \frac{p}{q} \log_a M.$$

Example 1. Expand, using the properties of logarithms:

(i) $\log_a (47 \times 83.6)$ (ii) $\log_a (3.27 \div 29.5)$ (iii) $\log_a \dfrac{62.9 \times 219.5}{41.5}$.

Solution. (i) $\log_a (47 \times 83.6) = \log_a 47 + \log_a 83.6.$

(ii) $\log_a (3.27 \div 29.5) = \log_a 3.27 - \log_a 29.5.$

(iii) $\log_a \dfrac{62.9 \times 219.5}{41.5} = \log_a 62.9 + \log_a 219.5 - \log_a 41.5.$

Example 2. Solve each of the following:

(i) $n = \log_2 (8 \times 32)$ (ii) $m = \log_3\left(\frac{243}{9}\right)$ (iii) $p = \log_7 (343)^3$.

Solution.

(i) $n = \log_2 (8 \times 32)$.

$\therefore\ n = \log_2 8 + \log_2 32$

$= 3 + 5$

$= 8.$

(ii) $m = \log_3 \frac{234}{9}$.

$\therefore\ m = \log_3 243 - \log_3 9$

$= 5 - 2$

$= 3.$

(iii) $p = \log_7 (343)^3$.

$\therefore\ p = 3 \log_7 343$

$= 3 \times 3$

$= 9.$

Example 3. Using the following table of powers of 3 solve

$$x = 243 \times 81.$$

POWERS OF 3	
$3^0 = 1$	$3^6 = 729$
$3^1 = 3$	$3^7 = 2187$
$3^2 = 9$	$3^8 = 6561$
$3^3 = 27$	$3^9 = 19683$
$3^4 = 81$	$3^{10} = 59049$
$3^5 = 243$	$3^{11} = 177147$

Solution.

METHOD 1

$x = 243 \times 81.$

$\therefore\ x = 3^5 \times 3^4$

$= 3^9$

$= 19683.$

METHOD 2

$x = 243 \times 81.$

$\therefore\ \log_3 x = \log_3 243 + \log_3 81$

$= 5 + 4$

$= 9.$

$\therefore\ x = 3^9$

$= 19683.$

Example 4. Simplify, using the properties of logarithms:

(i) $\log_2 25 + \log_2 5$ (ii) $\log_4 15 - \log_4 7$

(iii) $\log_3 55 + \log_3 16 - \log_3 44$ (iv) $3 \log_6 5 - \frac{1}{2} \log_6 7$.

Solution.

(i) $\log_2 25 + \log_2 5 = \log_2 (25 \times 5)$

$= \log_2 125$.

(ii) $\log_4 15 - \log_4 7 = \log_4 \frac{15}{7}$.

(iii) $\log_3 55 + \log_3 16 - \log_3 44 = \log_3 \frac{55 \times 16}{44} = \log_3 20.$

(iv) $3 \log_6 5 - \frac{1}{2} \log_6 7 = \log_6 \frac{5^3}{7^{\frac{1}{2}}} = \log_6 \frac{5^3}{\sqrt{7}}$.

Exercise 8-3

(B)

1. Expand, using the properties of logarithms:

(i) $\log_{10} \frac{48}{17.9}$ (ii) $\log_5 (53.4 \times 27.8 \times 36.9)$

(iii) $\log_a (xyz)$ (iv) $\log_{10} \left(\frac{53 \times 46.8}{91.6}\right)$

(v) $\log_a \frac{pqr}{st}$ (vi) $\log_a \left(\frac{1}{pq}\right)$

2. Expand, using the properties of logarithms:

(i) $\log_{10} \sqrt[3]{\frac{48}{37.6}}$ (ii) $\log_{10} (63.5)^{\frac{2}{3}}$ (iii) $\log_{10} 2\pi \sqrt{\frac{l}{g}}$

(iv) $\log_{10} \frac{7\sqrt{x^3}}{\sqrt[3]{17y}}$ (v) $\log_a \frac{x^p}{\sqrt[q]{y^2}}$ (vi) $\log_a \frac{x^3 \times y^{\frac{1}{2}}}{\sqrt[3]{z^4}}$

3. Solve each of the following:

(i) $n = \log_2 \left(\frac{8}{32}\right)$ (ii) $x = \log_2 8^3$ (iii) $y = \log_3 (243)^4$

4. If $\log_{10} 2 = 0.3010$, $\log_{10} 3 = 0.4771$, and $\log_{10} 10 = 1$, solve each of the following:

(i) $n = \log_{10} 5$ (ii) $x = \log_{10} 12$ (iii) $y = \log_{10} \sqrt[5]{12}$

(iv) $z = \log_{10} \frac{2}{3}$ (v) $w = \log_{10} \frac{9}{8}$ (vi) $p = \log_{10} 90$

5. Simplify each of the following, using the properties of logarithms:

(i) $\log_a 2x + 3(\log_a x - \log_a y)$

(ii) $\frac{1}{2} \log_2 25 - \log_2 7 + \log_2 14 - \frac{2}{3} \log_2 27$

(iii) $\frac{1}{2} \log_a x - \frac{2}{3} \log_a y$ (iv) $\frac{1}{2} \log_4 9 - \log_4 5 + 3 \log_4 10$

6. Make a table of the powers of 2 and using it calculate the following:

(i) 64×16 (ii) $1024 \div 64$

7. Make a table of the powers of 5 and using it calculate the following:
(i) 3125×625 (ii) $78125 \div 625$

(C)

The following theorem provides a relationship for changing a logarithm from base a to another base b.

THEOREM. *If* $M \in {}^+R$ *and* $a, b \in {}^+R$, $a \neq 1$, $b \neq 1$, *then*

$$\log_a M = \log_b M \times \frac{1}{\log_b a}.$$

Proof. Let $M = a^x$, then $x = \log_a M$. (definition)

$$M = a^x,$$

$$\leftrightarrow \log_b M = x \log_b a \quad \text{(theorem 3)}$$

$$\leftrightarrow \log_b M = \log_a M \times \log_b a \quad \text{(substitution)}$$

$$\leftrightarrow \log_a M = \log_b M \times \frac{1}{\log_b a}.$$

8. Change each of the following logarithms to an equivalent logarithm with the base indicated:

(i) $\log_2 14$ to a logarithm with base 3;
(ii) $\log_7 156$ to a logarithm with base 5;
(iii) $\log_6 57.9$ to a logarithm with base 10;
(iv) $\log_4 1587$ to a logarithm with base 10;
(v) $\log_p K$ to a logarithm with base q;
(vi) $\log_u A$ to a logarithm with base v.

8·4 Systems of logarithms. Logarithms were invented by the seventeenth century Scotch mathematician John Napier. He was concerned with inventing a scheme whereby multiplication could be made less laborious.

Although a system of logarithms may be built using as a base any positive real number except 1, there are only two systems commonly used. The one uses 10 as a base and is called the *System of Common Logarithms.* A common logarithm is designated by the abbreviation "*log*"; the subscript denoting the base 10 is not written. Thus $\log_{10} 50$ is written log 50. Common logarithms are used to facilitate computation. The other system which uses the irrational number e (2.71828 . . .) as a base is called *The Napierian System* or *System of Natural Logarithms.* A logarithm to base e is designated by the abbreviation "*ln*"; thus $\log_e 50$ is written *ln* 50. Natural logarithms are important in the study of advanced mathematics. The remainder of this chapter is concerned with common logarithms.

8•5 The characteristic and the mantissa of a logarithm. *Fig. 8-2* illustrates the graph of the exponential function defined by

$$y = 10^x,\ x \in R,$$

and its inverse function defined by

$$x = 10^y,\ x \in {}^+R$$

or $$y = \log x,\ x \in {}^+R.$$

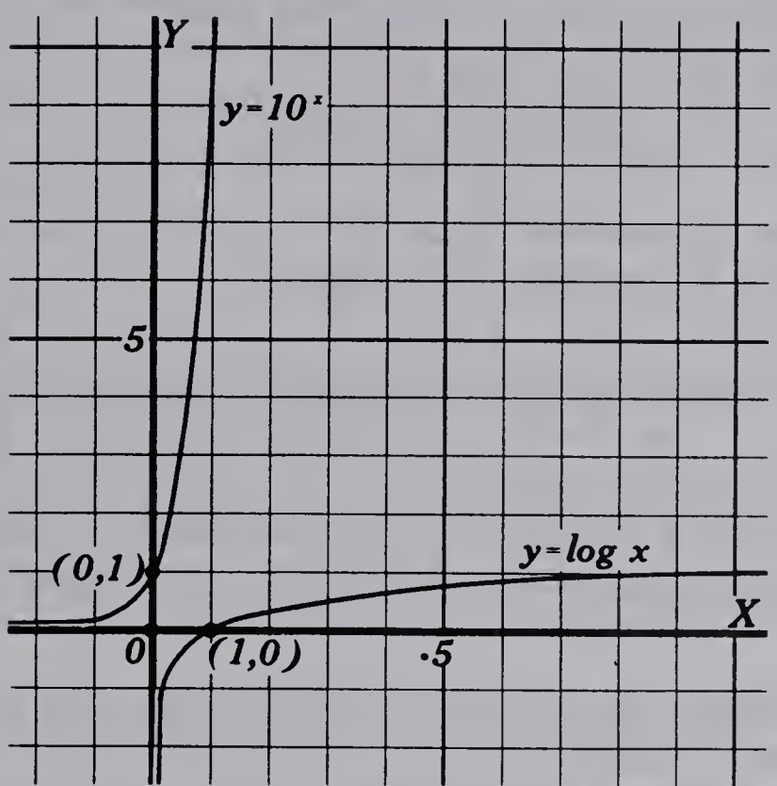


Fig. 8-2

Fig. 8-3 illustrates the part of the graph of the logarithmic function for the domain $0 < x \leqq 10,\ x \in R$.

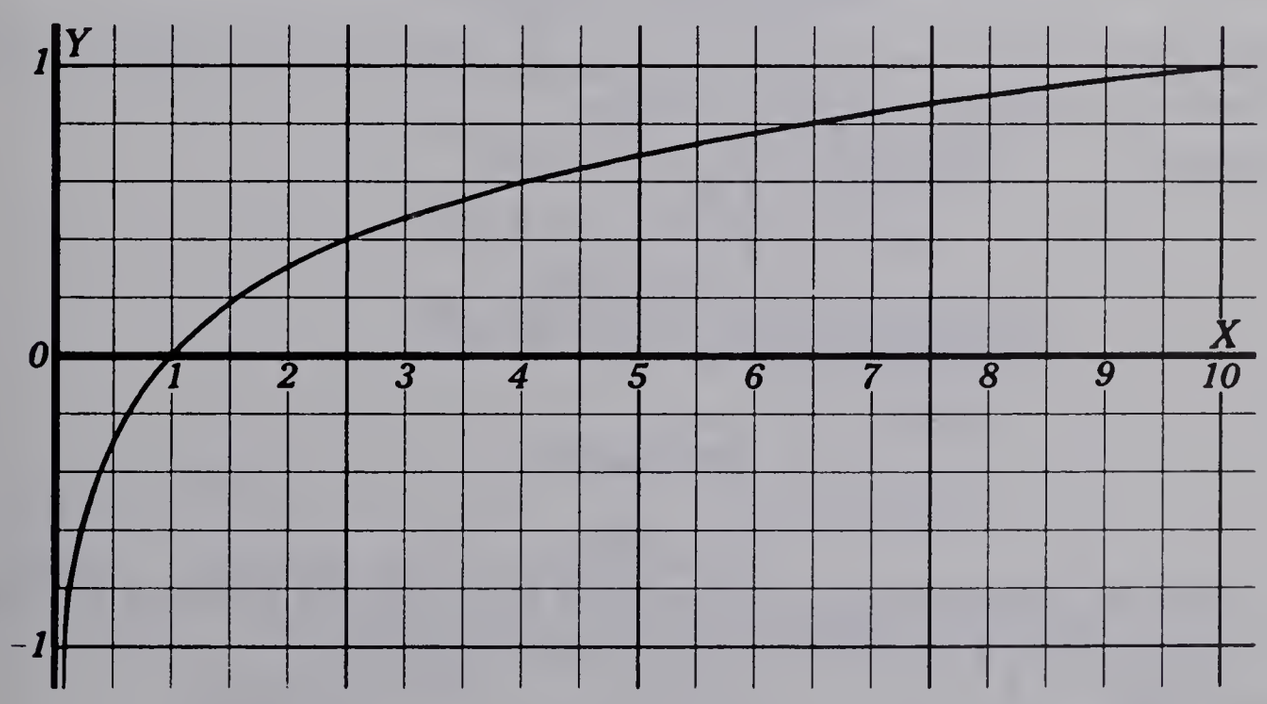


Fig. 8-3

From the graph it is seen that:

(i) $\log 1 = 0$; (ii) $\log 10 = 1$;

(iii) if $x \in R$ and $1 \leqq x \leqq 10$, then $0 \leqq \log x \leqq 1$.

Thus the logarithm of any real number between 1 and 10 is a real number between 0 and 1.

Example 1. Express log 469 in terms of log 4.69.

Solution.
$$\begin{aligned} \log 469 &= \log (4.69 \times 10^2) \\ &= \log 10^2 + \log 4.69 \\ &= 2 + \log 4.69. \end{aligned}$$

2 is called the *characteristic* of log 469. Log 4.69, a real number between 0 and 1, is called the *mantissa* of log 469.

In general,

if $x \in {}^+R$, and x is expressed in scientific or standard notation as

$$x = X \cdot 10^n, \quad (1 \leqq X < 10, n \in I),$$

that is, $\log x = n + \log X$,

then the integer n is defined to be the characteristic of $\log x$, and the number $\log X$ (which satisfies $0 \leqq \log X < 1$) is called the mantissa of $\log x$.

Example 2. Express log 0.00469 in terms of log 4.69 and indicate the characteristic and mantissa.

Solution.
$$\begin{aligned} \log 0.00469 &= \log (4.69 \times 10^{-3}) \\ &= \log 10^{-3} + \log 4.69 \\ &= -3 + \log 4.69. \end{aligned}$$

characteristic mantissa

Example 3. Find the characteristic of log 3769 by first expressing the number in scientific (standard) notation.

Solution.
$$\begin{aligned} \log 3769 &= \log (3.769 \times 10^3) \\ &= \log 10^3 + \log 3.769 \\ &= 3 + \log 3.769. \end{aligned}$$

The characteristic of log 3769 is 3.

Exercise 8-4

(B)

Find the characteristic of the logarithm of each of the following numbers by first expressing them in scientific (standard) notation.

1. 672 **2.** 89.3 **3.** 0.672

4. 0.00463 **5.** 0.156 **6.** 23.76

7. 5.84	**8.** 0.0000076	**9.** 111.1
10. 0.0583	**11.** 4963	**12.** 0.9006
13. 1.5	**14.** 9.9	**15.** 1.0073
16. 4518	**17.** 0.00516	**18.** 0.0589

***8·6* Finding the mantissa of a logarithm.** The mantissa of a logarithm may be obtained from a table of logarithms such as that on page 390 – 391 This table contains the mantissas, rounded off to four digits, of the logarithms of real numbers, rounded off to three digits. The following is an extract from this table.

	0	1	2	3	4	5	6	7	8	9
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235

The following examples illustrate how to use the table.

Example 1. Find: (i) log 5.16 (ii) log 51.6 (iii) log 0.0516.

Solution. (i) Log 5.16 is found at the intersection of the horizontal row marked 51 and the vertical column headed 6. Since the logarithm of a number between 1 and 10 is a number between 0 and 1

$$\log 5.16 \doteq 0.7126.$$

The decimal point is not included in the table.

$$\begin{aligned} \text{(ii) } \log 51.6 &= \log (5.16 \times 10^{1}) \\ &= 1 + \log 5.16 \\ &\doteq 1 + .7126. \end{aligned}$$

$$\begin{aligned} \text{(iii) } \log 0.0516 &= \log (5.16 \times 10^{-2}) \\ &= -2 + \log 5.16 \\ &\doteq -2 + .7126. \end{aligned}$$

Example 2. Find: (i) log 32.6 (ii) log 0.00928.

Solution. Using the table on page 391–392;

$$\begin{aligned} \text{(i) } \log 32.6 &= \log (3.26 \times 10^{1}) \\ &= 1 + \log 3.26 \\ &\doteq 1 + .5132. \end{aligned}$$

$$\begin{aligned} \text{(ii) } \log 0.00928 &= \log (9.28 \times 10^{-3}) \\ &= -3 + \log 9.28 \\ &\doteq -3 + .9675. \end{aligned}$$

Exercise 8-5

(B)

Find:

1. log 6.54	**2.** log 65.4	**3.** log 0.00654	**4.** log 83.6
5. log 7.16	**6.** log 0.234	**7.** log 5.42	**8.** log 0.105
9. log 0.000462	**10.** log 404	**11.** log 0.0111	**12.** log 3.14

***8·7* Finding a number given its logarithm.** From the logarithm table

$$\log 3.67 \doteq 0.5647.$$

The number 3.67 whose logarithm is 0.5647 is called the *antilogarithm* (antilog) of 0.5647. Thus,

$$\text{antilog } 0.5647 \doteq 3.67.$$

Similarly

$$\because \log 10 = 1, \qquad \therefore \text{antilog } 1 = 10.$$
$$\because \log 10^2 = 2, \qquad \therefore \text{antilog } 2 = 10^2.$$
$$\because \log 10^5 = 5, \qquad \therefore \text{antilog } 5 = 10^5.$$
$$\because \log 10^{-2} = -2, \qquad \therefore \text{antilog } (-2) = 10^{-2}.$$
$$\because \log 10^{-1} = -1, \qquad \therefore \text{antilog } (-1) = 10^{-1}.$$

When the logarithm of a positive real number is known, the number, rounded to three digits, may be obtained from the logarithm table. The method of obtaining the antilogarithm of a given logarithm is illustrated in the following examples.

Example 1. Find: (i) antilog 0.7917 (ii) antilog (2 + .7917).

Solution. (i) Locate 7917 in the body of the table. It is located at the intersection of the horizontal row marked 61 and the column headed 9.

$$\text{Thus,} \quad \text{antilog } 0.7917 \doteq 6.19.$$

$$\text{(ii)} \quad \text{antilog } (2 + .7917) \doteq 6.19 \times 10^2$$
$$\doteq 619.$$

If the mantissa of a given logarithm is a number between two of the number entries in the body of the logarithm table, select from the table the mantissa which is the closer approximation to the given mantissa; obtain the three digits of the antilogarithm for this mantissa from row number and column heading.

Example 2. Find antilog (− 3 + .9168).

Solution. Since the mantissa 0.9170 is the closest approximation in the table to the mantissa 0.9168, therefore take

$$\text{antilog } (-3 + .9168) \doteq 8.26 \times 10^{-3} \doteq 0.00826.$$

Exercise 8-6

(B)

Find the antilogarithm of each of the following:

1. 0.7412	**2.** $2 + .9133$	**3.** $-1 + .7356$
4. $-2 + .0682$	**5.** $1 + .4871$	**6.** 0.7853
7. $-1 + .9969$	**8.** $2 + .9581$	**9.** $-2 + .3263$
10. $-3 + .5159$	**11.** $3 + .3892$	**12.** $3 + .0682$
13. 0.9603	**14.** $1 + .8230$	**15.** $-1 + .7540$
16. $2 + .6669$	**17.** $-2 + .5620$	**18.** $-3 + .3890$

8·8 Multiplication and division using logarithms.

If $M, N \in {}^+R$, *then*

$$\log MN = \log M + \log N.$$

Example 1. Calculate 4.36×0.00417.

Solution.

$$\begin{aligned}
\text{Let } x &= 4.36 \times 0.00417 \\
&= 4.36 \times 4.17 \times 10^{-3}. \\
\log x &= \log 4.36 + \log (4.17 \times 10^{-3}). \\
\log 4.36 &\doteq \quad 0.6395. \\
\log (4.17 \times 10^{-3}) &\doteq \underline{-3 + .6201} \\
\log x &\doteq -3 + 1.2596 \\
&\doteq -2 + .2596. \\
x &\doteq \text{antilog } (-2 + .2596). \\
&\doteq 1.82 \times 10^{-2} \\
&\doteq 0.0182.
\end{aligned}$$

If $M, N \in {}^+R$, *then*

$$\log \frac{M}{N} = \log M - \log N.$$

Example 2. Calculate $0.652 \div 0.000216$.

Solution.

$$\begin{aligned}
\text{Let } x &= \frac{6.52 \times 10^{-1}}{2.16 \times 10^{-4}}. \\
\log x &= \log (6.52 \times 10^{-1}) - \log (2.16 \times 10^{-4}). \\
\log (6.52 \times 10^{-1}) &\doteq -1 + .8142 \\
\log (2.16 \times 10^{-4}) &\doteq \underline{-4 + .3345} \\
\log x &\doteq \quad 3 + .4797. \\
x &\doteq \text{antilog } (3 + .4797) \\
&\doteq 3.02 \times 10^{3} \\
&\doteq 3020.
\end{aligned}$$

Example 3. Calculate $\dfrac{0.362 \times 891 \times 0.0421}{32.6}$.

Solution. Let $x = \dfrac{(3.62 \times 10^{-1})(8.91 \times 10^{2})(4.21 \times 10^{-2})}{(3.26 \times 10^{1})}$.

$$\log x = \log (3.62 \times 10^{-1}) + \log (8.91 \times 10^{2}) + \log (4.21 \times 10^{-2}) - \log (3.26 \times 10^{1}).$$

$$\begin{aligned}
\log (3.62 \times 10^{-1}) &\doteq -1 + .5587 \\
\log (8.91 \times 10^{2}) &\doteq 2 + .9499 \\
\log (4.21 \times 10^{-2}) &\doteq -2 + .6243 \\
&\qquad\qquad 1 + .1329 \\
\log (3.26 \times 10^{1}) &\doteq 1 + .5132 \\
\log x &\doteq -1 + .6197. \\
x &\doteq \text{antilog } (-1 + .6197) \\
&\doteq 4.17 \times 10^{-1} \\
&\doteq 0.417.
\end{aligned}$$

Exercise 8-7

(B)

Calculate each of the following:

1. 61.2×4.21 **2.** $235 \times .136$ **3.** $0.063 \times 21.3 \times 7.56$

4. $10.1 \times 0.0434 \times 1.41$ **5.** $\dfrac{67.9}{3.16}$ **6.** $\dfrac{125}{0.0431}$

7. $\dfrac{0.00539}{0.214}$ **8.** $\dfrac{1}{48.9}$

9. $\dfrac{91 \times 23.4}{86.4 \times 18.2}$ **10.** $\dfrac{37.7 \times 0.0023 \times 0.0517}{49.6 \times 0.00514}$

11. $\dfrac{124}{123 \times 12.3 \times 1.23 \times 0.123 \times 0.0123}$

12. $\dfrac{1}{6.28 \times 328 \times 0.00102}$

13. A rectangular block is 32.7 cm. long and 24.4 cm. wide. If its volume is 8960 c.c., find its thickness to the nearest tenth centimetre.

14. The quantity of heat, H calories, produced in T sec. by an electric current of I amperes and V volts is given by the formula $H = K\,I\,V\,T$, where $K = 0.240$. Find H if $I = 1.56$, $V = 108$, and T $= 15.4$.

8·9 Powers and roots using logarithms.

If $M \in {}^{+}R$, *and* $p \in R$, *then*

$$\log M^p = p \log M.$$

Example 1. Calculate $(1.04)^{12}$.

Solution.

$$\begin{aligned} \text{Let } x &= (1.04)^{12}. \\ \log\ x &= 12 \log 1.04 \\ &\doteq 12(0.0170) \\ &\doteq 0.2040. \\ x &\doteq \text{antilog } 0.2040 \\ &\doteq 1.60. \end{aligned}$$

Example 2. Calculate $\sqrt{69.2}$.

Solution.

$$\begin{aligned} \text{Let } x &= \sqrt{69.2} \\ &= (6.92 \times 10^1)^{\frac{1}{2}}. \\ \log\ x &= \tfrac{1}{2} \log (6.92 \times 10^1) \\ &\doteq \tfrac{1}{2}(1 + .8401) \\ &\doteq \tfrac{1}{2}(1.8401) \\ &\doteq 0.920. \\ x &\doteq \text{antilog } 0.920 \\ &\doteq 8.32. \end{aligned}$$

Example 3. Calculate $\sqrt{0.216}$.

Solution.

$$\begin{aligned} \text{Let } x &= (2.16 \times 10^{-1})^{\frac{1}{2}}. \\ \log x &= \tfrac{1}{2} \log (2.16 \times 10^{-1}) \\ &\doteq \tfrac{1}{2}\,(-1 + .3345). \end{aligned}$$

In order to divide $-1 + .3345$ by 2, express $-1 + .3345$ as $-2 + 1.3345$, since the characteristic of a common logarithm must be an integer.

$$\begin{aligned} \log\ x &\doteq \tfrac{1}{2}\,(-2 + 1.3345) \\ &\doteq -1 + .6673. \\ x &\doteq \text{antilog } (-1 + .6673) \\ &\doteq 4.65 \times 10^{-1} \\ &\doteq 0.465. \end{aligned}$$

Example 4. Calculate $\sqrt[3]{0.437}$.

Solution.

$$\begin{aligned} \text{Let } x &= (4.37 \times 10^{-1})^{\frac{1}{3}}. \\ \log\ x &= \tfrac{1}{3} \log (4.37 \times 10^{-1}) \\ &\doteq \tfrac{1}{3}\,(-1 + .6405). \end{aligned}$$

$$\log x \doteq \tfrac{1}{3}(-3 + 2.6405)$$
$$\doteq -1 + .8802.$$
$$x \doteq \text{antilog}(-1 + .8802)$$
$$\doteq 7.59 \times 10^{-1}$$
$$\doteq 0.759.$$

Example 5. Calculate $\dfrac{(72.4)^2(0.326)}{\sqrt[3]{0.0479}}$.

Solution. Let $x = \dfrac{(7.24 \times 10^1)^2(3.26 \times 10^{-1})}{(4.79 \times 10^{-2})^{\frac{1}{3}}}$.

$$\log x = 2\log(7.24 \times 10^1) + \log(3.26 \times 10^{-1}) - \tfrac{1}{3}\log(4.79 \times 10^{-2}).$$

$$2\log(7.24 \times 10^1) \doteq 2(1 + .8597) = 3 + .7194$$
$$\log(3.26 \times 10^{-1}) \doteq -1 + .5132$$
$$3 + .2326$$

$$\tfrac{1}{3}\log(4.79 \times 10^{-2}) \doteq \tfrac{1}{3}(-2 + .6803)$$
$$\doteq \tfrac{1}{3}(-3 + 1.6803) = -1 + .5601$$
$$(3 + .6725)$$

$$x \doteq \text{antilog}(3 + .6725)$$
$$\doteq 4.70 \times 10^3$$
$$\doteq 4700.$$

Exercise 8-8

(B)

Calculate each of the following:

1. $(3.14)^2$ **2.** $\sqrt[3]{8.52}$ **3.** $\sqrt{0.376}$

4. $\sqrt[3]{(0.575)^2}$ **5.** $126^{\frac{3}{4}}$ **6.** $\sqrt{\dfrac{621}{182}}$

7. $\sqrt[4]{0.000414^3}$ **8.** $\sqrt[5]{0.00213}$ **9.** $\dfrac{323 \times (6.28)^5}{(4.67)^5}$

10. $\sqrt{\dfrac{0.791 \times 6.28}{0.0174}}$ **11.** $\dfrac{(4.69)^{\frac{3}{2}} \times (1.75)^2}{\sqrt{375}}$ **12.** $\sqrt[3]{\dfrac{9.37}{0.0731}}$

13. In an electric circuit the relation among the power, P watts, the voltage, V volts, and the resistance, R ohms, is expressed by

$$P = \frac{V^2}{R}.$$

Calculate P if $V = 110$, and $R = 62700$.

14. The relation between the area of a circular region, A square units, and the radius, r linear units, is expressed by

$$A = \pi r^2.$$

Calculate the area if $r = 21.2$, $\pi = 3.14$.

15. The relation between the time of oscillation of a pendulum, t seconds, and the length, l cm., is given by

$$t = 2\pi \sqrt{\frac{l}{g}}.$$

π and g are constants. Calculate t if $l = 39.2$, $\pi = 3.14$, and $g = 981$.

16. The relation between the area of a triangular region, A sq. units, and the lengths of the sides, a units, b units, and c units, is given by

$$A = \sqrt{s(s-a)(s-b)(s-c)}, \text{ where } s = \frac{a+b+c}{2}.$$

Calculate A if $s = 26.0$, $a = 15.3$, $b = 14.8$, $c = 21.9$.

17. The relation between the volume of a sphere, V cu. in., and the radius, r in., is given by $V = \frac{4}{3}\pi r^3$. Calculate V if $r = 12.4$ and $\pi = 3.14$.

8·10 **Linear interpolation (supplementary).** If we wish to find the logarithm of a number of four or more digits from the four place logarithm table, we may use the *method of interpolation*. A graphical study of the interpolation process for finding log 5.163 follows. *Fig. 8-4* shows the graph of $y = \log x$, $1 \leqq x \leqq 10$, $x \in R$. AB is the interval of the graph for $5.160 \leqq x \leqq 5.170$, $x \in R$.

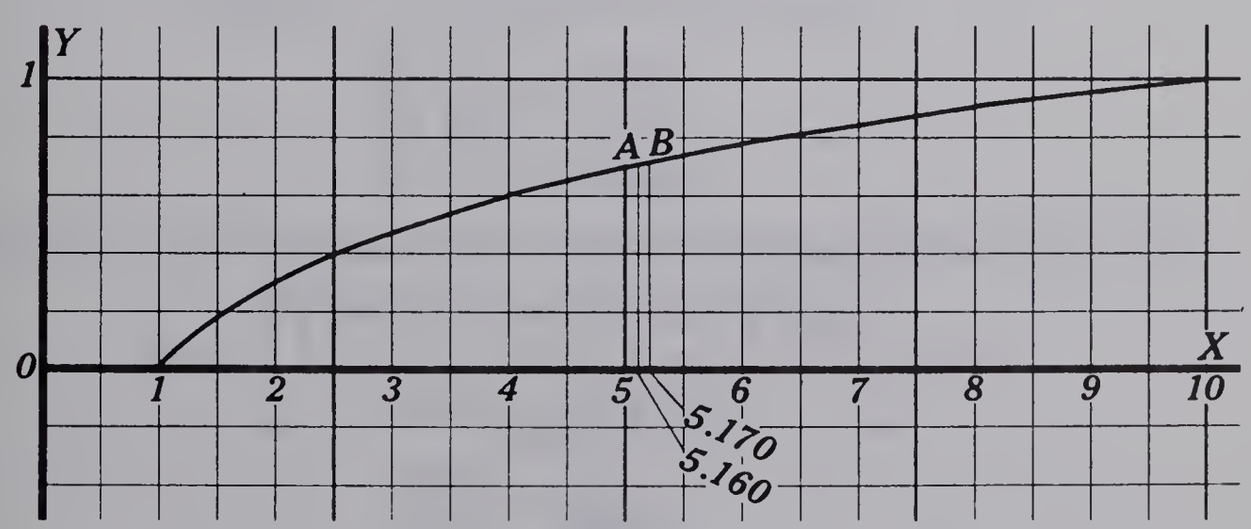


Fig. 8-4

It is evident from the graph that such a small interval as AB is very nearly a straight line segment. If a segment of the curve is considered to be linear, it may be represented by the line segment AB as in *Fig. 8-5*.

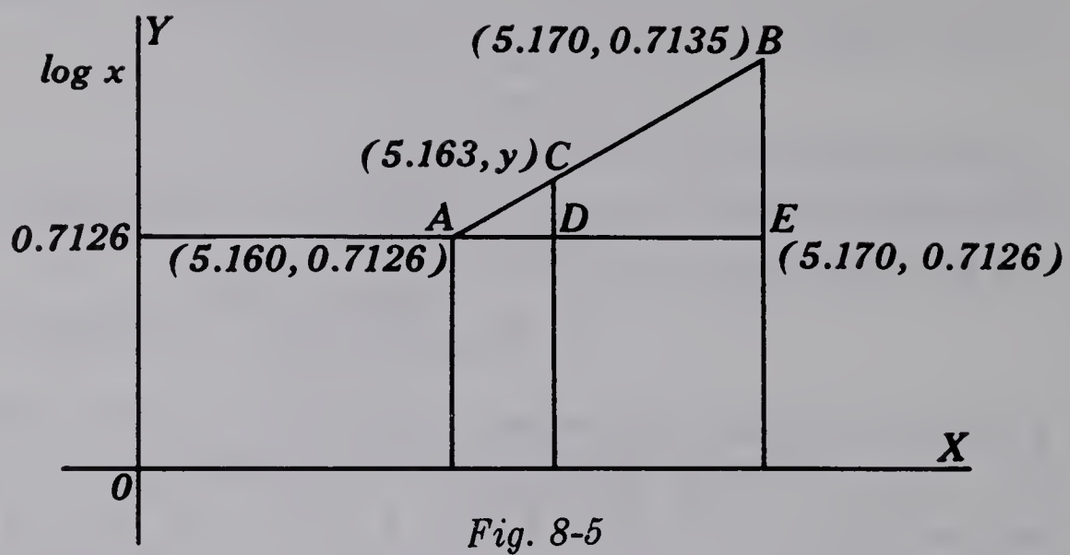


Fig. 8-5

From *Fig. 8-5*

$$\because \triangle AEB \sim \triangle ADC,$$

$$\therefore \quad \frac{DC}{AD} = \frac{EB}{AE}.$$

From the tables,

$$\log 5.16 \doteq 0.7126$$

$$\text{and } \log 5.17 \doteq 0.7135.$$

$$\therefore \frac{5.170 - 5.160}{5.163 - 5.160} \doteq \frac{0.7135 - 0.7126}{DC}.$$

$$\therefore DC \doteq \frac{0.0009 \times 0.003}{0.010}.$$

$$\therefore DC \doteq 0.00027.$$

$$\therefore y \doteq 0.7126 + 0.00027$$

$$\doteq 0.71287$$

$$\doteq 0.7129,$$

$$\therefore \log 5.163 \doteq 0.7129.$$

The interpolation process to find log 5.163 may be arranged as follows:

$$\begin{array}{rcl} 0.010 & 0.003 & \log 5.160 \doteq 0.7126 \\ & & \log 5.163 \doteq \quad d \\ & & \log 5.170 \doteq 0.7135 \end{array} \quad 0.0009$$

$$\therefore \frac{d}{0.0009} \doteq \frac{0.003}{0.010}$$

$$\therefore d \doteq \frac{0.0027}{10}$$

$$\doteq 0.003.$$

$$\therefore \log 5.163 \doteq 0.7129.$$

Exercise 8-9

(B)

Find the logarithm of each of the following by using linear interpolation:

1. 3.142	**2.** 529.8	**3.** 0.1937	**4.** 0.002465
5. 47.06	**6.** 0.0003258	**7.** 1.018	**8.** 0.09076	**9.** 1583

8•11 Finding an antilogarithm by interpolation (supplementary).

Example 1. Find antilog 0.3682.

Solution. 3682 is a number between the numbers 3674 and 3692 found in the logarithm table in the row marked 23. Thus

$$0.0018\left[0.0008\left[\begin{array}{l}\text{antilog } 0.3674 \doteq 2.33\\ \text{antilog } 0.3682 \doteq \text{——}\\ \text{antilog } 0.3692 \doteq 2.34\end{array}\right]x\right]0.01$$

$$\therefore \quad \frac{x}{0.01} \doteq \frac{0.0008}{0.0018}.$$

$$\therefore \quad x \doteq \frac{0.01 \times 8}{18} \doteq 0.004.$$

$$\therefore \quad \text{antilog } 0.3682 \doteq 2.334.$$

Example 2. Find antilog $(-2 + .9857)$.

Solution.

$$0.0005\left[0.0003\left[\begin{array}{l}\text{antilog } 0.9854 \doteq 9.67\\ \text{antilog } 0.9857 \doteq \text{——}\\ \text{antilog } 0.9859 \doteq 9.68\end{array}\right]x\right]0.01$$

$$\therefore \quad \frac{x}{0.01} \doteq \frac{0.0003}{0.0005}.$$

$$\therefore \quad x \doteq 0.006.$$

$$\therefore \quad \text{antilog } (-2 + .9857) \doteq 9.676 \times 10^{-2}$$

$$\therefore \quad \text{antilog } (-2 + .9857) \doteq 0.09676.$$

Exercise 8-10

(B)

Find the antilogarithm of each of the following by interpolation:

1. 0.8218	**2.** $1 + .7743$	**3.** $-2 + .4275$
4. $-1 + .1800$	**5.** $2 + .0880$	**6.** $-3 + .3194$
7. $3 + .5591$	**8.** 0.2073	**9.** $1 + .0270$

***8•12* Logarithmic and exponential equations (supplementary).** A logarithmic equation is an equation which contains the logarithm of the variable.

Example 1. Solve $4 \log x + 3.7960 = 4.6990 + \log x$.

Solution.

$$\begin{aligned} & 4 \log x + 3.7960 = 4.6990 + \log x \\ \leftrightarrow \quad & 4 \log x - \log x = 4.6990 - 3.7960 \\ \leftrightarrow \quad & 3 \log x = 0.9030 \\ \leftrightarrow \quad & \log x = 0.3010 \\ \leftrightarrow \quad & x = \text{antilog } .3010. \\ \therefore \quad & x \doteq 2.00. \end{aligned}$$

Example 2. Solve $500 = 276 \log \dfrac{d}{0.05}$.

Solution.

$$500 = 276 \log \frac{d}{0.05}.$$

$$\therefore \ 1.8116 \doteq \log \frac{d}{0.05} \quad \text{(division by 276)}$$

$$\therefore \ \text{antilog } 1.8116 \doteq \frac{d}{0.05}.$$

$$\therefore \ 6.48 \times 10^1 \doteq \frac{d}{0.05}.$$

$$\therefore \ 64.8 \doteq \frac{d}{0.05}.$$

$$\therefore \ d \doteq 3.24.$$

An exponential equation is an equation in which the variable appears as an exponent.

Example 3. Solve $4^x = 256$.

Solution.

$$\begin{aligned} & 4^x = 256 \\ \leftrightarrow \quad & x \log 4 = \log 256 \\ \leftrightarrow \quad & x = \frac{\log 256}{\log 4}. \\ \therefore \quad & x \doteq \frac{2 + .4082}{0.6021}. \\ \therefore \quad & x \doteq 4. \end{aligned}$$

Note: Since $4^4 = 256$, therefore the exact solution is $x = 4$.

Example 4. Solve $5^{x-3} = 52$

Solution.

$$5^{x-3} = 52$$

$$\leftrightarrow \quad (x-3)\log 5 = \log 52$$

$$\leftrightarrow \quad x - 3 = \frac{\log 52}{\log 5}$$

$$\leftrightarrow \quad x = \frac{\log 52}{\log 5} + 3.$$

$$\therefore \quad x \doteq \frac{1 + .7160}{0.6990} + 3.$$

$$\therefore \quad x \doteq 5.4.$$

Exercise 8-11

(B)

Solve:

1. $\log y + 4 \log y = 10$

2. $\log 2x + 2 \log x = 10$

3. $\log 10p + \log p = 1$

4. $\log x^2 - \log 2x = 2$

5. $200 = 276 \log \dfrac{d}{0.03}$

6. $5^x = 200$

7. $x^5 = 100$

8. $3^{x-2} = 10$

9. $4^{2x} = 63^2$

10. $\log x^3 + 4 \log x = 8$

11. $\log 2x^2 - \log 3x = \log 12$

12. $5^{3x+2} = 95.2$

***8·13* Interest.** An investor thinks of interest as income from invested money. A debtor has a different viewpoint, he thinks of interest as money paid for the use of money which he has borrowed. The following terms are used in investment transactions.

(i) *Principal* ($\$P$): The sum of money (capital) originally invested.

(ii) *Interest period:* The stated unit of time for calculation of interest; the interest period is considered to be one year unless otherwise specified.

(iii) *Term:* The duration (time) of the transaction.

(iv) *Amount* $\$A$: The sum of money equal to the principal together with the interest due at any time after the investment of the principal.

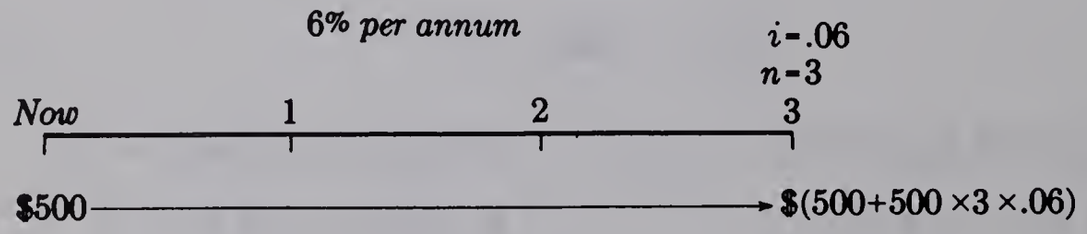
(v) *Interest rate* per period, i: The ratio of the interest earned in one interest period to the principal. For example, if the principal is \$1,000 and the interest is \$60 for one interest period, the interest rate i is given by

$$i = \frac{60}{1000}$$

$$= \frac{6}{100} \text{ (or 6\% or .06).}$$

***8•14* Simple interest.** If it is agreed that the interest on \$$P$ for a term of n interest periods is n times the interest on \$$P$ for 1 interest period, then the interest is *simple interest.*

Example 1. Calculate to the nearest dollar the amount of \$500 at simple interest for a term of 3 years at 6% per annum.



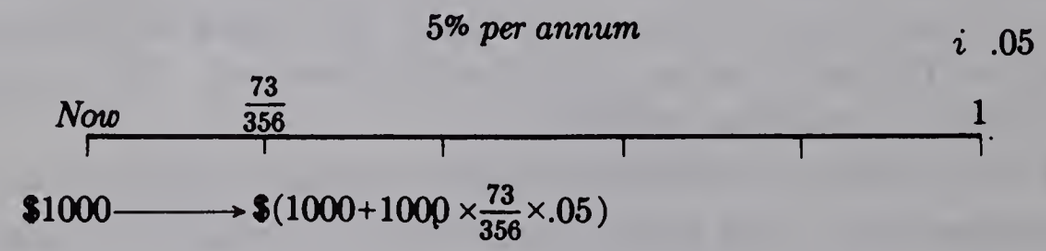
Solution. It is usually advisable to display the problem on a line diagram.

From the diagram,
$$A = 500 + 500 \times .06 \times 3$$
$$= 500 + 90$$
$$= 590.$$

$\therefore$ the amount is \$590.

Example 2. Calculate to the nearest cent the amount of \$1,000 at simple interest at 5% per annum for a term of 73 days.

Solution.



From the diagram,
$$A = 1000.00 + 1000.00 \times .05 \times \frac{73}{365}$$
$$= 1000.00 + 10.00$$
$$= 1010.00.$$

$\therefore$ the amount is \$1010.00.

In general, if

(i) the principal is $\$P$;

(ii) the interest earned is $\$I$;

(iii) the amount is $\$A$;

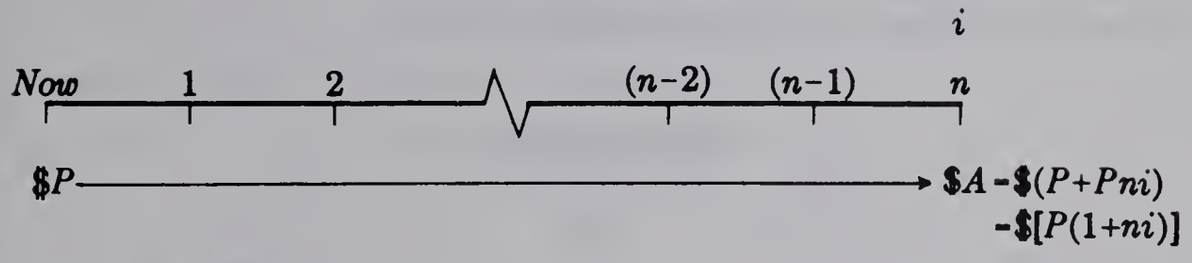
(iv) the interest rate per period is i;

(v) the number of interest periods is n;

then $$I = Pni,$$

and $$A = P + Pni$$
$$= P(1 + ni).$$

This is illustrated on the following line diagram.



8·15 Present value of an amount. A given principal sum of $\$P$ at simple interest i, for n periods will amount to $\$A$ where

$$A = P(1 + ni). \qquad (1)$$

We may think of $\$A$ as the *future value* of a sum of money whose *present value* is $\$P$.

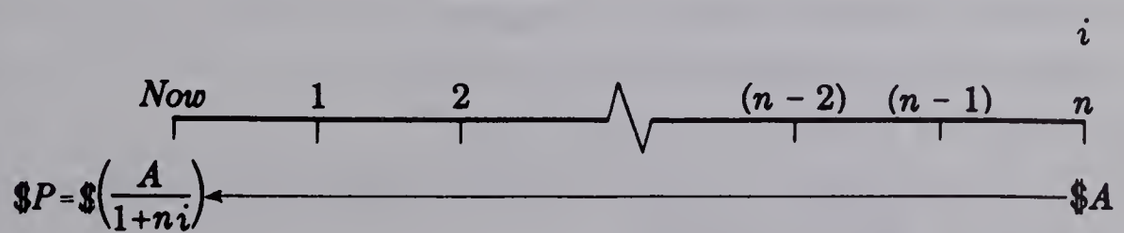
We may answer the question: "What sum of money ($\$P$) must be placed at interest (i) today to obtain a specified amount ($\$A$) n periods from now," by solving equation (1) for P.

$$\because\ A = P(1 + ni),$$

$$\therefore\ P = \frac{A}{1 + ni},$$

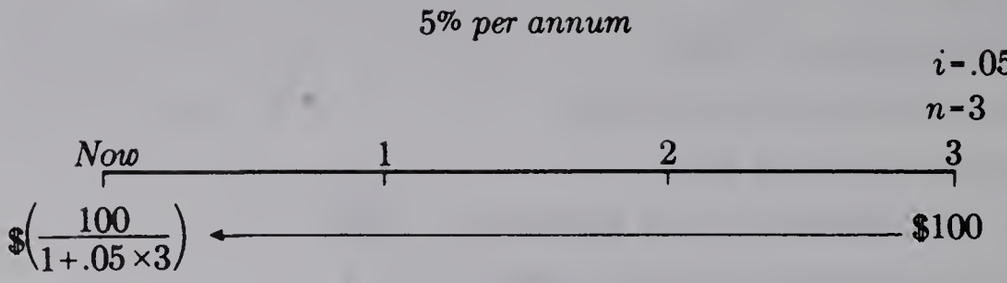
which may be written, $P = A(1 + ni)^{-1}$.

This is illustrated in the following line diagram:



Example 1. Find to the nearest dollar the present value of \$100 at simple interest of 5% per annum due in 3 years.

Solution.



$$\left(\frac{100}{1 + .05 \times 3}\right) = \left(\frac{100}{1.15}\right)$$

$$= 86.96$$

The present value to the nearest dollar is $87.

Exercise 8-12

(B)

1. If $1,000 earns $36.60 interest in one year, find the rate per year.
2. If $100 earns $5.50 interest in one year, find the rate per year.
3. If $250 earns $13.75 interest in one year, find the rate per year.

Find the amount of each of the following to the nearest cent; begin each solution with a line diagram.

4. $1250 at 5% per annum for 2 years.
5. $5000 at 6% per annum for 146 days.
6. $2850 at 6% per annum for 170 days.
7. $75 at 1% per month for 3 months.
8. $700 at 8% per annum for 4 months.

Find the present value of each of the following to the nearest cent; begin each solution with a line diagram.

9. $500 due in 2 years at 5% per annum.
10. $1200 due in 90 days at 7% per annum.
11. $1000 due in 6 months at 8% per annum.
12. $75 due in 3 months at 2% per month.
13. Find to the nearest cent the principal that will amount to $350 in $1\frac{1}{4}$ years at $3\frac{1}{4}$% per annum.
14. Find to the nearest cent the principal that will amount to $1000 in $1\frac{1}{2}$ years at 6% per annum.

15. If a man borrowed \$225 and repayed \$255 at the end of 6 months, what rate of interest per annum did he pay?

16. If the interest on a given principal for 9 months at 4% per annum is \$18.75, find the principal and the amount.

17. Find the time required for \$635.42 to amount to \$685.46 at $5\frac{1}{4}\%$ per annum.

***8·16* Compound interest.** Find the amount of an investment of \$100 invested for 5 years at 6% per annum, if the interest due each year is added to the principal and thereafter earns interest.

Copy and complete the following table to show the yearly growth of the amount. Show the corresponding amount at 6% per annum simple interest. Compare your table with the one on page 430.

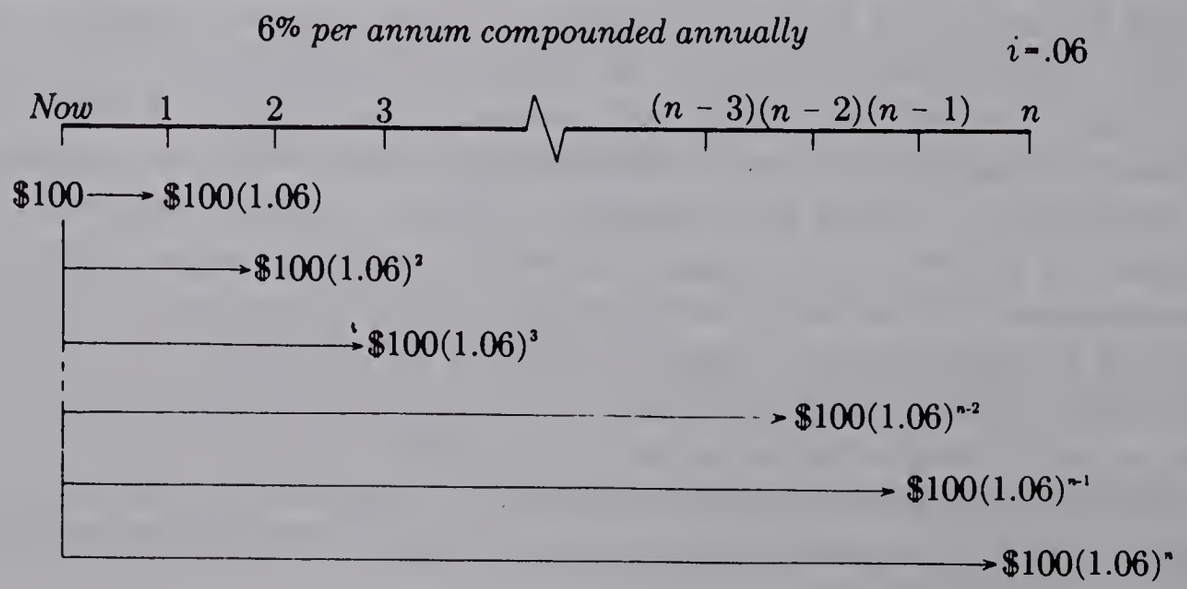
(1) INTEREST PERIOD	(2) PRINCIPAL AT THE BEGINNING OF INTEREST PERIOD IN DOLLARS	(3) INTEREST IN DOLLARS	(4) AMOUNT AT END OF INTEREST PERIOD IN DOLLARS	(5) AMOUNT AT 6% SIMPLE INTEREST IN DOLLARS
1	100	$100 \times .06 = 6$	106	106
2	106	$106 \times .06 = 6.36$	112.36	112
3	112.36			
4				
5				

The amount at the end of 5 years (in column 4) in the above example is called the *compound amount* of \$100 at 6% per annum compounded annually. The difference between the compound amount and the original principal is called the *compound interest.* The process of computing interest and adding it to the existing principal is referred to as *compounding* or *converting the interest into principal.* The time between successive conversions of interest to principal is called the *conversion period.* The compound amount at the end of each period is greater than the corresponding amount at simple interest, except for the first interest period. Investments are usually made in such a way that the interest at the end of each period is reinvested to obtain the advantage of compound interest.

The following table shows how to obtain an expression for the amount of \$100 at 6% compounded annually for any number of interest periods of 1 year each from 1 to n.

INTEREST PERIOD	PRINCIPAL AT THE BEGINNING OF INTEREST PERIOD IN DOLLARS	INTEREST IN DOLLARS	COMPOUND AMOUNT AT END OF INTEREST PERIOD IN DOLLARS
1	100	$100(.06)$	$100 + 100(.06) = 100(1 + .06) = 100(1.06)$
2	$100(1.06)$	$100(1.06)(.06)$	$100(1.06) + 100(1.06)(.06) = 100(1.06)(1+.06) = 100(1.06)^2$
3	$100(1.06)^2$	$100(1.06)^2(.06)$	$100(1.06)^2 + 100(1.06^2)(.06) = 100\ (1.06)^3$
•	•	•	•
•	•	•	•
•	•	•	•
5	$100(1.06)^4$	$100(1.06)^4(.06)$	$100(1.06)^5$
•	•	•	•
•	•	•	•
•	•	•	•
n	$100(1.06)^{n-1}$	$100(1.06)^{n-1}(.06)$	$100(1.06)^n$

The growth, period by period, is illustrated in the following line diagram.



The amount $100(1.06)^5$ may be found or calculated using logarithms or, more conveniently, by using the Amount Table on page 396- 397. This table gives the amount of \$1 at the indicated rate per period for 1, 2, 3, . . . 40 periods. The value $(1.06)^5$ is found at the intersection of the row marked 5 and column headed 6%.

Thus $(1.06)^5 \doteq 1.33823$.

$\therefore\ 100(1.06)^5 \doteq 133.823$.

The amount of \$100 invested at 6% per annum compounded annually for 5 years is \$133.82 to the nearest cent.

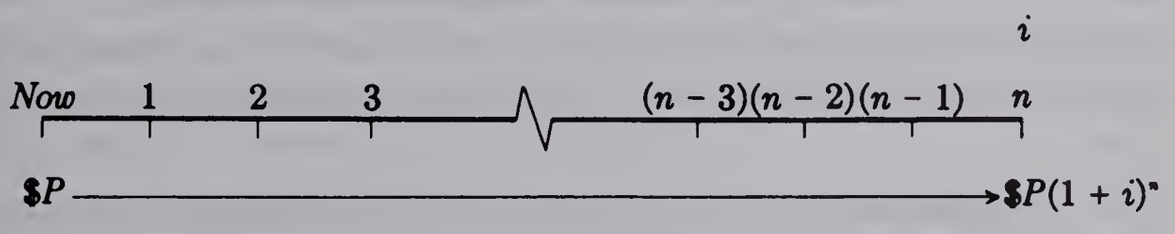
We say the principal of \$100 *accumulates* at interest to the amount \$133.82. The *accumulation factor* is $(1.06)^5$.

In general, if (i) the principal is $\$P$;
(ii) the compound amount is $\$A$;
(iii) the interest rate per period is i;
(iv) the number of interest periods is n;
and the conversion period is the same as the interest period, then

$$A = P(1 + i)^n.$$

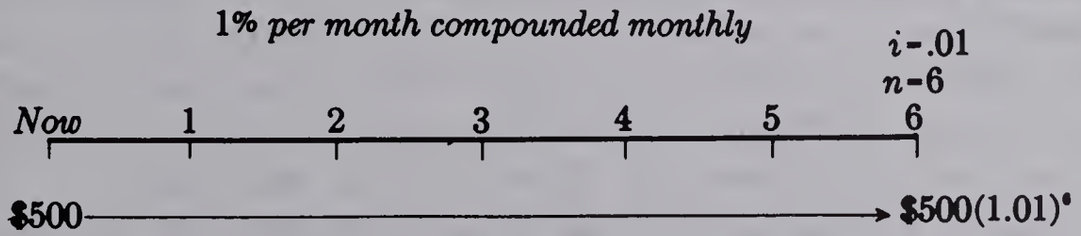
The accumulation factor is $(1 + i)^n$.

This is illustrated in the following line diagram.



Example 1. Find, to the nearest cent, the compound amount of \$500 invested at 1% per month compounded monthly for 6 months.

Solution.



From the diagram, $A = 500(1.01)^6$.

From the table, $A \doteq 500(1.06152)$,

$A \doteq 530.760$.

$\therefore$ the compound amount is \$530.76 to the nearest cent.

Exercise 8-13

(B)

Find the compound amount of each of the following; begin each solution with a line diagram.

1. \$350 in 4 years at 5% per annum compounded yearly. (Answer to the nearest cent.)
2. \$2000 in 5 years at 4% per half year compounded semi-annually. (Answer to the nearest 10 cents).
3. \$1500 in 9 months at 2% per month compounded monthly. (Answer to the nearest cent.)
4. \$800 in 18 months at 3% per quarter year compounded quarterly. (Answer to the nearest cent.)
5. D dollars in y years at 6% per annum compounded yearly.
6. K dollars in x years at rate r per annum compounded yearly.
7. Find:

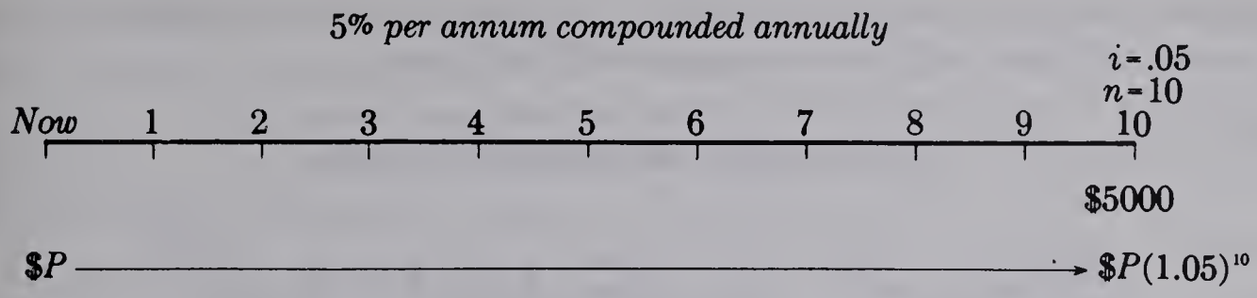
   (i) the compound amount to the nearest 10 cents at the end of 10 years, if a principal of \$1000 is invested over the first half of the term at 5% per annum compounded annually and over the second half of the term at 3% per annum compounded annually.

   (ii) the compound amount to the nearest 10 cents if the principal had been invested at 4% per annum, compounded annually over the entire term.
8. If the population of a city is 100,000 and is increasing at the rate of 10% per year, find the population 10 years later to the nearest ten thousand.

***8•17* Present value of a compound amount.** On your eighth birthday your grandfather, who wished to ensure that \$5000 would be available on your eighteenth birthday for your college education, invested an amount of money at 5% per annum compounded annually to amount to \$5000 when you are 18 years old. How much did he invest?

Our experience suggests that \$5000 due ten years hence is equivalent *now* to a sum which is somewhat less than \$5000 if money is worth 5% compounded annually. This sum is called the *present value* of \$5000 due ten years hence at 5% per annum compounded annually.

The problem is illustrated in the following line diagram on page 217. If we represent the sum to be invested by \$$P$, then



From the diagram, $P(1.05)^{10} = 5000.$

$$\therefore\ P = \frac{5000}{(1.05)^{10}}$$

$$\text{or } P = 5000(1.05)^{-10}.$$

$\frac{1}{(1.05)^{10}}$ or $(1.05)^{-10}$ may be found by using logarithms, or more conveniently, by using the *present value table* on page 398 - 399 This table gives the present value of \$1 due in 1, 2, 3, . . . 40 periods at the indicated rates of interest (i) per period. $(1.05)^{-10}$ is found at the intersection of the row marked 10 and the column headed 5%.

Thus $(1.05)^{-10} \doteq 0.61391$

and $5000(1.05)^{-10} \doteq 3069.55.$

$\therefore$ the sum to be invested is \$3070 to the nearest dollar.

The process of finding the present value of an amount is the undoing or inverse process to that of finding the compound amount. It is sometimes referred to as *discounting a sum.* In the above problem the sum discounted is \$5000 and the *discount factor* is $(1.05)^{-10}$.

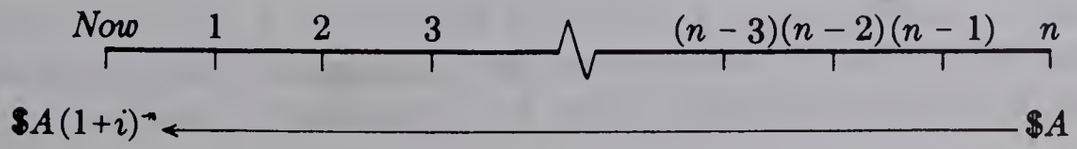
In general, if (i) the amount to be discounted is \$$A$;
(ii) the present value of the amount is \$$P$;
(iii) the interest rate per period is i;
(iv) the number of interest periods is n;
and the conversion period is the same as the interest period, then

$$P = \frac{A}{(1+i)^n},$$

$$\text{or } P = A(1+i)^{-n}.$$

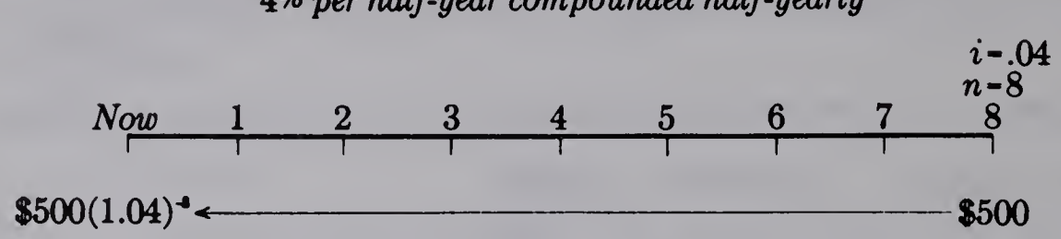
The discount factor is $(1 + i)^{-n}$.

This is usually illustrated on a line diagram as follows:



Example. Find, to the nearest dollar, the present value of \$500 due in 4 years at 4% per half year compounded half-yearly.

Solution.



From the diagram, $P = 500\ (1.04)^{-8}$.

From the tables, $P \doteq 500 \times 0.73069$

$\doteq 365.35$.

$\therefore$ the present value is \$365 to the nearest dollar.

Exercise 8-14

(B)

Find the present value of each of the following; begin your solution with a line diagram.

1. \$500 due 6 years hence at 4% per annum compounded yearly. (Answer to the nearest cent.)
2. \$1000 due 4 years hence at 3% per half year compounded half-yearly. (Answer to the nearest 10 cents.)
3. \$1200 due 8 months hence at 2% per month compounded monthly. (Answer to the nearest 10 cents.)
4. \$600 due 3 years hence at 4% per quarter-year compounded quarterly. (Answer to the nearest cent.)
5. \$1500 due in 90 days at 2% per month compounded monthly. (Answer to the nearest 10 cents.)
6. What sum of money, to the nearest ten dollars, should be invested now at $4\frac{1}{2}$% per annum compounded yearly to enable a person to purchase five years from now an automobile which will cost \$3500?
7. A house is offered for sale on the condition that \$10,000 be paid now, and \$2000 be paid at the end of each year for 5 years. If money earns 6% per annum compounded yearly, find the cash price of the house to the nearest ten dollars.
8. A man owes \$500 due in 2 years and \$800 due in 4 years. If money is worth 4% per half-year compounded half-yearly, what cash settlement, to the nearest dollar, could he make now?

8•18 Rate per conversion period. In the calculation of compound interest, the significant interest rate is the rate per conversion period. However, a rate is usually quoted as a rate per year with the frequency of conversion also given. For example, if compound interest is to be computed on the basis of 3% per half-year compounded semi-annually, it will be quoted as 6% per annum compounded semi-annually. Thus, in solving a problem it may be necessary to change the given rate per annum into a rate per conversion period, and to change the time in years to a number of conversion periods.

Complete the following table; compare your solutions with those on page 430.

INTEREST RATE PER ANNUM (%)	HOW COMPOUNDED	RATE PER CONVERSION PERIOD (i)	NUMBER OF YEARS	NUMBER OF CONVERSION PERIODS (n)	COMPOUND AMOUNT OF \$1 AFTER n PERIODS
4	yearly	.04	2	2	$(1.04)^2$
4	$\frac{1}{2}$ yearly	.02	2	4	$(1.02)^4$
3	$\frac{1}{2}$ yearly		$5\frac{1}{2}$		
4	quarterly		3		
5	$\frac{1}{2}$ yearly		4		
4	quarterly		$6\frac{1}{4}$		
8	quarterly		$4\frac{1}{2}$		
6	monthly		2		
9	$\frac{1}{2}$ yearly		5		

8•19 Equivalent annual rate. A Loan Company quotes an interest rate of 1% per month compounded monthly on small loans. What is the *equivalent annual rate compounded annually*?

A loan of \$1 for 1 year (12 months) at 1% per month compounded monthly requires a repayment of

$$\$(1 + .01)^{12} \quad \text{or} \quad \$(1.01)^{12}.$$

A loan of \$1 for 1 year at a rate i per annum compounded annually requires a repayment of

$$\$(1 + i).$$

If the two rates are *equivalent*, then

$$1 + i = (1.01)^{12}.$$

$$\therefore \quad 1 + i \doteq 1.12683. \qquad \text{(from the table)}$$

$$\therefore \quad i \doteq 0.127.$$

Thus the equivalent annual rate compounded annually is approximately 12.7%; somewhat more than $(1 \times 12)\%$ or 12%.

Exercise 8-15

(B)

Find the compound amount of each of the following to the nearest ten cents; begin each solution with a line diagram.

1. \$4000 for 3 years at 8% per annum compounded semi-annually.
2. \$2500 for 20 months at 12% per annum compounded monthly.
3. \$1500 for $6\frac{1}{2}$ years at 6% per annum compounded semi-annually.
4. \$2000 for 5 years at 7% per annum compounded annually.
5. \$3000 for 3 years at 8% per annum compounded quarterly.

Find the present value of each of the following to the nearest ten cents; begin each solution with a line diagram.

6. \$3800 due in 4 years if money is worth 8% per annum compounded semi-annually.
7. \$1250 due in 15 months if money is worth 12% per annum compounded monthly.
8. \$1800 due in 2 years if money is worth 8% per annum compounded quarterly.
9. \$1300 due in $2\frac{1}{2}$ years if money is worth 5% per annum compounded quarterly.
10. At the birth of a child, what sum should his parents invest for him to provide \$1000 at age 16, if money earns 4% per annum compounded semi-annually?
11. If a bank pays interest at the rate of 3% per annum compounded semi-annually, how much must a man deposit now to have \$4000 in an account 15 years hence?
12. Find, for each of the following, the equivalent annual rate compounded annually:
    (i) 2% per month compounded monthly;
    (ii) 3% per quarter compounded quarterly;
    (iii) 6% per half-year compounded half-yearly;
    (iv) 2.5% per two months compounded bi-monthly;
    (v) 3% per half-year compounded monthly.

***8·20* Graphical comparison of simple interest and compound interest.** The amount, \$$S$, of \$1 invested at 6% per period for n periods at simple interest is given by

$$S = 1 + .06n$$

which is the defining equation of a linear function.

The compound amount, $\$C$, of \$1 invested at 6% per period compounded each period for n periods is given by

$$C = (1.06)^n$$

which is the defining equation of an exponential function.

The graphs of the two functions for the domain $0 \leqq n \leqq 20$ are illustrated in *Fig. 8-6*. The graphs illustrate the comparative growth of the amounts over the 20 periods. The exponential growth of the amount at compound interest exceeds the linear (constant) growth of the amount at simple interest except for time less than one period. The insert in *Fig. 8-6* illustrates that, for time less than the first interest period, the amount of \$1 at compound interest of 6% is less than the amount of \$1 at simple interest of 6%. This may be tested by finding and comparing $(1.06)^{\frac{1}{2}}$ (use logarithms) and $1 + \frac{1}{2}(.06)$

n	1	2	3	4	5	6	7	8	9	10
S	1.06	1.12	1.18	1.24	1.30	1.36	1.42	1.48	1.54	1.60
C	1.06	1.124	1.191	1.262	1.338	1.418	1.504	1.594	1.689	1.791
n	11	12	13	14	15	16	17	18	19	20
S	1.66	1.72	1.78	1.84	1.90	1.96	2.02	2.08	2.14	2.20
C	1.898	2.012	2.133	2.261	2.397	2.540	2.693	2.854	3.026	3.207

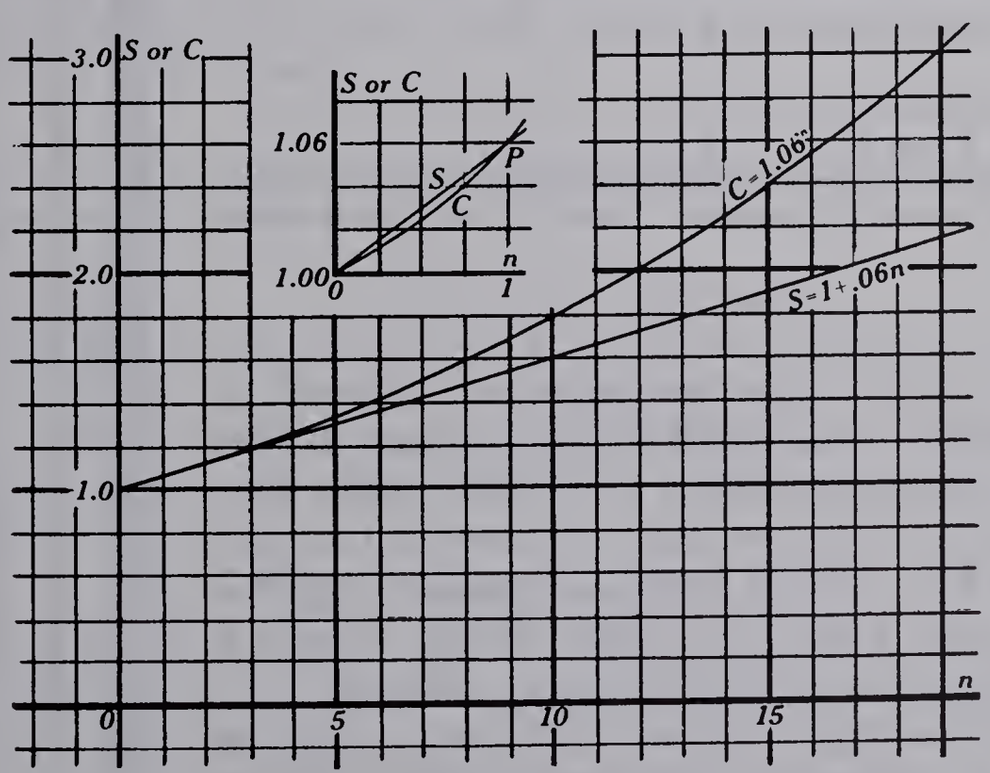


Fig. 8-6

***8•21* The slide rule (supplementary).** Shortly after John Napier discovered logarithms an English mathematician, John Oughtred (1574-1660) realized that this discovery could be used to perform multiplication and division mechanically. In 1622, he made a simple *slide rule* for multiplication and division by using two sticks marked with logarithmic scales. The construction of a slide rule and its method of use is discussed in the following paragraphs.

The following table gives the common logarithms, to three decimal places, of the integers from 1 to 10.

I	1	2	3	4	5	6	7	8	9	10
Log	0	.301	.477	.602	.699	.778	.845	.903	.954	1

Multiplying 4 and 2 is equivalent to adding the logarithms .602 and .301 and finding the number, 8, that corresponds to the sum of these logarithms, $\log (4 \times 2) \doteq .602 + .301 = .903$. This addition may be done mechanically by using a pair of (10 inch) scales on separate rules, marked as illustrated in *Fig. 8-7*, and capable of sliding against each other.

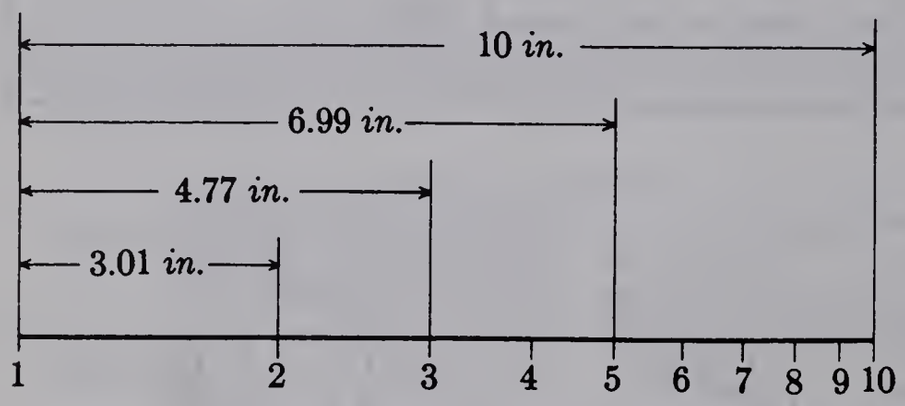


Fig. 8-7

Marks 1, 2, 3, . . ., 10 are located on the scale at *distances from one end which are proportional to their logarithms.* Thus, since $\log 1 = 0$, 1 (left index) is at the left end; since $\log 2 \doteq .301$, 2 is marked at the point 3.01 inches from the left index; since $\log 3 \doteq .477$, 3 is marked at the point 4.77 inches; finally, since $\log 10 = 1$, 1 (right index) is placed at the right end.

When two of these scales are used side by side, the logarithms of two numbers can be easily added and the sum read as the product of the two numbers.

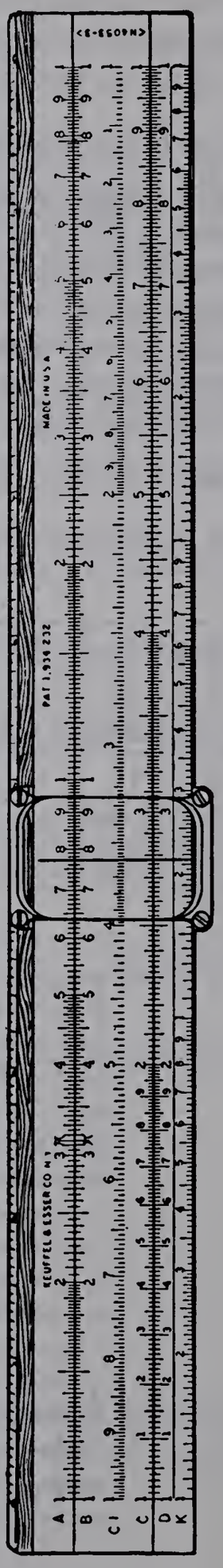
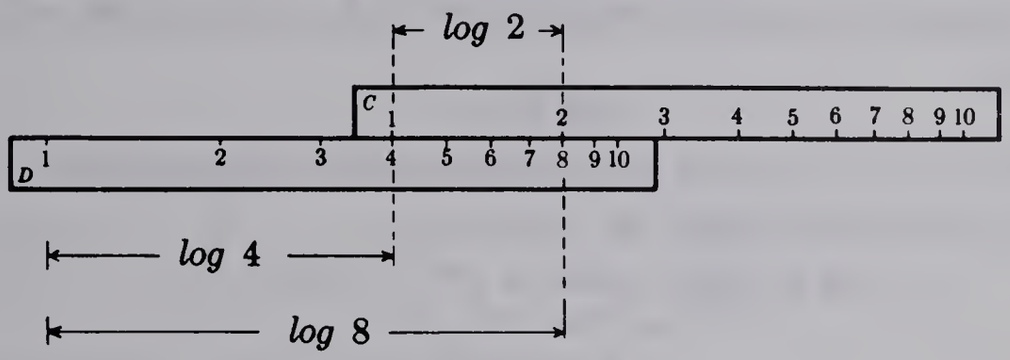


Fig. 8-8

Fig. 8-8 is a picture of a commercial type slide rule called a Mannheim slide rule in honour of its inventor. It contains a number of logarithmic scales used for various purposes. The *stock* contains scales named A, D, and K on its upper surface. The *slider*, which is free to slide in grooves in the centre of the stock, contains scales named B, CI, and C. To assist in setting the scales accurately a runner or *cursor* with a cross wire or hair line is provided. The C and D scales are the ones usually used for multiplication and division.

To multiply 4 and 2, set the hair line on the number 4 on the D scale; move the C scale till the left index, 1, is under the hair line; move the hair line to the number 2 on the C scale; the product 8 appears under the hair line on the D scale. This is illustrated in *Fig. 8-9*



$\log 4 + \log 2 = \log 8$ $\qquad$ $\log 8 - \log 2 = \log 4$

$4 \times 2 = 8$ $\qquad$ $8 \div 2 = 4$

Fig. 8-9

To divide 8 by 2, the process is simply reversed; the hair line is set opposite the dividend 8 on the D scale; the C scale is moved until the divisor, 2, is under the hair line; the quotient appears on the D scale opposite the left index of the C scale.

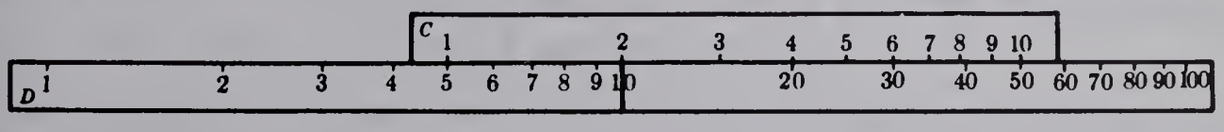


Fig. 8-10

If we attempt to find the product 5×6 (or the sum $\log 5 + \log 6$) we discover that 6 on the C scale is past the right index of the D scale. If we were to add a second scale, *Fig. 8-10*, to the right end of the D scale, the 6 on the C scale will coincide with the 3 (or 30) on the second part of the D scale, because

$$\log 5 + \log 6 = \log 10 + \log 3 = \log 30.$$

Study of *Fig. 8-10* leads to the conclusion that it is not necessary to have an extension of the D scale to obtain the product in examples of this type. If we place the *right index* of the C scale opposite 5 on the D scale, *Fig. 8-11*, we read the product on the D scale opposite the 6 on the C scale.

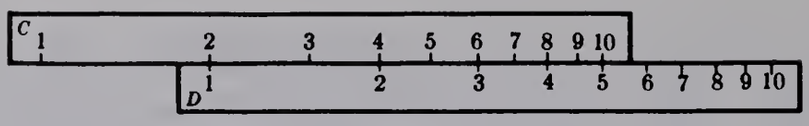


Fig. 8-11

To divide 30 by 6 the process is reversed; 6 on the C scale is placed opposite 3 (or 30) on the D scale; the quotient 5 is read on the D scale opposite the right index of the C scale.

The logarithm scales on a slide rule are calibrated so that three digits may be read accurately. Thus the slide rule is really a mechanical three digit logarithm table. It enables one to obtain quickly a three digit estimate of the answer to quite difficult computations.

One difficulty in the use of the slide rule is the placing of the decimal point in the result of a series of operations. If we used the slide rule to find the product

$$584 \times 72.5$$

we would read the digits 423 from the slide rule as the product.

To place the decimal point we observe that

$$\begin{aligned} 584 \times 72.5 &= 5.84 \times 10^2 \times 7.25 \times 10 \\ &\doteq 6 \times 7 \times 10^3 \\ &\doteq 42 \times 10^3. \end{aligned}$$

Therefore the product is 42,300 to three digit accuracy.

The slide rule may also be used to find the *squares and square roots of numbers*. We know

$$\log x^2 = 2 \log x.$$

To square a number its logarithm must be multiplied by 2. Thus to find the square of 3 on the slide rule it is only necessary to double the distance to 3 on the C or D scale and find 9, that is

$$\log 3 + \log 3 = \log 9.$$

The operation of doubling the distance to any number on scales C or D is conveniently done by using the A scale, *Fig. 8-8*. Since A is a double scale, the logarithmic distance to any number on the A scale represents double the logarithmic distance to the number opposite it on scale D. Thus (in *Fig. 8-8*) if the hair line is set over 3 on scale D, its square, 9, appears under the hair line on scale A. Check each of the following in *Fig. 8-8*.

(i) 4 on the A scale is opposite 2 on the D scale;
(ii) 81 on the A scale is opposite 9 on the D scale;
(iii) 7 on the D scale is opposite 49 on the A scale;
(iv) 2.78 on the D scale is opposite 7.70 on the A scale.

Some slide rules have scales for finding cubes and cube roots; for solving trigonometry problems; and for solving many other special problems.

Practice Exercise 8-16

(B)

Express in logarithmic form:

1. $5^3 = 125$ **2.** $3^0 = 1$ **3.** $27^{\frac{2}{3}} = 9$

4. $r^p = q$ **5.** $8^{0.6667} \doteq 4$ **6.** $10^{0.7782} \doteq 6$

Express in exponential form:

7. $\log_8 2 = \frac{1}{3}$ **8.** $\log_9 3 = 0.5$ **9.** $\log_{125} 5 \doteq 0.3333$

10. $\log_{10} 2 \doteq 0.3010$ **11.** $\log_{10} 3 \doteq 0.4771$ **12.** $\log_q p = r$

Practice Exercise 8-17

(B)

Find the value of x in each of the following:

1. $\log_{343} x = -\frac{2}{3}$ **2.** $\log_{81} x = -\frac{3}{4}$

3. $\log_{625} x = \frac{1}{4}$ **4.** $\log_{216} x = -\frac{2}{3}$

Find the value of each of the following:

5. $7^{\log_7 343}$ **6.** $3^{\log_3 81}$

7. $p^{\log_p q}$ **8.** $a^{\log_a b}$

Express each of the following as a single logarithm:

9. $2 \log_p q + \frac{1}{5} \log_p r - 3 \log_p s$

10. $\frac{2}{3} \log_v x - (\frac{1}{2} \log_v z + 2 \log_v w)$

11. $\frac{1}{2} (2 \log_a b + \frac{1}{3} \log_a c - 3 \log_a d)$

Practice Exercise 8-18

(B)

Find the logarithm of each of the following:

1. 34.8 **2.** 3.48 **3.** 0.000348

4. 0.0294 **5.** 9.87 **6.** 0.659

Find the antilogarithm of each of the following:

7. 1.9826 **8.** $-2 + .2210$ **9.** $2 + .2825$

10. 0.4071 **11.** $-3 + .4855$ **12.** $-1 + .3391$

Practice Exercise 8-19

(B)

Calculate:

1. $0.00416 \times 3.79 \times 21.6$
2. $47.9 \times 32.8 \times 0.517$
3. $0.000909 \times 37.9 \times 6.53$
4. $0.0995 \times 0.0119 \times 43.6$
5. $0.397 \div 24.8$
6. $3.19 \div 0.00516$
7. $27.6 \div 9.46$
8. $0.0425 \div 39.3$
9. $\dfrac{4.13 \times 88.2}{741 \times 0.0545}$
10. $\dfrac{0.00611 \times 31.1}{0.949 \times 111}$

Practice Exercise 8-20

(B)

Calculate:

1. $(0.734)^2$
2. $(3.86)^3$
3. $\sqrt{0.834}$
4. $\sqrt[3]{0.477}$
5. $\sqrt[3]{(57.5)^2}$
6. $\sqrt{4.32 \times 19.1}$
7. $\dfrac{41.8}{\sqrt{37.4 \times 0.00519}}$
8. $\dfrac{1.09 \times 27.9}{\sqrt{0.555}}$
9. $\dfrac{\sqrt[3]{0.0968}}{\sqrt{0.124} \times 39.6}$
10. $\sqrt{\dfrac{\sqrt{0.0743}}{0.00751}}$

Review Exercise 8-21

(B)

Using the properties of logarithms verify each of the following:

1. $\log_3 135 = 3 + \log_3 5$
2. $\log_3 243 = 5$
3. $\log_2 10 = 1 + \log_2 5$
4. $\log_2 196 = 2 + 2 \log_2 7$

Find the value of y in each of the following:

5. $\log_8 y = -\frac{4}{3}$
6. $9^y = 243$

Find the value of each of the following:

7. $5^{\log_5 125}$
8. $6^{\log_7 49}$
9. $a^{\log_a r}$
10. $\log_2 (32)^{\frac{4}{7}}$

Simplify:

11. $\log_{10} 17 + \frac{1}{3} \log_{10} 23 - 2 \log_{10} 6$

12. $3 \log_a b + \frac{1}{2} (\log_a c - \log_a d)$

Calculate:

13. 31.6×69.4

14. 0.00763×29.8

15. 0.905×0.175

16. 19.9×9.01

17. $9.04 \div 0.632$

18. $479 \div 83.4$

19. $0.00748 \div 0.0415$

20. $0.0931 \div 0.00892$

21. $(1.78)^2$

22. $(9.95)^2$

23. $(2.05)^3$

24. $(0.787)^2$

25. $\sqrt{0.576}$

26. $\sqrt[3]{0.419}$

27. $\sqrt{0.0519}$

28. $\sqrt[3]{0.000462}$

29. $(5.91)^{\frac{2}{5}}$

30. $\sqrt[3]{(1.05)^4}$

31. $\dfrac{194}{\sqrt{0.69}}$

32. $\dfrac{6.47}{27.6 \times 0.877}$

33. $\dfrac{(1.83)^3 (7.35)^2}{\sqrt{634}}$

34. $\sqrt{\dfrac{(5.01)^2 \times 13.6}{\sqrt{0.0436}}}$

35. Find the mean proportional between 3.27 and 9.48.

36. If the volume of a cylinder is 36.50 cu. in. and its length is 5.24 in., find its radius to the nearest tenth inch.

37. If the volume of a sphere is 2000 cu. in., find its radius to the nearest inch.

38. Find to the nearest dollar the compound amount of \$5000 at 4% per annum compounded semi-annually for $6\frac{1}{2}$ years.

39. If a debt of \$400 falls due in 10 years, what is the present value of the debt, to the nearest dollar, if money is worth 6% per annum compounded semi-annually?

40. At the birth of his daughter, a father set up a fund for her university education. He invested \$1500 at 5% per annum compounded semi-annually. If the fund is allowed to accumulate, to what will it amount on her 18th birthday? State the compound amount to the nearest dollar.

41. Find the sum of money invested at 6% per annum compounded semi-annually which will amount to \$1000 in $3\frac{1}{2}$ years. State the sum to the nearest dollar.

Chapter IX

THE QUADRATIC FUNCTION AND ITS APPLICATIONS

9•1 The quadratic function. Suppose a ball is thrown vertically upward with an initial velocity of 100 feet per second. If there is no gravitational force acting on the ball and no air resistance, the height h feet of the ball from the release point in time t seconds is given by the relation

$$f = \{(t, h) \mid h = 100\,t,\ t \in {}^{+}R\}. \tag{1}$$

f is a linear function and its graph is shown in *Fig. 9-1.*

t	0	3	5
h	0	300	500

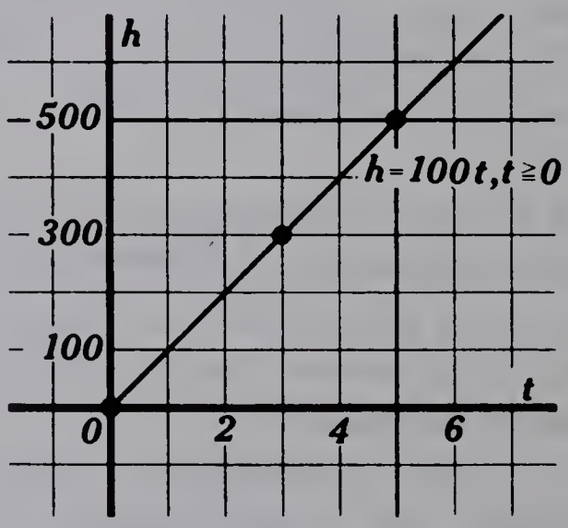


Fig. 9-1

We know from physics that if the gravitational force on a body falling freely is considered, then the distance s feet through which the body falls in time t seconds is given by the relation

$$g = \{(t, s) \mid s = 16t^2,\ t \in {}^{+}R\}. \tag{2}$$

This relation tells us that if there is no upward force on the ball and if the ball falls freely, it is 16 feet below the release point after 1 second, 64 feet below the release point after 2 seconds and so on.

Combining (1) and (2), neglecting air resistance and noting that the gravitational force exerted on the ball is opposite to the initial velocity, the distance d feet of the ball from the release point in time t seconds is given by the relation

$$p = \{(t, d) \mid d = 100t - 16t^2, t \in {}^+R\}$$

or $$p = \{(t, d) \mid d = -16t^2 + 100t, t \in {}^+R\}.$$

By the closure property of R under addition and multiplication, there is one and only one d for each $t \in {}^+R$. It follows that p is a function with

t	0	1	2	3	4	5	6	7
d	0	84	136	156	144	100	24	−84

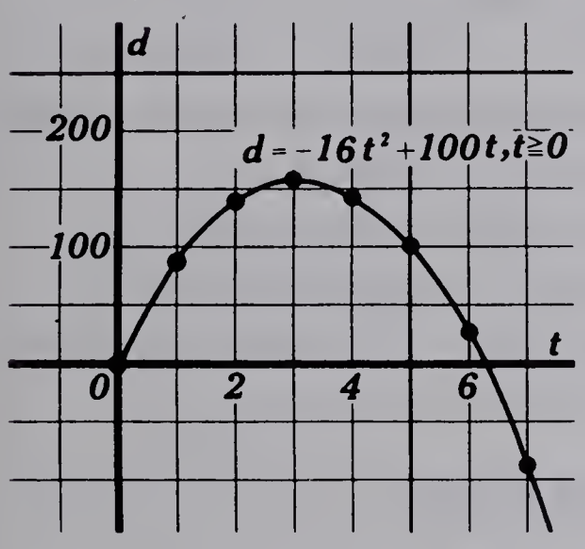


Fig. 9-2

domain ${}^+R$. Since the greatest degree of the variable t in the defining sentence is the second, therefore p is said to be a *quadratic function.*

A table of values and the graph of the function is shown in *Fig. 9-2*

In this chapter we shall study *quadratic functions* and *quadratic equations.* We shall discover, among other things, methods which we can apply to calculate the maximum value of d, and the number of seconds required for the ball to reach the maximum height.

9•2 The general quadratic function. If a, b, c are real constants with $a \neq 0$, then any sentence of the form

$$y = ax^2 + bx + c, x, y \in R$$

defines a quadratic relation on R. Since R is closed under the operations of addition and multiplication, there is one and only one $(ax^2 + bx + c) \in R$ for each $x \in R$, that is, one and only one $y \in R$, for each $x \in R$.

Thus,

$$q = \{(x, y) \mid y = ax^2 + bx + c, x \in R\}$$

where $a \neq 0$, is called the *general quadratic function* on R.

Exercise 9-1

(A)

Determine which of the following are quadratic functions by comparing each with the general quadratic function:

1. $f = \{(x, y) \mid y = 2x^2 - 3x - 5, x \in R\}$
2. $m = \{(x, y) \mid 3x^2 - 4x + 6 = y, x \in R\}$
3. $g = \{(x, y) \mid y = 3x - 5, x \in R\}$
4. $l = \{(x, y) \mid 2y^2 - y + 2 = x, x, y \in R\}$
5. $k = \{(a, b) \mid b = a^2, a \in R\}$
6. $s = \{(a, b) \mid a = b^2, a, b \in R\}$
7. $e = \{(x, y) \mid y = x^2, x \in R\}$
8. $r = \{(x, y) \mid x = y^2, x, y \in R\}$

(B)

9. If $f = \{(x, y) \mid y = 2x^2 + x + 1, x \in R\}$, find
   (i) $f(0)$ (ii) $f(1)$ (iii) $f(-1)$ (iv) $f(10)$
10. If $g = \{(a, b) \mid b = 4a^2 - 3, a \in R\}$, find
    (i) $g(-2)$ (ii) $g(0)$ (iii) $g(2)$ (iv) $g(4)$

For each of the following quadratic functions determine the values indicated:

11. $f = \{(x, y) \mid y = 2x^2 - x - 15, x \in R\}$; $f(3), f(-\frac{5}{2})$
12. $g = \{(x, y) \mid y = -2x^2 - x - 15, x \in R\}$; $g(3), g(-\frac{5}{2})$
13. $a = \{(x, y) \mid y = x^2 - 6x + 9, x \in R\}$; $a(3), a(2), a(4)$
14. $b = \{(x, y) \mid y = -x^2 - 6x - 9, x \in R\}$; $b(-3), b(-2), b(-4)$

9·3 The role of *a* in the general quadratic function.

Discovery Exercise 9-2

(B)

Write solutions for the following problems and compare them with those on page 430.

For each of the following quadratic functions:

(i) *determine whether the graph is symmetric with respect to the x-axis or y-axis;*

(ii) *make a table of values;* (iii) *sketch the graph.*

(The graphs of (i), (ii), and (iii) of each question should be sketched with reference to the same set of coordinate axes.)

1. (i) $q_1 = \{(x, y) \mid y = x^2, x \in R\}$
   (ii) $q_2 = \{(x, y) \mid y = 2x^2, x \in R\}$
   (iii) $q_3 = \{(x, y) \mid y = 3x^2, x \in R\}$

(The graphs of (i), (ii), and (iii) of each question should be sketched on the same set of coordinate axes.)

2. (i) $q_4 = \{(x, y) \mid y = -x^2 + 4, x \in R\}$

(ii) $q_5 = \{(x, y) \mid y = -2x^2 + 4, x \in R\}$

(iii) $q_6 = \{(x, y) \mid y = -3x^2 + 4, x \in R\}$

3. (i) $q_7 = \{(x, y) \mid y = x^2 + 2x - 3, -4 \leqq x \leqq 4, x \in R\}$

(ii) $q_8 = \{(x, y) \mid y = 2x^2 + 2x - 3, -4 \leqq x \leqq 4, x \in R\}$

(iii) $q_9 = \{(x, y) \mid y = -2x^2 + 2x - 3, -4 \leqq x \leqq 4, x \in R\}$

4. Use the solutions to questions 1 to 3 to answer the following.

In the defining sentence of the general quadratic function:

(i) if $a > 0$, does the graph open upward or downward?

(ii) if $a < 0$, does the graph open upward or downward?

(iii) if $a > 0$, what is the effect on the graph of making $|a|$ greater?

(iv) if $a < 0$, what is the effect on the graph of making $|a|$ greater?

These examples suggest the following conclusions:

(i) *the graphs of quadratic functions have the same general shape (a parabola);*

(ii) *in the general quadratic function*

$$q = \{(x, y) \mid y = ax^2 + bx + c, x \in R\}:$$

(a) *if $a > 0$, the corresponding parabola opens upward;*

(b) *if $a < 0$, the corresponding parabola opens downward;*

(c) *the greater $|a|$ becomes, the more rapidly the graph rises (or falls); that is, the more rapidly the value, $q(x)$, of the function changes with x.*

The statements (a) and (b) are proved in Section 9·10.

9·4 Graphs of quadratic functions.

Example 1. Sketch the graph of the quadratic function

$$q = \{(x, y) \mid y = -2x^2 - 4x + 2, -3 \leqq x \leqq 1, x \in R\}$$

for the domain indicated. Find the y-intercept and estimate the x-intercepts from the graph.

Solution.

(i) *Table of Values*

x	-3	-2	-1	0	1
y (or $-2x^2 - 4x + 2$)	-4	2	4	2	-4

(ii) *Graph*

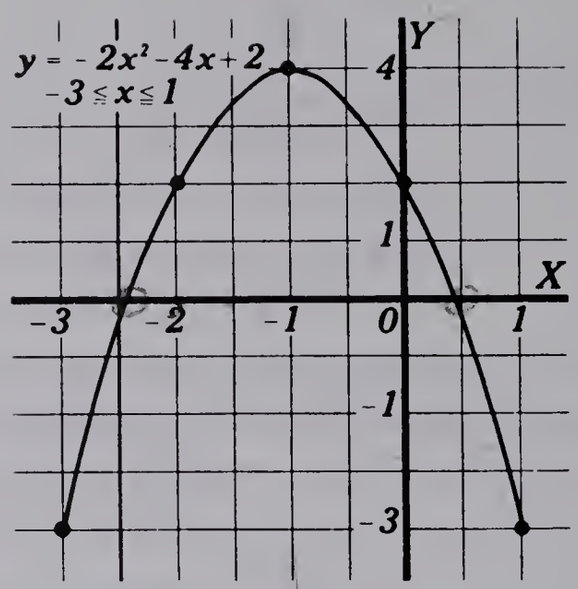


Fig. 9-3

(iii) *y-intercept.* From the table of values the y-intercept is 2.

(iv) *Estimated x-intercepts.* From the graph the x-intercepts are -2.4 and 0.4 approximately.

Note that $a = -2 < 0$ in the preceding example, and the parabola opens downward. The graph in *Fig. 9-3* appears to be symmetric with respect to the vertical line with equation $x = -1$. This line is called the *axis of symmetry* of the parabola. The point of intersection of a parabola with its axis of symmetry is called the *vertex* of the parabola. The vertex of the parabola of *Fig. 9-3* is the point with coordinates $(-1, 4)$.

For any quadratic function $q = \{(x, y) \mid y = ax^2 + bx + c, x \in R\}$, the equation $ax^2 + bx + c = 0$ is called *the corresponding quadratic equation.* The real roots (if any) of this equation are the x-intercepts of the graph of the quadratic function q. As in Example 1, these roots may be estimated approximately from the graph. In Section 9·5 a more satisfactory (algebraic) method of obtaining the roots of the corresponding quadratic equation is given.

Write a solution for the following problem and compare your solution with that on page 433.

For the quadratic function

$$q = \{(x, y) \mid y = 3x^2 - 12x + 4, -3 \leqq x \leqq 7, x \in R\}:$$

(a) sketch the graph;

(b) state the coordinates of the apparent vertex and the equation of the axis of symmetry of the parabola;

(c) estimate the roots of the corresponding equation from the graph.

Exercise 9-3

(B)

By plotting points (for the domain indicated), sketch the graphs of the following quadratic functions. For each, state the coordinates of the apparent vertex and the equation of the axis of symmetry of the parabola; note the sign of a and whether the parabola opens upward or downward. Finally, estimate the roots of the corresponding quadratic equation (the x-intercepts) from the graph.

1. $q = \{(x, y) \mid y = -3x^2 + 6x + 24, -4 \leqq x \leqq 6, x \in R\}$
2. $q = \{(x, y) \mid y = 2x^2 - 12x + 10, -1 \leqq x \leqq 7, x \in R\}$
3. $q = \{(x, y) \mid y = x^2 - x - 2, -3 \leqq x \leqq 4, x \in R\}$
4. $q = \{(x, y) \mid y = -x^2 + 6x - 10, -1 \leqq x \leqq 7, x \in R\}$
5. $q = \{(x, y) \mid y = 2x^2 - 8x + 1, -2 \leqq x \leqq 6, x \in R\}$
6. $q = \{(x, y) \mid y = -3x^2 + 6x - 4, -2 \leqq x \leqq 4, x \in R\}$

9·5 The solution of the corresponding quadratic equation. For the general quadratic function,

$$q = \{(x, y) \mid y = ax^2 + bx + c, x \in R\},$$

the equation

$$ax^2 + bx + c = 0 \qquad (1)$$

is called the *corresponding quadratic equation*. The problem of determining the x-intercepts of the graph of a quadratic function is the same as that of finding the roots of the corresponding quadratic equation. If the factors of the left side of the equation are obvious, the roots may be determined as in Section 1·7. If the factors of the left side of the equation are not evident, then the solution can be obtained by the method of *completing the square* as follows.

Example. By completing the square, solve the following quadratic equations:

(i) $x^2 + 4x + 2 = 0$

(ii) $4x^2 + 12x + 5 = 0$.

Solution.

(i) $x^2 + 4x + 2 = 0$

$\leftrightarrow \quad x^2 + 4x = -2$

$\leftrightarrow \quad x^2 + 4x + 4 = -2 + 4$ [Complete the square on the L.S. by adding the square of $\frac{1}{2}$ the coefficient of x to each side of the equation.]

$\leftrightarrow \quad (x + 2)^2 = 2$

$\leftrightarrow \quad x + 2 = \pm\sqrt{2}$

$\leftrightarrow \quad x = -2 \pm \sqrt{2}.$

(ii) $4x^2 + 12x + 5 = 0$

$\leftrightarrow \quad x^2 + 3x + \frac{5}{4} = 0$ (Divide by the coefficient of x^2.)

$\leftrightarrow \quad x^2 + 3x = -\frac{5}{4}$

$\leftrightarrow \quad x^2 + 3x + \frac{9}{4} = \frac{9}{4} - \frac{5}{4}$ [Complete the square on the L.S. by adding the square of $\frac{1}{2}$ the coefficient of x to each side of the equation.]

$\leftrightarrow \quad \left(x + \frac{3}{2}\right)^2 = 1$

$\leftrightarrow \quad x + \frac{3}{2} = \pm 1$

$\leftrightarrow \quad x = -\frac{3}{2} \pm 1$

$\leftrightarrow \quad x = -\frac{1}{2}$ or $x = -\frac{5}{2}$.

Solve the following quadratic equations by the method indicated, and compare your solutions with those given on page 434.

1. $x^2 + 3x + 2 = 0$; by factoring

2. $2x^2 + 5x - 3 = 0$; by completing the square

3. $-4x^2 + 7x - 2 = 0$; by completing the square

Exercise 9-4

(B)

Solve the following quadratic equations by factoring:

1. $3x^2 - 5x = 0$

2. $3x^2 - 11x - 14 = 0$

3. $z^2 - 3z = 10$

4. $x^2 - 11x + 28 = 0$

5. $2x^2 = 7x + 15$

6. $y(y - 8) = 20$

Solve by completing the square:

7. $3x^2 + 6x - 1 = 0$

8. $y^2 - 15y - 4 = 0$

9. $3x(3x - 2) = 6x - 5$

10. $4z^2 - 4z - 1 = 0$

11. $6u^2 + 7u - 3 = 0$

12. $x^2 - 2x - 11 = 0$

13. $x^2 - 2x + 9 = 0$

14. $3z^2 - 9z - 4 = 0$

15. $p(2p - 4) = 5$

16. $5y^2 - 15y + 9 = 0$

17. $(x - 2)^2 + 3x - 5 = 0$

18. $(3x - 2)^2 + (x + 1)^2 = 0$

(C)

Solve the following equations using any method:

19. $3x^2 = 4(x + 4) - 1$

20. $(3u-1)^2-2(3u-1)-35=0$

21. $(y - 2)(y + 1) = 4(y + 1)$

22. $10x^2 - 19x + 9 = 0$

23. $x^2 - 2ax = 15a^2$, $(a \in R)$

24. $x^2 + 4bx - 5b^2 = 0$, $(b \in R)$

25. $(x^2 - x)^2 = 8(x^2 - x) - 12$

9·6 Quadratic equations with literal coefficients (supplementary). To solve a quadratic equation with numerical coefficients, such as

$$(x - 2)(x + 1) = 4(x - 1),$$

it is usually convenient to reduce it first to the standard form $ax^2+bx+c=0$. The example just stated reduces to $x^2 - 5x + 2 = 0$, which can then be solved by completing the square.

This procedure may or may not be advisable for quadratic equations with literal coefficients as the following examples illustrate.

Example 1. Solve for x:

$$x^2 - a^2 = (x - a)(b + c)\ ,\ a, b, c \in R.$$

Solution.

$$\begin{aligned}
& x^2 - a^2 = (x - a)(b + c) \\
\leftrightarrow \quad & (x - a)(x + a) = (x - a)(b + c) \\
\leftrightarrow \quad & (x - a)[(x + a) - (b + c)] = 0 \\
\leftrightarrow \quad & (x - a)[x - (b + c - a)] = 0 \\
\leftrightarrow \quad & x - a = 0 \quad \text{or} \quad x - (b + c - a) = 0 \\
\leftrightarrow \quad & x = a \quad \text{or} \quad x = b + c - a.
\end{aligned}$$

Example 2. Solve for x:

$$x(x + 8c) = c(2x - 7c),\ c \in R.$$

Solution.

$$\begin{aligned}
& x(x + 8c) = c(2x - 7c) \\
\leftrightarrow \quad & x^2 + 6cx + 7c^2 = 0 \\
\leftrightarrow \quad & x^2 + 6cx + 9c^2 = 9c^2 - 7c^2 \\
\leftrightarrow \quad & (x + 3c)^2 = 2c^2 \\
\leftrightarrow \quad & x + 3c = \sqrt{2}c \quad \text{or} \quad x + 3c = -\sqrt{2}c \\
\leftrightarrow \quad & x = -(3 - \sqrt{2})c \quad \text{or} \quad x = -(3 + \sqrt{2})c.
\end{aligned}$$

Example 3. Solve for x:

$$acx^2 - bcx = adx - bd,\ \ a, b, c, d \in R,\ a \neq 0,\ c \neq 0.$$

Solution.

For $a \neq 0, c \neq 0$,

$$\begin{aligned} & acx^2 - bcx = adx - bd \\ \leftrightarrow \quad & cx(ax - b) = d(ax - b) \\ \leftrightarrow \quad & cx(ax - b) - d(ax - b) = 0 \\ \leftrightarrow \quad & (cx - d)(ax - b) = 0 \\ \leftrightarrow \quad & cx - d = 0 \quad \text{or} \quad ax - b = 0 \\ \leftrightarrow \quad & x = \frac{d}{c} \quad \text{or} \quad x = \frac{b}{a}. \end{aligned}$$

Exercise 9-5

(B)

Solve the following quadratic equations:

1. $x^2 - mx = mn - nx,\ m, n \in R$ **2.** $(y - c)(y - d) = cd,\ c, d \in R$

3. $x^2 - ax - bx + ab = 0,\ a, b \in R$ **4.** $ax^2 + bx = 0,\ a, b \in R,\ a \neq 0$

5. $x^2 - 5ax + 6a^2 = 0,\ a \in R$ **6.** $ax^2 - a^2x - x + a = 0,\ a \in R,\ a \neq 0$

7. $3z^2 + m^2 = mz + 3m^2z,\ m \in R$

8. $acx^2 + (ad - bc)x - bd = 0,\ a, b, c, d \in R,\ ac \neq 0$

9. $a(bx^2 + dx) = -c(bx + d),\ a, b, c, d \in R,\ a \neq 0,\ b \neq 0$

9·7 Maximum or minimum values, range, and graphs of quadratic functions.

Example 1. For the quadratic function

$$q = \{(x, y) \mid y = x^2 - 4x,\ x \in R\}:$$

(i) determine the intercepts of the graph;

(ii) determine the range of the function;

(iii) determine the equation of the axis of symmetry, and the coordinates of the vertex of the graph;

(iv) make a table of values and sketch the graph.

Solution.

(i) *Intercepts.*

Let $y = 0$ in $y = x^2 - 4x$.

$\therefore \quad x^2 - 4x = 0$

$\leftrightarrow \quad x(x - 4) = 0$

$\leftrightarrow \quad x = 0$ or $x = 4$.

$\therefore$ the x-intercepts are 0 and 4.

Let $x = 0$ in $y = x^2 - 4x$.

$\therefore \quad y = 0$.

$\therefore$ the y-intercept is 0.

(ii) *Range.* $y = x^2 - 4x$

$\leftrightarrow y = (x^2 - 4x + 4) - 4$ (completing the square)

$\leftrightarrow y = (x - 2)^2 - 4$.

$\therefore\ y \geqq -4.$ ($\because\ (x-2)^2 \geqq 0$ for all $x \in R$.)

The function q therefore has a *minimum value of* -4, occurring for $x = 2$. It follows from this that the range of q must at least be a subset of $\{y \mid y \geqq -4,\ y \in R\}$. In Section 9·10 it will be shown that *if any quadratic function has a minimum value* y_0, *then its range consists of all* $y \in R$ *for which* $y \geqq y_0$. (Similarly, it will be shown that *if a quadratic function has a maximum value* y_1, *then its range consists of all* $y \in R$ *for which* $y \leqq y_1$.) Accepting these facts without proof for now, and throughout this section, it follows that for the function q defined by $y = x^2 - 4x$, $x \in R$,

the range of q is $\{y \mid y \geqq -4,\ y \in R\}$.

(iii) *Axis of symmetry and vertex.* Since q has a minimum value of -4 occurring for $x = 2$, it seems reasonable to assume (on the basis of the example and exercises of Section 9·4) that:

the equation of the axis of symmetry is $x = 2$;

the coordinates of the vertex of the graph are $(2, -4)$.

In Section 9·10 it will be proved *for any quadratic function* q *that if it has a minimum (or maximum) value of* y_0 *occurring for* $x = x_0$, *then the graph (parabola) of* q *will have the line with equation* $x = x_0$ *as axis of symmetry, and the point with coordinates* (x_0, y_0) *as vertex.* This fact will be used throughout this section without proof.

(iv) *Table of values and graph.*

x	2	3	4	5
y	-4	-3	0	5

Note that only values of $x \geqq 2$ are used in the table of values. Points on the graph (*Fig. 9-4*) with $x < 2$ are obtained from these by reflection in the line $x = 2$.

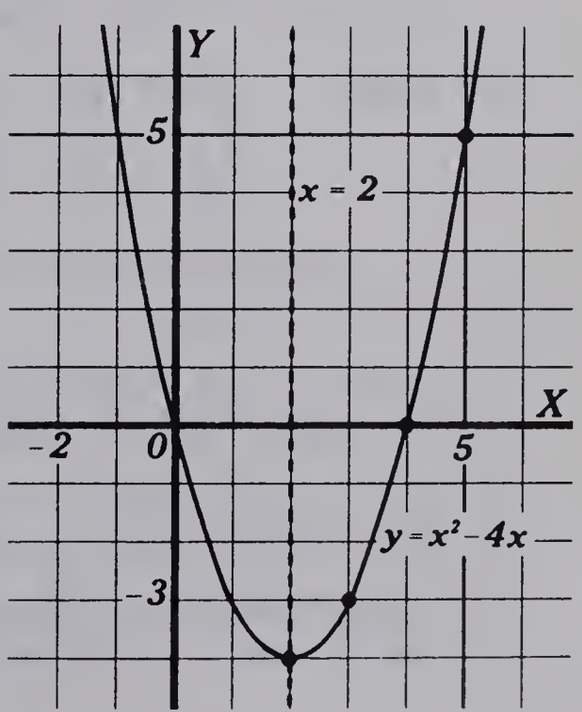


Fig. 9-4

Example 2. For the quadratic function with defining sentence

$$y = -2x^2 + 4x - 3,\ x \in R:$$

(i) determine the intercepts of the graph;

(ii) determine the range of the function;

(iii) state the equation of the axis of symmetry of the graph;
(iv) state the coordinates of the vertex of the graph;
(v) make a table of values and sketch the graph.

Solution.

(i) *Intercepts.* Let $y = 0$ in $y = -2x^2 + 4x - 3$.

$$\therefore \quad -2x^2 + 4x - 3 = 0$$

$$\leftrightarrow \quad x^2 - 2x + \frac{3}{2} = 0$$

$$\leftrightarrow \quad x^2 - 2x = -\frac{3}{2}$$

$$\leftrightarrow \quad (x-1)^2 = -\frac{3}{2} + 1 \qquad \text{(completing the square)}$$

$$\leftrightarrow \quad (x-1)^2 = -\frac{1}{2}.$$

$\because$ $(x-1)^2 \geqq 0$ for all $x \in R$, the last equation has no real roots.
$\therefore$ the graph has no x-intercepts.

Let $x = 0$ in $y = -2x^2 + 4x - 3$.

$\therefore$ $y = -3$.

$\therefore$ the y-intercept is -3.

(ii) *Range.*

$$y = -2x^2 + 4x - 3$$
$$\leftrightarrow y = -2(x^2 - 2x) - 3$$
$$\leftrightarrow y = -2(x^2 - 2x + 1) - 1, \quad \text{(completing the square)}$$
$$\leftrightarrow y = -2(x-1)^2 - 1.$$
$$\therefore y \leqq -1.$$

($\because$ $-2(x-1)^2 \leqq 0$ for all $x \in R$.)
$\therefore$ the function has a maximum value of -1, occurring for $x = 1$.
$\therefore$ the range is $\{y \mid y \leqq -1, y \in R\}$.

(iii) *Axis of symmetry.* The equation of the axis of symmetry is $x = 1$.

(iv) *Vertex.* The coordinates of the vertex are $(1, -1)$.

(v) *Table of values and graph.*

x	1	2	3
y	-1	-3	-9

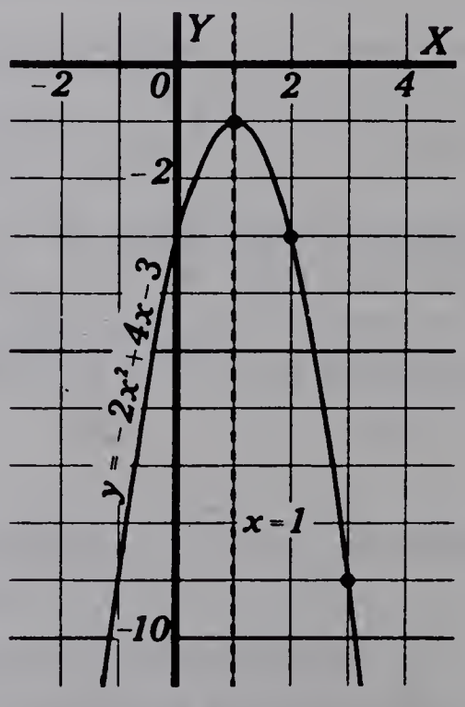


Fig. 9-5

Solve the following problems and compare your solutions with those on page 434.

For each of the quadratic functions in questions 1 and 3:

(*i*) *determine the intercepts of the graph;*

(*ii*) *determine the range of the function;*

(*iii*) *state the equation of the axis of symmetry of the graph;*

(*iv*) *state the coordinates of the vertex of the graph;*

(*v*) *make a table of values and sketch the graph.*

(The graphs of (a), (b), and (c) of each question should be sketched with respect to the same set of coordinate axes.)

1. (a) $q_1 = \{(x, y) \mid y = x^2 - 2x, x \in R\}$

(b) $q_2 = \{(x, y) \mid y = x^2 - x, x \in R\}$

(c) $q_3 = \{(x, y) \mid y = x^2 + 2x, x \in R\}$

2. Use the solutions to question 1 to answer the following.

In the defining sentence $y = ax^2 + bx + c$, $x \in R$ of the general quadratic function:

(i) does a change in the value of b alone affect the size and shape of the graph?

(ii) does a change in the value of b alone affect the location of the axis of symmetry, or the ordinate of the vertex?

(iii) does a change in the value of b alone affect the intercepts?

3. (a) $q_4 = \{(x, y) \mid y = -2x^2 + 4x - 2, x \in R\}$

(b) $q_5 = \{(x, y) \mid y = -2x^2 + 4x, x \in R\}$

(c) $q_6 = \{(x, y) \mid y = -2x^2 + 4x + 3, x \in R\}$

4. Use the solutions to question 3 to answer the following.

In the defining sentence $y = ax^2 + bx + c$, $x \in R$ of the general quadratic function:

(i) does a change in the value of c alone affect the size and shape of the graph?

(ii) does a change in the value of c alone affect the location of the axis of symmetry, or the ordinate of the vertex?

(iii) does a change in the value of c alone affect the intercepts?

(iv) what does c represent on the graph?

Exercise 9-6

(B)

For each of the following quadratic functions:

(*i*) *determine the intercepts of the graph;* (*ii*) *determine the range;*

(*iii*) *state the equation of the axis of symmetry of the graph;*

(*iv*) *state the coordinates of the vertex of the graph;*

(*v*) *make a table of values and sketch the graph.*

1. $q_1 = \{(x, y) \mid y = x^2 - 6x + 8, x \in R\}$
2. $q_2 = \{(x, y) \mid y = x^2 - 4x + 3, x \in R\}$
3. $q_3 = \{(x, y) \mid y = 4x^2 - 4x - 15, x \in R\}$
4. $q_4 = \{(x, y) \mid y = x^2 - 10x + 9, x \in R\}$
5. $q_5 = \{(x, y) \mid y = -x^2 + 2\sqrt{2}x - 3, x \in R\}$
6. $q_6 = \{(u, v) \mid v = -3u^2 - 2u, u \in R\}$
7. $q_7 = \{(x, y) \mid y = (2x - 1)^2 - (x + 2)^2, x \in R\}$
8. $q_8 = \{(t, s) \mid s = (2t - 1)(2t + 3), t \in R\}$

(C)

9. Show that if the point $P(-1 + k, y)$, $k \in R$, is on the graph of the function defined by $y = -2x^2 - 4x + 2$, then so is the point $P'(-1 - k, y)$. Why does this prove that the line with equation $x = -1$ is the axis of symmetry of the graph?

9·8 Maximum and minimum value problems. Any quadratic function q has a unique maximum value (if $a < 0$) or a unique minimum value (if $a > 0$). This maximum or minimum value and the value of x for which it occurs can be determined by the process of completing the square. These facts are of the utmost importance in a great variety of problems of both theoretical and practical application.

Example 1. A ball is thrown vertically upward with an initial velocity of 100 feet per second. Neglecting air resistance, it was shown in Section 9·1 that the distance d feet of the ball from the release point in time t seconds is given by

$$d = -16t^2 + 100t\,, \quad t \in {}^+R.$$

Determine the maximum height attained by the ball, and the number of seconds required to attain this maximum height.

Solution. $\because\ d = -16t^2 + 100t$

$$= -16\left(t^2 - \frac{25}{4}t\right)$$

$$= -16\left[t^2 - \frac{25}{4}t + \left(\frac{25}{8}\right)^2\right] + 16\left(\frac{25}{8}\right)^2 \qquad \text{(completing the square)}$$

$$= -16\left(t - \frac{25}{8}\right)^2 + \frac{625}{4},$$

$$\therefore\ d \leqq \frac{625}{4} \qquad (\because\ -16\left(t - \frac{25}{8}\right)^2 \leqq 0 \text{ for all } t \in R).$$

$\therefore$ the maximum height attained by the ball is $156\frac{1}{4}$ feet, and it takes $\frac{25}{8}$ seconds or $3\frac{1}{8}$ seconds to attain this maximum height.

Example 2. A rectangular parking lot is to be enclosed by 1000 yards of wire fence. What is the maximum area which may be enclosed? What are the dimensions of the sides of the lot if it encloses this maximum possible area?

Solution.

Represent the number of yards in one side of the lot by x, $x \in {}^+R$, $x < 500$. Since the total perimeter is 1000 yards, the number of yards in the adjacent side of the lot is $500 - x$.

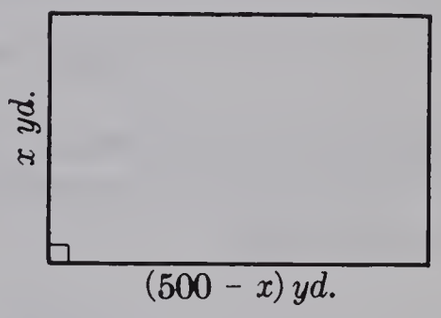


Fig. 9-6

$\therefore$ if $A(x)$ is the number of square yards enclosed by the fence, then

$$\begin{aligned} A(x) &= x(500 - x) \\ &= -x^2 + 500x \\ &= -(x^2 - 500x + 250^2) + 250^2 \quad \text{(completing the square)} \\ &= -(x - 250)^2 + 250^2. \end{aligned}$$

$\therefore\ A(x) \leqq 250^2$.

$\therefore$ the maximum area enclosed is 250^2 or 62,500 square yards, and the dimensions of the sides of the lot enclosing the maximum area are 250 yards by 250 yards.

Example 3. A rectangular parking lot is to be placed with one side along a straight street where no fence is required. If 1000 yards of wire are available for fencing the other three sides of the lot, what is the maximum area the lot may have, and what are the dimensions of the lot of maximum area?

Solution.

Represent the number of yards of fence required for a side perpendicular to the street by x, $x \in {}^+R$, $x < 500$.

Then the number of yards of fence in the side parallel to the street is $1000 - 2x$.

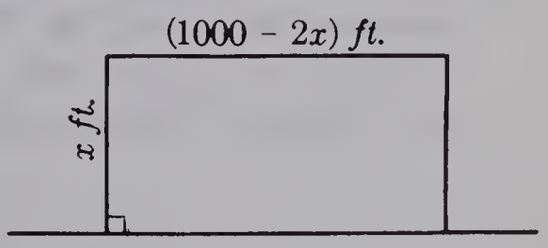


Fig. 9-7

$\therefore$ if $A(x)$ is the number of square yards enclosed by the fence, then

$$A(x) = x(1000 - 2x).$$

Complete the solution of this problem, and compare your solution with that on page 437.

Example 4. Find two positive numbers whose product is 36 and whose sum is a minimum.

Solution.

Let x and y denote any two positive real numbers whose product is 36.

$$\because \quad xy = 36$$
$$\therefore \quad y = \frac{36}{x}.$$

So denoting the sum of the two numbers by $S(x)$, it follows that

$$S(x) = x + \frac{36}{x}, \quad x \in {}^+R.$$

Although the function S is not a quadratic function, the minimum value of S may still be determined by a process of completing the square. To do this, write

$$S(x) = (\sqrt{x})^2 + \frac{36}{(\sqrt{x})^2}$$
$$= \left(\sqrt{x} - \frac{6}{\sqrt{x}}\right)^2 + 12, \quad x \in {}^+R.$$
$$\therefore \quad S(x) \geqq 12 \quad \text{for all } x > 0, \text{ and}$$
$$S(x) = 12 \quad \leftrightarrow \quad \sqrt{x} = \frac{6}{\sqrt{x}}$$
$$\leftrightarrow \quad x = 6.$$

The required numbers are 6 and $\frac{36}{6}$, or 6.

Solve the following problems and compare your solutions with those on page 437.

1. Find the real number which exceeds its square by the largest possible amount.
2. $\triangle ABC$ has a right angle at B and has sides AB, BC with measurements of 45 ft. and 60 ft. respectively. Find the dimensions of the rectangle of largest area which can be inscribed in the triangle and have adjacent sides on AB and BC.

Exercise 9-7

(A)

State an expression in x for each of the following:

1. The area of a rectangular lot with perimeter 1200 yards and the number of yards in the length of one side represented by x, where $0 < x < 600$, $x \in {}^+R$.

State an expression in x for each of the following:

2. The product of two positive real numbers whose sum is 9.

3. The area of a rectangular lot fenced on three sides with 500 ft. of fencing, where the number of feet in the length of each of the equal fenced sides is x, $0 < x < 250$, $x \in {}^{+}R$.

4. The absolute value of the difference between a real number x and its square:
(i) if the square of the number is greater than the number;
(ii) if the number is greater than its square.

State the maximum or minimum value of the function f, and the value (or values) of x for which this occurs:

5. $f(x) = -(x-3)^2 + 11,\ x \in R$

6. $f(x) = 3\left(x - \frac{4}{x}\right)^2 - 5,\ x \in R,\ x \neq 0$

7. $f(x) = -2(x-7)^2 - 4,\ x \in R$

8. $f(x) = 6\left(\sqrt{x} - \frac{6}{\sqrt{x}}\right)^2 + 5,\ x \in {}^{+}R$

9. $f(x) = 6\left(\sqrt{|x|} - \frac{6}{\sqrt{|x|}}\right)^2 + 5,\ x \in R,\ x \neq 0$

10. $f(x) = 2(|x| - 7)^2 + 4,\ x \in R$

11. $f(x) = (x^2 + 2)^2 - 4,\ x \in R$

(B)

Complete the square and determine the maximum or minimum value of the function, f, and the value (or values) of x for which this occurs:

12. $f(x) = 2x^2 - x - 10,\ x \in R$

13. $f(x) = -2x^2 - 6x + 3,\ x \in R$

14. $f(x) = (2x+5)(x-3),\ x \in R$

15. $f(x) = x^2 + \frac{1}{x^2},\ x \in R,\ x \neq 0$

16. $f(x) = x^2 + (1-x)^2,\ x \in R$

17. $f(x) = 4x + \frac{36}{x},\ x \in {}^{+}R$

18. $f(x) = 3x + \frac{1}{3x} - 5,\ x \in {}^{+}R$

19. A rectangular field is to be enclosed by a fence and divided into two smaller plots by a fence parallel to one of the sides. Find the dimensions of the largest such field if 1200 ft. of fence is available. What is the area of this field?

20. If a farmer harvests his crop today he will have 1200 bushels worth \$2.00 per bushel. Every week he waits, the crop increases by 100 bushels, but the price drops 10 cents per bushel. When should he harvest the crop?

21. The base of an isosceles triangle is 20 ft. and its altitude is 40 ft. Find the dimensions of the inscribed rectangle of maximum area if two of its vertices are on the base of the triangle. What is the maximum area?

22. Find two positive real numbers whose sum is 13 and whose product is a maximum.

23. Find two positive real numbers whose sum is 13, if the sum of their squares is a minimum.

24. Prove that the sum of any positive real number and its reciprocal is at least 2.

25. A rectangular field, 3200 square yards in area, is to be fenced off along the straight bank of a river, no fence being needed along the bank. What dimensions should the field have if the amount of fencing is to be a minimum?

26. A printed page is to contain 432 square centimetres of actual printed matter. There is to be a margin 4 cm. wide along the sides and 3 cm. wide along the top and bottom. What should be the dimensions of the page if the amount of paper used is to be a minimum?

27. If 400 people will attend a moving picture theatre when the admission price is 80 cents, and if the attendance decreases by 40 for each 10 cents added to the price, what price of admission will yield the greatest gross receipts?

28. A projectile is shot straight up from a height of 6 ft. with an initial velocity of 192 ft. per second. Its height in feet after t seconds is given by $h(t) = 6 + 192t - 16t^2$, $t \epsilon^+ R$. After how many seconds does the projectile reach its maximum height? What is the maximum height? After how many seconds will the height be 518 ft.?

(C)

29. A man wishes to enclose two separate lots with 300 yards of fencing, one lot being a square and the other a rectangle twice as long as it is wide. Determine the dimensions of each lot if the total area to be enclosed is a minimum.

30. A window is to be designed in the form of a rectangle surmounted by a semicircle, the diameter of which coincides with the upper base of the rectangle. If the total perimeter of the window is to be 30 feet, determine its dimensions to admit the greatest amount of light.

31. A ship A is 50 miles north of another ship B and is sailing south at the rate of 10 miles per hour. If B is sailing west at the rate of 20 miles per hour, when is the distance between the ships a minimum, and what is this minimum distance? (Hint: use the theorem of Pythagoras to find the distance between the ships after t hours. Then use the fact that the distance will be a minimum when its square is a minimum and conversely.)

32. Would a circular lot enclose a greater area than the maximum rectangular area in Example 2, page 241? Determine the area for a circular lot.

33. It can be proved that among all closed curves of a given perimeter, the circle encloses the maximum area. Assuming this to be true, prove that a semicircular lot in Example 3, page 241, would give the maximum area possible.

34. Verify the identity $(x + y)^2 - (x - y)^2 = 4xy$, $x, y \in R$. Hence prove that if the sum of two real numbers is constant, then their product is a maximum when the numbers are equal, while if the product of two positive real numbers is constant, their sum is a minimum when they are equal.

9·9 The general quadratic function (supplementary). In Section 9·2 the general quadratic function was introduced, namely

$$q = \{(x, y) \mid y = ax^2 + bx + c, x \in R\},$$

where the coefficients a, b, c are real numbers with $a \neq 0$. Some specific examples of quadratic functions were discussed in Sections 9·3, 9·4, and 9·7. In succeeding sections, the same analysis will be applied to the *general* quadratic function. The following points related to the general quadratic function q, and to its graph, will be discussed:

(i) the *range* of q, and the *maximum* or *minimum* value of q;

(ii) the *axis of symmetry*, and the *vertex* of the graph of q;

(iii) the *corresponding general quadratic equation*

$$ax^2 + bx + c = 0,$$

and its roots.

Items (i) and (ii) are discussed in the next section, while a more detailed discussion of (iii) is given in the following sections.

9·10 The range of the general quadratic function, the equation of the axis of symmetry, and the coordinates of the vertex of the graph (supplementary).

The defining equation

$$y = ax^2 + bx + c\,, \quad x \in R, \; a \neq 0$$

of the general quadratic function q may be rewritten as follows:

$$y = ax^2 + bx + c$$

$$\leftrightarrow \quad ax^2 + bx + c = y$$

$$\leftrightarrow \quad x^2 + \frac{b}{a}x = \frac{y}{a} - \frac{c}{a}$$

$$\leftrightarrow \quad x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{y}{a} - \frac{c}{a} + \frac{b^2}{4a^2} \quad \text{(completing the square)}$$

$$\leftrightarrow \quad \left(x + \frac{b}{2a}\right)^2 = \frac{y}{a} - \frac{4ac - b^2}{4a^2}$$

$$\leftrightarrow \quad x + \frac{b}{2a} = \pm\sqrt{\frac{y}{a} - \frac{4ac - b^2}{4a^2}}.$$

$$\therefore \quad x \in R \quad \leftrightarrow \quad \frac{y}{a} - \frac{4ac - b^2}{4a^2} \geqq 0, y \in R.$$

Two cases require consideration.

(i) $a > 0$

For $a > 0$,

$$\frac{y}{a} - \frac{4ac - b^2}{4a^2} \geqq 0$$

$$\leftrightarrow \quad y \geqq \frac{4ac - b^2}{4a}.$$

$$\therefore \quad \text{the range of } q \text{ is } \left\{y \mid y \geqq \frac{4ac - b^2}{4a}, \ y \in R\right\}.$$

Thus for $a > 0$, the range has a *minimum value* $\dfrac{4ac - b^2}{4a}$.

(ii) $a < 0$

For $a < 0$,

$$\frac{y}{a} - \frac{4ac - b^2}{4a^2} \geqq 0$$

$$\leftrightarrow \quad y \leqq \frac{4ac - b^2}{4a}.$$

$$\therefore \quad \text{the range of } q \text{ is } \left\{y \mid y \leqq \frac{4ac - b^2}{4a}, \ y \in R\right\}.$$

Thus for $a < 0$, the range has a *maximum value* $\dfrac{4ac - b^2}{4a}$.

In each case the maximum or minimum value occurs when

$$\left(x + \frac{b}{2a}\right)^2 = 0$$

or

$$x = -\frac{b}{2a}.$$

For each y in the range of the function q

$$\frac{y}{a} - \frac{4ac - b^2}{4a} \geqq 0 .$$

$\therefore$ for y in the range of q

$$ax^2 + bx + c = y \leftrightarrow x + \frac{b}{2a} = \pm \sqrt{\frac{y}{a} - \frac{4ac - b^2}{4a^2}} \qquad (1)$$

$$\leftrightarrow x = -\frac{b}{2a} - \sqrt{\frac{y}{a} - \frac{4ac}{4a^2}} \text{ or } x = -\frac{b}{2a} + \sqrt{\frac{y}{a} - \frac{4ac - b^2}{4a^2}} .$$

$\therefore$ for each y in the domain of q there are two points with coordinates (x, y) on the graph which are located symmetrically about the line with equation $x = -\frac{b}{2a}$.

$\therefore$ the equation of the axis of symmetry of the graph of the general quadratic function is

$$x = -\frac{b}{2a}. \qquad (2)$$

Since the vertex of a parabola was defined to be the point of intersection of the parabola with its axis of symmetry, the ordinate of the vertex may be found by setting $x = -\frac{b}{2a}$ in equation (1).

$\therefore$ the coordinates of the vertex of the graph of the general quadratic function are

$$\left(-\frac{b}{2a}, \frac{4ac - b^2}{4a}\right)$$

Exercise 9-8

(A)

For the quadratic function

$q = \{(x, y) \mid y = ax^2 + bx + c, x \in R\}$, $(a, b, c \in R, a \neq 0)$:

1. State the equation of the axis of symmetry of the graph.
2. State the coordinates of the vertex of the graph.
3. State the range of q, if $a > 0$.
4. State the range of q, if $a < 0$.
5. If q has a maximum value, state this value and state whether $a > 0$ or $a < 0$ in this case.
6. If q has a minimum value, state this value and state whether $a > 0$ or $a < 0$ in this case.

(B)

For each of the following quadratic functions q, determine the range, maximum or minimum value of q, the equation of the axis of symmetry and coordinates of the vertex of the graph of q, and the roots of the equation $q(x)=0$. Using these results and a table of values, sketch the graph of q.

7. $q = \{(x, q(x)) \mid q(x) = 2x^2 - 10x + 13, x \in R\}$

8. $q = \{(x, q(x)) \mid q(x) = -4x^2 + 8x + 5, x \in R\}$

9. $q = \{(x, q(x)) \mid q(x) = 3x^2 + 10x, x \in R\}$

10. $q = \{(x, q(x)) \mid q(x) = \sqrt{3}x^2 - 6x + 3\sqrt{3}, x \in R\}$

11. $q = \{(x, q(x)) \mid q(x) = -2x^2 + 9x - 41, x \in R\}$

12. $q = \{(x, q(x)) \mid q(x) = 5x^2 - 8x + 1, x \in R\}$

13. $q = \{(x, q(x)) \mid q(x) = 2x^2 + 4x + 9, x \in R\}$

14. $q = \{(x, q(x)) \mid q(x) = -x^2 - 7x - 12, x \in R\}$

9•11 Discovery of the general quadratic formula.

Discovery Exercise 9-9

(B)

Write solutions for the following problems and compare your solutions with those on page 438.

Solve the following quadratic equations by the method of completing the square. In each case state the condition which must be satisfied if the equation is to have (i) *real roots,* (ii) *real and equal roots.*

1. $x^2 + 3x + c = 0$

2. $x^2 + bx + c = 0$

3. $3x^2 + bx + c = 0$

4. $ax^2 + bx + c = 0, \ (a \neq 0).$

9•12 The general quadratic formula.

The general quadratic equation is

$$ax^2 + bx + c = 0, \quad (a, b, c \in R, \ a \neq 0).$$

The real roots of this equation may be found by the method of completing the square.

For $a \neq 0$,

$$ax^2 + bx + c = 0$$

$$\leftrightarrow \quad x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

$$\leftrightarrow \quad x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = \left(\frac{b}{2a}\right)^2 - \frac{c}{a}$$

$$\leftrightarrow \quad \left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}.$$

Two cases now arise.

(i) If $b^2 - 4ac < 0$, then

$\left(x + \frac{b}{2a}\right)^2 \geqq 0$ for all $x \in R$, and $\frac{b^2 - 4ac}{4a^2} < 0$, and therefore the equation $ax^2 + bx + c = 0$ has no real roots.

(ii) If $b^2 - 4ac \geqq 0$, then

$$\left(\frac{x + b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

$$\leftrightarrow \quad x + \frac{b}{2a} = \frac{\sqrt{b^2 - 4ac}}{2a} \quad \text{or} \quad x + \frac{b}{2a} = \frac{-\sqrt{b^2 - 4ac}}{2a}$$

$$\leftrightarrow \quad x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{or} \quad x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

The number $(b^2 - 4ac)$ is called the *discriminant* of the general quadratic equation $ax^2 + bx + c = 0$, $a \neq 0$.

The results of this discussion are summarized in the following theorem.

THEOREM. *For the general quadratic equation*

$$ax^2 + bx + c = 0, \ (a, b, c \in R, a \neq 0):$$

(i) *if* $b^2 - 4ac < 0$, *there are no real roots*;

(ii) *if* $b^2 - 4ac \geqq 0$, *there are the two real roots*

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a};$$

if $b^2 - 4ac > 0$ *the roots are real and unequal*;

if $b^2 - 4ac = 0$ *the roots are real and equal.*

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

is called the *general quadratic formula* and may be used to solve any quadratic equation for which $b^2 - 4ac \geqq 0$.

Example. Use the general quadratic formula to solve the equations:

(i) $5x^2 + 6x - 1 = 0$ (ii) $2x^2 = 18x + 5$

(iii) $x(2x - 3) = 2x - 6$ (iv) $4x^2 - 4\sqrt{2}x + 2 = 0$.

Solution. (i) Compare $5x^2 + 6x - 1 = 0$

with $ax^2 + bx + c = 0$.

$a = 5, \quad b = 6, \quad c = -1$.

$\therefore$ by the quadratic formula,

$$x = \frac{-6 \pm \sqrt{36 - 4(5)(-1)}}{10}$$

$$= \frac{-6 \pm \sqrt{56}}{10}$$

$$= \frac{-6 \pm 2\sqrt{14}}{10}$$

$$= \frac{-3 \pm \sqrt{14}}{5}.$$

(ii) $2x^2 = 18x + 5$

$\leftrightarrow \quad 2x^2 - 18x - 5 = 0.$

Compare $2x^2 - 18x - 5 = 0$

with $ax^2 + bx + c = 0$.

$a = 2, \quad b = -18, \quad c = -5$.

$\therefore$ by the quadratic formula,

$$x = \frac{18 \pm \sqrt{(-18)^2 - 4(2)(-5)}}{4}$$

$$= \frac{18 \pm \sqrt{364}}{4}$$

$$= \frac{18 \pm 2\sqrt{91}}{4}$$

$$= \frac{9 \pm \sqrt{91}}{2}.$$

(iii) $x(2x - 3) = 2x - 6$

$\leftrightarrow \quad 2x^2 - 5x + 6 = 0.$

$a = 2, \quad b = -5, \quad c = 6$

$\therefore$ the discriminant, $b^2 - 4ac$

$= 25 - 48 < 0.$

$\therefore$ the equation has no real roots.

(iv) $$4x^2 - 4\sqrt{2}x + 2 = 0 .$$

$$a = 4, \quad b = -4\sqrt{2}, \quad c = 2.$$

$\therefore$ by the quadratic formula,

$$x = \frac{4\sqrt{2} \pm \sqrt{32 - (4)(4)(2)}}{8}$$

$$= \frac{4\sqrt{2}}{8}$$

$$= \frac{\sqrt{2}}{2}.$$

The discriminant is zero, and the equation has two equal roots.

Although a general formula is now available for the real roots of any quadratic equation, in many cases it is easier to obtain the roots by factoring. If the expression $ax^2 + bx + c$ has obvious linear factors, solve the quadratic equation by factoring; otherwise use the general quadratic formula.

Exercise 9-10

(A)

1. What is the discriminant of the general quadratic equation $ax^2 + bx + c = 0$ $(a, b, c \in R, a \neq 0)$?
2. State how the discriminant determines the character of the roots of a quadratic equation.

(B)

For each of the following quadratic equations, determine the value of the discriminant and state whether the equation has no real roots, real and unequal roots, or real and equal roots:

3. $4x^2 - 4x + 1 = 0$
4. $4x^2 - x - 5 = 0$
5. $z^2 + z + 1 = 0$
6. $7u^2 - 5u - 2 = 0$
7. $x^2 + \sqrt{2}x + \frac{1}{4} = 0$
8. $x^2 - 2ax + a^2 = 0, (a \in R)$
9. $x^2 - ax - 1 = 0, (a \in R)$
10. $\sqrt{3}x^2 - 2x - \sqrt{3} = 0$
11. $3y^2 + \pi y + 4 = 0$
12. $9x^2 - 12x + 4 = 0$

Solve by the general quadratic formula:

13. $6x^2 - 7x + 2 = 0$
14. $2x^2 - 6x - 1 = 0$
15. $2x^2 = 13(x - 1) + 3$
16. $1200y^2 = 10y + 1$

17. $3x^2 = 5(x-1)^2$ **18.** $z^2 - 41z + 420 = 0$

19. $5z^2 - 12z + 2 = 0$ **20.** $x^2 + 2bx - c^2 = 0$, $(b, c \in R)$

21. $x^2 - 6ax + 3a^2 = 0$, $(a \in R)$

22. $2ax^2 + (a-2)x = 1$, $(a \in R, a \neq 0)$

23. $\pi u^2 + (\pi^2 - 1)u - \pi = 0$ **24.** $x(x - \sqrt{2} + 4) = 4(x+1)$

***9·13* Problems involving the solution of quadratic equations.** In this section some examples are given of problems which may be solved by setting up a quadratic equation and solving this equation.

If a problem is solved by means of a quadratic equation, it does not follow that both roots of the equation will be *admissible solutions* of the original problem. For example, a length can not be negative, and the number of men in a row can not be fractional, negative, or irrational. A root which must be excluded due to the conditions stated in the problem is called an *inadmissible root.* Each admissible root must be verified by showing that it satisfies the original statement of the problem.

Example 1. The sum of two real numbers is 19, and their product is -66. Find the numbers.

Solution. Represent one of the real numbers by x.
The sum of the two numbers is 19.
$\therefore$ represent the second number by $19 - x$.
The product of the two numbers is $x(19-x)$.
But the product is -66.

$\therefore$ For $x \in R$, $x(19-x) = -66$,
$\leftrightarrow \quad x^2 - 19x - 66 = 0$
$\leftrightarrow \quad (x-22)(x+3) = 0$
$\leftrightarrow \quad x = 22$ or $x = -3$.

If $x = 22$,	If $x = -3$,
then $19 - 22 = -3$.	then $19 - (-3) = 22$.

The two numbers are 22 and -3.

Verification. Sum of numbers $= 22 + (-3) = 19$.
Product of numbers $= (22)(-3) = -66$.

Solve the following problems and compare your solutions with those on page 439.

1. A theatre contains 1015 seats. If the number of rows is 6 more than the number of seats in each row, how many rows are there?

2. A rectangular swimming pool has dimensions 30 ft. and 100 ft. If the area is to be doubled by increasing the width and the length by the same amount, what are the new dimensions?

Exercise 9-11

(B)

1. The difference between two real numbers is 15, and their product is 100. Find the numbers.
2. If the sum of two real numbers is 20 and the sum of their squares is 218, find the numbers.
3. Two boys agree to each mow half a lawn 40 feet by 60 feet. How wide a strip around the outside must the first boy cut to do his share?
4. The sum of the first n consecutive positive integers 1, 2, . . . , n can be shown to be $\frac{1}{2}n(n+1)$. Find n if the sum is 2211.
5. The lengths of the three sides of a right triangle are consecutive integers. Find the sides.
6. Find three consecutive even natural numbers whose sum is $\frac{1}{6}$ the product of the first two.
7. Find three consecutive natural numbers the sum of whose squares is 677.

***9•14* The graphical interpretation of the roots of a quadratic equation.** The x-intercepts of the graph of the general quadratic function

$$q = \{(x, y) \mid ax^2 + bx + c = 0, x \in R\}, \text{ where } a \neq 0,$$

are the roots of the general quadratic equation

$$ax^2 + bx + c = 0\,, \quad a \neq 0.$$

If $b^2 - 4ac < 0$, there are no real roots and the graph of q has no x-intercepts. In this case the graph of q is entirely above or entirely below the x-axis (*Fig. 9-8*).

If $b^2 - 4ac > 0$, there are two real roots,

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ and}$$

$$x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

and hence two x-intercepts. In this case the graph of q meets the x-axis in two distinct points (*Fig. 9-9*).

If $b^2 - 4ac = 0$, $x_1 = x_2$ and there is only one x-intercept. In this case the graph of q meets the x-axis in a single point or two *coincident* points (*Fig. 9-10*).

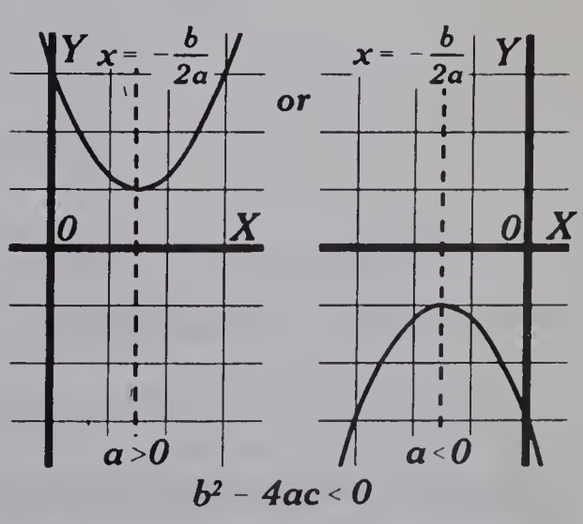


Fig. 9-8

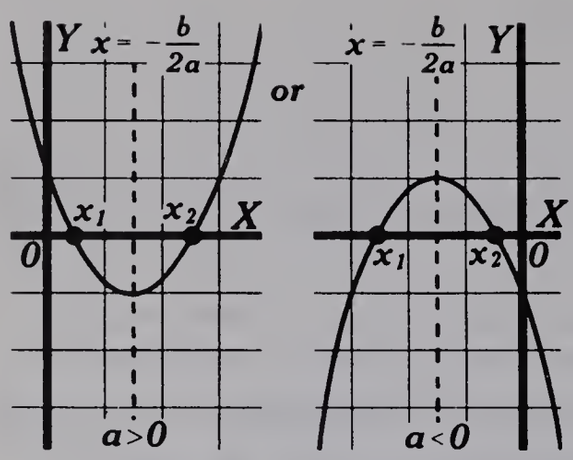


Fig. 9-9

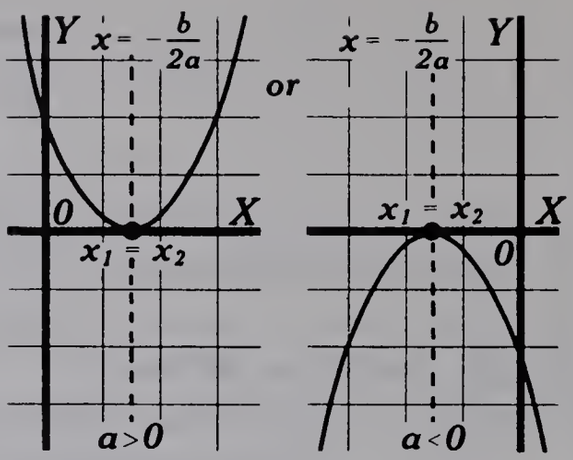


Fig. 9-10

***9·15* Introduction to complex numbers.** The quadratic equation

$$x^2 + 6x + 10 = 0$$

has the discriminant

$$b^2 - 4ac = -4,$$

which is less than 0 .

Therefore the equation has no real roots.

But $x^2 + 6x + 10 = 0$

$\leftrightarrow$ $(x + 3)^2 = -1$, (completing the square).

There is no real number $(x + 3)$ such that $(x + 3)^2 = -1$.

Suppose we extend the set of real numbers by inventing a new number represented by i, such that $i^2 = -1$.

Then $(x + 3)^2 = -1$

would become $(x + 3)^2 = i^2$

and would have the two "roots"

$$x = -3 \pm i.$$

Similarly, the quadratic equation

$$x^2 + x + \frac{5}{16} = 0$$

or $\left(x + \frac{1}{2}\right)^2 = -\frac{1}{16}$, (completing the square).

would become $\left(x + \frac{1}{2}\right)^2 = \frac{1}{16}i^2$,

and would have the two "roots"

$$x = -\frac{1}{2} \pm \frac{1}{4}i.$$

The above discussion suggests that the real number system R may be *extended* so that every quadratic equation will have two roots in the

extended number system. This is a similar procedure to that used in previous extensions of the number systems. For example, since equations such as $3x + 2 = 0$ had no root in the system of integers, I, the system I was extended to the system of rationals Q. In Q the equation $3x + 2 = 0$ has the root $-\frac{2}{3}$. Similarly the equation $x^2 - 2 = 0$ had no root in Q. The system Q was extended to the real number system R. In R the equation $x^2 - 2 = 0$ has roots $\pm\sqrt{2}$. The form of the "roots" in the two examples discussed above suggests a system of numbers associated with a set of numerals of the form

$$a + bi, \text{ where } a \in R,\ b \in R,\ i^2 = -1.$$

These numbers are called *complex numbers.* The set of complex numbers is represented by C.

Thus $$C = \{a + bi \mid a, b \in R\}, \text{ where } i^2 = -1.$$

In the work which follows it is assumed that every quadratic equation has two roots in C.

If $b = 0$, the roots are real numbers.

If $b \neq 0$, the roots will be referred to as non-real complex numbers.

Solve the following quadratic equations by completing the square, and replacing (-1) *by* i^2 *when necessary. Compare your solutions with those on page* 439.

1. $3x^2 + 2x + 1 = 0$ **2.** $27z^2 - 6z + 2 = 0$ **3.** $27z^2 - 6z - 2 = 0$

Exercise 9-12

(B)

By completing the square and replacing (-1) *by* i^2 *when necessary, solve the following quadratic equations:*

1. $x^2 - 3x + 5 = 0$

2. $x^2 + x + 1 = 0$

3. $2x^2 - 4x + 3 = 0$

4. $4x^2 - 4x + 1 = 0$

5. $3x^2 - 2x + 1 = 0$

6. $\sqrt{3}x^2 - x - \sqrt{3} = 0$

7. $2x^2 - x + 1 = 0$

8. $2x^2 - x - 1 = 0$

***9·16* Equations solved as quadratic equations.** The equation

$$(x^2 - x)^2 - 8(x^2 - x) + 12 = 0$$

is not a quadratic equation in x; it is called a *quartic* equation, or equation of the fourth degree in x. Nevertheless, it is a quadratic equation in the variable $(x^2 - x)$ and can be solved either by using the general quadratic formula or by factoring, as follows:

For $y = x^2 - x$,

$$
\begin{aligned}
& (x^2 - x)^2 - 8(x^2 - x) + 12 = 0 \\
\leftrightarrow \quad & y^2 - 8y + 12 = 0 \\
\leftrightarrow \quad & (y - 2)(y - 6) = 0 \\
\leftrightarrow \quad & y - 2 = 0 \quad \text{or} \quad y - 6 = 0 \\
\leftrightarrow \quad & y = 2 \quad \text{or} \quad y = 6 \\
\leftrightarrow \quad & x^2 - x = 2 \quad \text{or} \quad x^2 - x = 6 \\
\leftrightarrow \quad & x^2 - x - 2 = 0 \quad \text{or} \quad x^2 - x - 6 = 0 \\
\leftrightarrow \quad & (x - 2)(x + 1) = 0 \quad \text{or} \quad (x - 3)(x + 2) = 0 \\
\leftrightarrow \quad & x = 2 \text{ or } x = -1 \quad \text{or} \quad x = 3 \text{ or } x = -2.
\end{aligned}
$$

Example 1. Solve $\dfrac{2x + 1}{x - 3} - \dfrac{x + 4}{x} = \dfrac{8}{5}$.

Solution.

Since division by zero is not defined, $\therefore x - 3 \neq 0, x \neq 0$.

For $x - 3 \neq 0$, $x \neq 0$,

$$
\begin{aligned}
& \frac{2x + 1}{x - 3} - \frac{x + 4}{x} = \frac{8}{5} \\
\leftrightarrow \quad & 5x(2x + 1) - 5(x - 3)(x + 4) = 8x(x - 3) \\
\leftrightarrow \quad & x^2 - 8x - 20 = 0 \\
\leftrightarrow \quad & (x + 2)(x - 10) = 0 \\
\leftrightarrow \quad & x = -2 \text{ or } x = 10
\end{aligned}
$$

The roots are -2 and 10.

Example 2. Solve $\dfrac{x + 3}{x} - x = 5 + \dfrac{3}{x}$.

Solutim.

Since division by zero is not defined, $\therefore x \neq 0$.

For $x \neq 0$,

$$
\begin{aligned}
& \frac{x + 3}{x} - x = 5 + \frac{3}{x} \\
\leftrightarrow \quad & x + 3 - x^2 = 5x + 3 \\
\leftrightarrow \quad & x^2 + 4x = 0 \\
\leftrightarrow \quad & x(x + 4) = 0 \\
\leftrightarrow \quad & x = 0 \text{ or } x = -4.
\end{aligned}
$$

Since $x \neq 0$, -4 is the root.

Example 3. Solve $(x^2 + 3x - 1)(x^2 + 3x - 2) = 6$.

Solution. This equation is a quartic equation in x; however it is a quadratic equation in $x^2 + 3x$.

For $y = x^2 + 3x$,

$$(x^2 + 3x - 1)(x^2 + 3x - 2) = 6$$
$$\leftrightarrow \quad (y - 1)(y - 2) = 6$$
$$\leftrightarrow \quad y^2 - 3y - 4 = 0$$
$$\leftrightarrow \quad (y - 4)(y + 1) = 0$$
$$\leftrightarrow \quad y = 4 \text{ or } y = -1$$
$$\leftrightarrow \quad x^2 + 3x - 4 = 0 \text{ or } x^2 + 3x + 1 = 0$$
$$\leftrightarrow x = -4 \text{ or } x = 1 \text{ or } x = \frac{-3 + \sqrt{5}}{2} \text{ or } x = \frac{-3 - \sqrt{5}}{2}.$$

Write solutions for the following problems and compare your solutions with those on page 440.

Solve:

1. $\dfrac{y^2 + 2}{y^2 - 1} + 2\left(\dfrac{y^2 - 1}{y^2 + 2}\right) = 3$

2. $(\log x)^2 - 5 \log x + 6 = 0$

3. $27(3^{2x}) - 242(3^x) - 9 = 0$

Example 4. Two cars each travel 330 miles. One of them travels 5 miles per hour faster than the other and covers the distance in a half-hour less time. Find the speeds of the two cars.

Solution. Represent the speed of the slower car by x miles per hour, $(x \in {}^+R)$, and the speed of the faster car by $(x + 5)$ miles per hour.

	SPEED IN M.P.H.	DISTANCE IN MILES	TIME IN HOURS
Slower car	x	330	$\dfrac{330}{x}$
Faster car	$x + 5$	330	$\dfrac{330}{x + 5}$

From the table, the difference in time is $\left(\dfrac{330}{x} - \dfrac{330}{x + 5}\right)$ hours.

But the difference in time is $\dfrac{1}{2}$ hour.

$$\therefore \frac{333}{x} - \frac{330}{x + 5} = \frac{1}{2}, \quad x \in {}^+R.$$

Complete this solution and compare it with that on page 440.

Exercise 9-13

(A)

For each of the following, state what condition must be taken in conjunction with equation (ii) in order that (i) and (ii) be equivalent.

1. (i) $x + \frac{1}{x} = 9$

(ii) $x^2 + 1 = 9x$

2. (i) $x - \frac{4}{x} = 3$

(ii) $x^2 - 4 = 3x$

3. (i) $-\frac{1}{x} + x = 2x - 2$

(ii) $-1 + x^2 = 2x^2 - 2x$

4. (i) $\frac{2}{x-3} = \frac{x^2 - 6x + 11}{x - 3}$

(ii) $2 = x^2 - 6x + 11$

5. (i) $\frac{x}{2} + x = \frac{3}{5}$

(ii) $15x = 6$

6. (i) $\frac{x^2 + 3x}{x^2 - 3x} + \frac{x^2 - 3x}{x^2 + 3x} = 7$

(ii) $y + \frac{1}{y} = 7$

(B)

Solve:

7. $\frac{2(x+9)}{5} + \frac{3}{x} = \frac{35}{6}$

8. $\frac{1}{u-1} - \frac{1}{u+1} = \frac{1}{24}$

9. $\frac{1}{x} + \frac{1}{x-3} = \frac{7}{3x-5}$

10. $t + t^{-1} = 2$

11. $\frac{y-3}{y-2} - \frac{y+4}{y} = \frac{3}{2}$

12. $x^4 - 13x^2 + 36 = 0$
(Hint: let $y = x^2$.)

13. $v^6 - 12v^3 + 35 = 0$

14. $(x + x^2)(1 + x + x^2) = 156$

15. $6\left(x + \frac{1}{x}\right)^2 - 35\left(x + \frac{1}{x}\right) + 50 = 0$

16. $\frac{x^2 + 2x}{3} + \frac{3}{x^2 + 2x} = 2$

17. $\frac{x^2 + 2}{x + 1} + \frac{x + 1}{x^2 + 2} = \frac{13}{6}$

18. $x^2 + \frac{1}{x^2} + x + \frac{1}{x} = 4$

19. $x(x-1)(x-2)(x-3) - 360 = 0$

20. $2^{2x} - 6(2^x) + 8 = 0$

21. $2(\log x)^2 - 15 \log x - 50 = 0$

22. $6(\log x)^2 - 7 \log x + 2 = 0$

23. $3^{2x} - 30(3^x) + 81 = 0$

24. A man drives to a town 286 miles away, and returns home, all in 12 hours. If his rate going was 8 miles per hour faster than his rate returning, find his average rate in each direction.

25. A manufacturer produces two types of clocks. One sells for \$4 more than the other. If a customer gets 5 more clocks for \$120 by buying the cheaper type, what is the price of each type?

(C)

Solve:

26. $4^x - 2(4^{-x}) = 1$ **27.** $5^x + 5^{1-x} = 6$

28. Two pipes are used for running water into a tank. When both pipes are used the tank can be filled in 48 minutes. The larger pipe alone will fill the tank in 40 minutes less time than the smaller one. Find the time required for each pipe separately to fill the tank.

***9•17* Equations involving radicals solved as quadratics.** The methods used to solve equations involving radicals may give rise to quadratic equations as illustrated in the following examples.

Example 1. Solve $\sqrt{4 - 3x} - x = 12$.

Solution.

$$\sqrt{4 - 3x} - x = 12 \qquad (1)$$

$$\leftrightarrow \quad \sqrt{4 - 3x} = x + 12 \qquad (2)$$ (This step isolates the term involving the radical.)

$$\rightarrow \quad 4 - 3x = (x+12)^2 \qquad (3)$$ (Note: This step is not reversible.)

$$\leftrightarrow \quad x^2 + 27x + 140 = 0 \qquad (4)$$

$$\leftrightarrow \quad (x + 7)(x + 20) = 0$$

$$\leftrightarrow \quad x + 7 = 0 \text{ or } x + 20 = 0$$

$$\leftrightarrow \quad x = -7 \text{ or } x = -20.$$

Since one step in the solution is not reversible, the last equation is not necessarily equivalent to the original equation. Thus it is necessary to verify in order to determine the root or roots.

Verification in (1). If $x = -7$,

L. S. $= \sqrt{4 - 3(-7)} - (-7)$ R. S. $= 12$.

$= \sqrt{25} + 7$

$= 5 + 7 = 12$. $\therefore$ -7 is a root of (1).

If $x = -20$,

L. S. $= \sqrt{4 - 3(-20)} - (-20)$ R. S. $= 12$.

$= \sqrt{64} + 20$

$= 8 + 20 = 28$. $\therefore$ -20 is not a root of (1).

-20 is a root of equation (3) but not of equation (2), because the step which produces (3) from (2) is not reversible.

In general, for any real numbers a and b:

if $a = b$, then $a^2 = b^2$;

however if $a^2 = b^2$, then $a = b$ or $a = -b$.

Thus *$a = b$ is not equivalent to $a^2 = b^2$.*

Hence, if in the solution of an equation any step involves the squaring of both sides of the equation, the roots of the original equation are determined only after verification by substitution into the original equation.

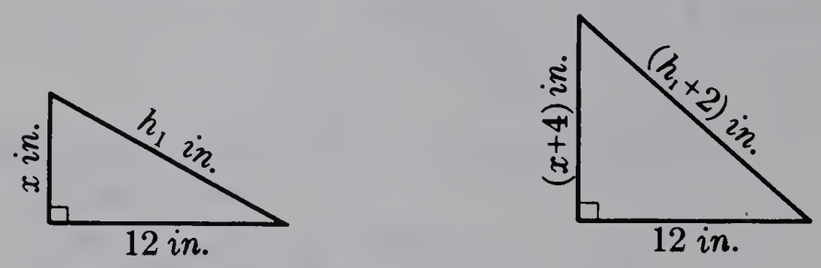
Write solutions to the following problems and compare your solutions with those on page 440.

Solve:

1. $r = 5 + \sqrt{1 + r}$ **2.** $\sqrt{4y - 3} = 1 + \sqrt{y + 1}$

Example 2. The longer of the two sides about the right angle of a right triangle is 12 inches long. If the shorter side were 4 inches longer, the hypotenuse would be 2 inches longer. Find the length of the shorter side.

Solution.



(*a*) (*b*)

Fig. 9-11

Represent the number of inches in the shorter side by x, $(x \in {}^+R)$, and the number of inches in the hypotenuse of the given right triangle by h_1.

If x is increased to $x + 4$, the length of the hypotenuse is increased to $(h_1 + 2)$ inches.

$h_1 = \sqrt{144 + x^2}$ (Theorem of Pythagoras)

$\therefore$ the length of the new hypotenuse is $(\sqrt{144 + x^2} + 2)$ inches.

But the length of the new hypotenuse is

$(\sqrt{144 + (x + 4)^2})$ inches. (Theorem of Pythagoras)

$\therefore \sqrt{144 + (x + 4)^2} = \sqrt{144 + x^2} + 2.$

Complete this solution and compare it with that on page 441.

Exercise 9-14

(A)

For each of the following pairs of equations, state whether the second is equivalent to the first; justify your answer:

1. $3x - 5 = 2x + 7$
$x - 12 = 0$

2. $\sqrt{x + 2} = 2x - 3$
$x + 2 = 4x^2 - 12x + 9$

3. $2 - x = x - 6$
$2x - x^2 = x^2 - 6x$

4. $\sqrt{x} = 3$
$x = 9$

5. $\sqrt{x} + 3 = x + 4$

$x + 6\sqrt{x} + 9 = x^2 + 8x + 16$

6. $x + \frac{1}{x} = 2$

$x^2 + 1 = 2x$

7. $x + 5 = \sqrt{x}$

$x^2 + 25 = x$

8. $\sqrt{x} + 5 = x + 2$

$x + 25 = x^2 + 4x + 4$

(B)

Solve:

9. $\sqrt{p - 3} = p - 5$

10. $2 + \sqrt{x^2 + 3} = 0$

11. $3x + \sqrt{6x + 37} + 11 = 0$

12. $\sqrt{2x - 3} - \sqrt{x + 2} = 1$

13. $\sqrt{2y + 3} - \sqrt{5y + 1} + 1 = 0$

14. $\sqrt{2x + 1} = 2 + \sqrt{4x + 6}$

15. $(x^{\frac{1}{2}} + (4x)^{\frac{1}{2}})^{-1} = 3$

16. $\sqrt{2x-2} - \sqrt{x-3} = \sqrt{x-15}$

17. $x^{-\frac{1}{2}} + (4x)^{-\frac{1}{2}} = 3$

18. $3\left(\frac{x + 2}{3x - 8}\right) = \sqrt{\frac{x + 4}{x - 4}}$

19. A positive real number exceeds its square root by 6. Find the number.

20. How long is each side of a square if a diagonal is 10 inches longer than a side?

21. How much must be added to the length of a rectangle 8 inches by 6 inches in order to increase the length of the diagonal by 2 inches?

22. If the perimeter of a rectangle is 34 ft. and the length of a diagonal is 13 ft., find the lengths of the sides.

(C)

23. The numerator of a fraction is x and the denominator is $\sqrt{x + 14}$. If the numerator were decreased by 18 and the denominator doubled, the value of the fraction would be decreased by $\frac{10}{3}$. Find x, $x \in R$.

24. $8x^3 - 19x^{\frac{3}{2}} - 27 = 0$

25. $x^{\frac{1}{3}} - (2x^2)^{\frac{1}{3}} = -4^{\frac{1}{3}}$

***9·18* Graphs defined by quadratic inequalities.** The graph of the quadratic function

$$q = \{(x, y) \mid y = x^2 + 3x - 4, x \in R\}$$

is a parabola opening upwards with x-intercepts -4 and 1, y-intercept -4, and vertex with coordinates $(-\frac{3}{2}, -\frac{25}{4})$.

The graph separates the xy-plane into three sets of points defined by:

(i) $q = \{(x, y) \mid y = x^2 + 3x - 4, x \in R\}$;

(ii) $A = \{(x, y) \mid y < x^2 + 3x - 4, x, y \in R\}$;

(iii) $B = \{(x, y) \mid y > x^2 + 3x - 4, x, y \in R\}$.

The graph of q is the parabola described above. The graph of A is the set such that the ordinate of a point in the set is less than the ordinate of the point on the parabola with the same abscissa. Thus the graph of A is the set of points *below* the parabola defined by q. The graph of B is the set of points *above* the parabola defined by q.

The graph of A is shaded in *Fig. 9-12*. (The parabola is indicated by a broken line since it is not part of A.)

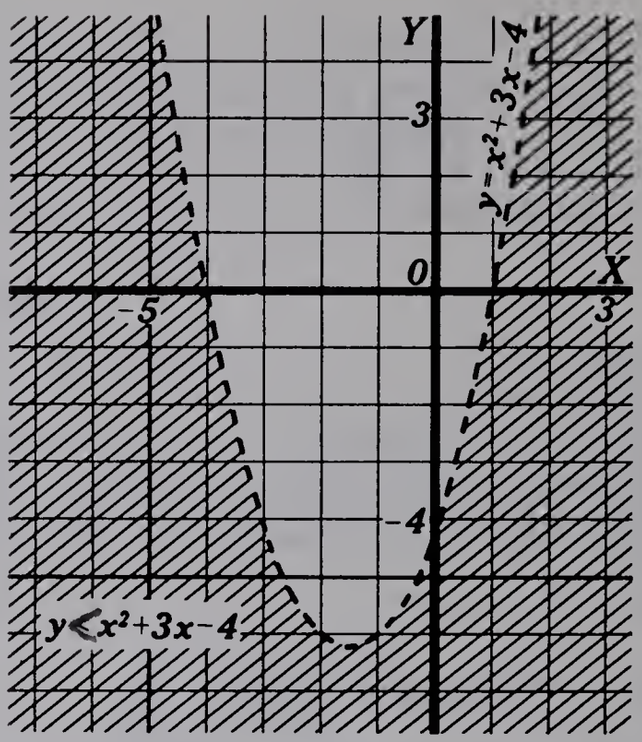


Fig. 9-12

Example 1. Sketch the graph of the relation

$$C = \{(u, v) \mid v - u \leqq -2u^2 + 3u + 1,\ u, v \in R\}.$$

Solution.

$$v - u \leqq -2u^2 + 3u + 1$$
$$\leftrightarrow \quad v \leqq -2u^2 + 4u + 1.$$

The graph of C is the set of points which lie on or below the graph of

$$q = \{(u, v) \mid v = -2u^2 + 4u + 1,\ u \in R\}.$$

The graph of q is a parabola opening downwards with u-intercepts such that $u = \dfrac{-4 \pm \sqrt{24}}{-4} = 1 \pm \frac{1}{2}\sqrt{6} \doteq -0.2$ or 2.2.

$$v = -2u^2 + 4u + 1$$
$$\leftrightarrow \quad v = -2(u^2 - 2u) + 1$$
$$\leftrightarrow \quad v = -2(u - 1)^2 + 3.$$

$\therefore$ q has a maximum value of 3, occurring for $u = 1$.

$\therefore$ the coordinates of the vertex of the parabola are (1, 3).

The graph of C is the shaded region of *Fig. 9-13*. (The parabola is indicated by a solid line since it is part of C.)

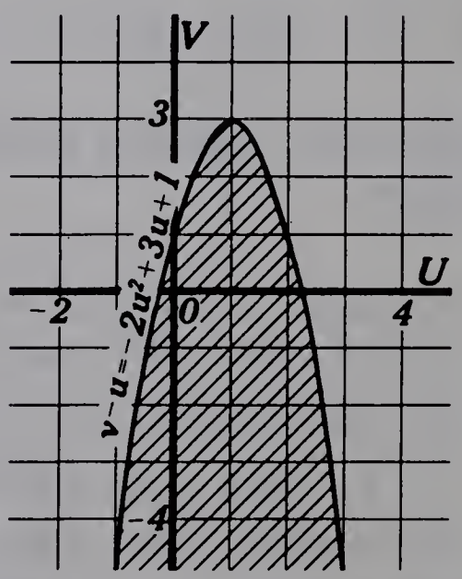


Fig. 9-13

Example 2. Sketch the graph of the relation

$$D = \{(x, y) \mid y \geqq x + 1 \text{ and } y < -4x^2 + 12x + 8,\ x, y \in R\}.$$

Solution. The graph of D consists of all points whose coordinates satisfy both inequations

(i) $y \geqq x + 1$

and (ii) $y < -4x^2 + 12x + 8$.

$\therefore$ the graph of D consists of all points which are

(i) above or on the line with equation $y = x + 1$,

and (ii) below the parabola with equation $y = -4x^2 + 12x + 8$.

The parabola opens downward and has y-intercept 8, and x-intercepts such that

$$x = \frac{-12 \pm \sqrt{272}}{-8} = \frac{3}{2} \mp \frac{1}{2}\sqrt{17}$$

$$\doteq -0.55 \text{ or } 3.55.$$

$$y = -4x^2 + 12x + 8$$

$$\leftrightarrow \quad y = -4(x^2 - 3x) + 8$$

$$\leftrightarrow \quad y = -4\left(x - \frac{3}{2}\right)^2 + 8 + 9$$

$$\leftrightarrow \quad y = -4\left(x - \frac{3}{2}\right)^2 + 17.$$

$\therefore$ the coordinates of the vertex of the parabola are $\left(\frac{3}{2}, 17\right)$.

The graph of D is shaded in *Fig. 9-14*.

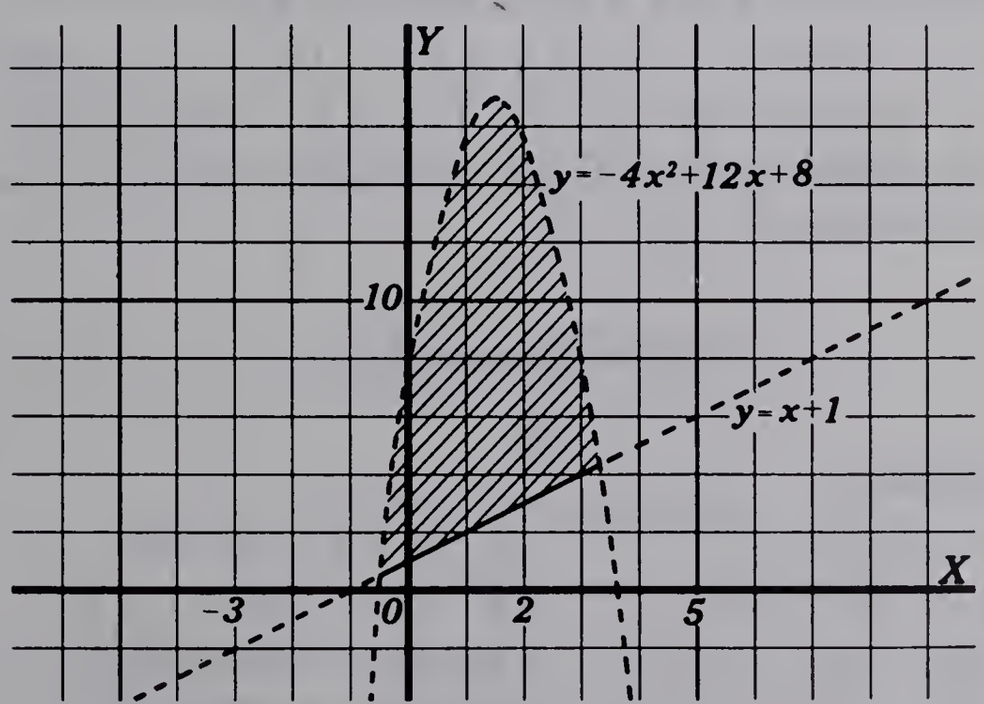


Fig. 9-14

Discuss, and sketch the graphs of the following relations. Compare your solutions with those on page 441.

1. $C = \{(x, y) \mid x^2 - 5x + 4 < y \leqq 3,\ x, y \in R\}$
   $\leftrightarrow C = \{(x, y) \mid x^2 - 5x + 4 < y \text{ and } y \leqq 3,\ x, y \in R\}$
2. $D = \{(r, s) \mid 2s > 2r^2 - 10r + 9,\ r, s \in R,\ 0 \leqq r \leqq 6\}$

Exercise 9-15

(B)

Sketch the graph of each of the following relations. If a boundary line is not to be included in the graph of the relation, indicate the boundary by a broken line.

1. $q_1 = \{(x, y) \mid y > x^2 + 3x - 4,\ x, y \in R\}$
2. $q_2 = \{(x, y) \mid y \leqq -3x^2 + 6x + 9,\ x, y \in R\}$
3. $q_3 = \{(u, v) \mid v + 2u \leqq 2u^2 - 10u + 10,\ u, v \in R\}$
4. $q_4 = \{(u, v) \mid 4u^2 - 6u \geqq 7u^2 - v - 9,\ u, v \in R\}$
5. $q_5 = \{(x, y) \mid y < 2x^2 - 8x + 1,\ x, y \in R\}$
6. $q_6 = \{(x, y) \mid x^2 < x + y + 2,\ x, y \in R\}$
7. $q_7 = \{(x, y) \mid 2 \leqq y < 2x^2 + 4x,\ x, y \in R\}$
8. $q_8 = \{(x, y) \mid x^2 < 7x - y + 5,\ x, y \in R,\ -1 < x \leqq 8\}$

(C)

9. $q_9 = \{(r, s) \mid 3r - s \leqq 4r^2 - 5r - 5,\ r, s \in R,\ s > -2\}$
10. $q_{10} = \{(r, s) \mid 2 > s - 6 > 6r^2 - 13r,\ r, s \in R\}$
11. $q_{11} = \{(x, y) \mid 3x^2 - 7x - 1 > y > 5x^2 - 12x - 2,\ x, y \in R\}$
12. $q_{12} = \{(x, y) \mid 2x^2 - 8x + 12 < y + x^2 \leqq 9x - 20,\ x, y \in R\}$
13. Find the coordinates of the points of intersection of the two boundary curves in problem 11.

Practice Exercise 9-16

(B)

Solve by factoring:

1. $x^2 - x - 42 = 0$
2. $y^2 - y - 72 = 0$
3. $25a^2 + 10a + 1 = 0$
4. $16m^2 + 8m + 1 = 0$
5. $2x^2 - x - 1 = 0$
6. $16l^2 - 15l - 1 = 0$
7. $2k^2 - 5k - 18 = 0$
8. $12c^2 - c - 20 = 0$
9. $-18a = a^2 + 81$
10. $17x - 1 = 16x^2$

Practice Exercise 9-17

(B)

Solve by completing the square:

1. $x^2 + x - 1 = 0$
2. $x^2 - x - 2 = 0$
3. $3x^2 + 18x + 17 = 0$
4. $2a^2 - a - 7 = 0$
5. $2c^2 - 3c - 5 = 0$
6. $-3x^2 + 7x - 2 = 0$
7. $16x - x^2 = 0$
8. $3e^2 - 14e - 2 = 0$
9. $3s^2 - 3s - 1 = 0$
10. $\frac{1}{3}m^2 - \frac{2}{3}m - 1 = 0$

$m^2 - 2m - 3$

Practice Exercise 9-18

(B)

Solve using the quadratic formula:

1. $2x^2 - x - 1 = 0$
2. $x^2 - 5x + 1 = 0$
3. $3y^2 + 10y + 5 = 0$
4. $2x^2 + 3x - 1 = 0$
5. $5v^2 - 6v - 12 = 0$
6. $2s^2 - 5s + 1 = 0$
7. $2x^2 - x + 1 = 0$
8. $2y^2 - 4y + 3 = 0$
9. $4 - 2a = a^2$
10. $10x^2 = 1 - 2x$

Practice Exercise 9-19

(B)

Find the maximum or minimum value of each of the following quadratic functions:

1. $f = \{(x, y) \mid y = 2x^2 - 2x + 2, x \in R\}$
2. $a = \{(x, y) \mid y = 2x^2 + 6x - 3, x \in R\}$
3. $b = \{(x, y) \mid y = (2x + 5)(x - 3), x \in R\}$
4. $c = \{(x, y) \mid y = 10 - 3(x + 2)^2, x \in R\}$
5. $d = \{(x, y) \mid y = 6 - 2x^2 + 3x, x \in R\}$
6. $e = \{(x, y) \mid y = (5 - 3x)(x + 5), x \in R\}$

Practice Exercise 9-20

(B)

Find the coordinates of the vertex of the graph of each of the quadratic functions defined by the equation in each of the following:

1. $y = x^2 - 1, x \in R$
2. $y = -x^2 - 1, x \in R$

3. $y = 3x^2 - x - 2,\ x \in R$
4. $y = -3x^2 - x - 2,\ x \in R$
5. $y = x^2 - x,\ x \in R$
6. $y = -x^2 - x,\ x \in R$
7. $y = x^2 - 2x + 2,\ x \in R$
8. $y = -x^2 - 2x + 2,\ x \in R$
9. $y = x^2 + \frac{1}{2}x + 4,\ x \in R$
10. $y = \frac{1}{3}x^2 + 2x - 5,\ x \in R$

Review Exercise 9-21

(A)

For the general quadratic function

$q = \{(x,\ y) \mid y = ax^2 + bx + c,\ x \in R\}$, *where* $a, b, c \in R, a \neq 0$:

1. If q has a maximum value which is positive, state the sign of a and of the discriminant $b^2 - 4ac$.
2. If q has the minimum value zero, state the sign of a, and of the discriminant.
3. If the graph of q is a parabola opening downwards, and if $b^2 - 4ac < 0$, is the y-intercept positive, negative, or zero?
4. If $a > 0$ and the y-intercept of the graph of q is negative, what is the sign of the discriminant?

(B)

For each of the following quadratic functions, determine the intercepts of the graph, the range of the function, the equation of the axis of symmetry, and the co-ordinates of the vertex of the graph. State whether the corresponding parabola opens upward or downward.

5. $q_1 = \{(x, y) \mid y = 4x^2 - 4x - 1,\ x \in R\}$
6. $q_2 = \{(x, y) \mid y = -x^2 + 10x - 16,\ x \in R\}$
7. $q_3 = \{(u, v) \mid v = 3u^2 + 6u - 9,\ u \in R\}$
8. $q_4 = \{(r, s) \mid s = 3r^2 - 6r + 15,\ r \in R\}$
9. $q_5 = \{(x, y) \mid y = (2x - 3)(2x + 5),\ x \in R\}$
10. $q_6 = \{(x, y) \mid y = 2x^2 - 360x + 16{,}102,\ x \in R\}$
11. $q_7 = \{(x, y) \mid y = 4x^2 - 4ax + a^2 - b^2,\ x \in R\}$, where $a, b \in R$
12. $q_8 = \{(x, y) \mid y = -x^2 - 2bx + a^2,\ x \in R\}$, where $a, b \in R$
13. $q_9 = \{(x, y) \mid y = x^2 + 2ax + 2bx + 2a^2 + 2b^2,\ x \in R\}$, where $a, b \in R, a \neq b$
14. Find the real number that exceeds $\frac{1}{4}$ of its square by the largest possible amount.
15. Show that $2x^2 - x - 10 \geqq -10\frac{1}{8}$ for all $x \in R$.

16. A rectangular lot is to be enclosed by a fence, and split into three lots by two other fences parallel to one of the sides. Find the maximum area of the lot if 660 yards of fencing are used.

17. Twenty inches of moulding are purchased to frame a rectangular picture. What is the maximum possible area, including the moulding?

18. If a car travelled 10 miles per hour faster, it would require 2 hours less time to travel 315 miles. What is the car's present speed?

19. A window is to be made in the shape of a rectangle surmounted by an equilateral triangle whose base is the upper side of the rectangle. If the perimeter of the window is 22 feet, find the width in order that the window have the maximum area.

20. A telephone exchange yields a net profit of \$12 a year for each subscriber if there are not more than 1000 subscribers. If the number of subscribers exceeds 1000, the profit on each telephone is decreased by as many cents as there are subscribers over 1000. What number of subscribers will give the maximum net profit for the exchange, and what is this profit?

21. Find two consecutive integers whose product is 272.

22. The perimeter of a rectangle is 20 inches and its area is 24 square inches. Find its dimensions.

23. One side of a right triangle is 10 units less than the hypotenuse, and the other side is 5 units less than the hypotenuse. Find the sides.

24. A number of footballs were purchased for \$150. If each football had cost \$5 more, 5 fewer would have been received. Find the actual price of each football.

25. The area of a square is trebled by adding 10 inches to one side and 12 inches to the other. Find the side of the square.

26. The length of a rectangle is 2 feet greater than its width. If the length is increased by 3 feet and the width by 1 foot, the area of the new rectangle will be twice the area of the old. What is the width and length of the original rectangle?

27. A swimming pool has two intake pipes. One will fill the pool in 10 hours and the other in 15 hours. If both pipes are open, how long will it take to fill the pool?

28. The number of lines of print on a page is 2 less than three times the average number of words per line. If the total number of words on a page is 408, find the average number of words per line.

29. A passenger train averages 16 m.p.h. more than a freight train on the same trip of 210 miles. If the passenger train covers the distance in $1\frac{1}{2}$ hours less time, find the average speed of the two trains.

30. A merchant bought goods listed at \$500 subject to two successive discounts of $x\%$, $x \in {}^{+}R$. If the second discount was \$32.55, find the value of x.

Solve each of the following equations:

31. $\dfrac{2x+1}{x-1} = \dfrac{x-1}{3} - x$

32. $x + \sqrt{x+2} = 10$

33. $(x^2 - 5x)(x^2 - 5x + 3) = 4$

34. $\sqrt{x-15} = 15 - \sqrt{x}$

35. $2x^{\frac{3}{2}} - 5x^{\frac{3}{4}} = 7$

36. $\dfrac{2}{x-3} = \dfrac{x^2 - 6x + 11}{x-3}$

37. $\sqrt{x^2 - 2x + 1} + \sqrt{x^2 - 2x - 6} = 7$

38. $6\sqrt{x^2 - 2x + 6} = 21 + 2x - x^2$

(C)

Sketch the graph of each of the following relations. If a boundary line is not to be included in the graph of the relation, indicate the boundary by a broken line.

39. $B = \{(x, y) \mid x^2 + 3x - 4 < y \leqq 1,\ x, y \in R\}$

40. $D = \{(x, y) \mid -3x^2 + 27x - 59 \leqq y \leqq 1,\ x, y \in R\}$

41. $E = \{(u, v) \mid v - 2u \leqq 5u^2 - 10u + 1,\ u, v \in R\}$

42. $N = \{(x, y) \mid 1 \leqq x - y < 11x - 4x^2 + 7,\ x, y \in R\}$

43. $P = \{(x, y) \mid 13 \leqq 10x - 4x^2 + 9 < y,\ x, y \in R\}$

44. $Q = \{(x, y) \mid 3x + 2y < 2x^2 - 5x + 7 < 10,\ x, y \in R\}$

45. $S = \{(x, y) \mid 2x - y \leqq -2x^2 + 10x - 3 \leqq y + 7,\ x, y \in R\}$

46. Find the coordinates of the points of intersection of the two boundary curves in problem 42.

Chapter X

INTRODUCTION TO THE THEORY OF QUADRATIC EQUATIONS

***10•1* Character of the roots of a quadratic equation.** In Chapter 9 we solved quadratic equations, including the general quadratic equation and some with complex roots. These solutions lead to the following theorem.

THEOREM. *The general quadratic equation*

$$ax^2 + bx + c = 0,\ a, b, c \in R,\ a \neq 0$$

with discriminant $D = b^2 - 4ac$ *has two complex roots.*

(i) *If* $D = 0$, *the roots are real and equal (coincident).*

(ii) *If* $D > 0$, *the roots are real and unequal (distinct).*

(iii) *If* $D < 0$, *the roots are unequal and non-real.*

In cases (i) and (ii), the roots of the general quadratic equation are given by the general quadratic formula:

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{or} \quad x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

In case (iii), the roots of the general quadratic equation are given by the general quadratic formula with $b^2 - 4ac$ replaced by $(4ac - b^2)i^2$, that is, $x_1 = \dfrac{-b + \sqrt{4ac - b^2}\, i}{2a}$ or $x_2 = \dfrac{-b - \sqrt{4ac - b^2}\, i}{2a}$, $i^2 = -1$.

The theorem shows that the *character of the roots* of a quadratic equation depends on the sign of the discriminant D.

Example 1. Determine, without solving the equations, the character of the roots of the equations:

(i) $x^2 - 2\sqrt{2}x + 2 = 0$ (ii) $6z^2 + z - 11 = 0$

(iii) $5y^2 + 7y + 2 = 0$ (iv) $2z^2 - 3z + 2 = 0$.

Solution. (i) $D = b^2 - 4ac$

$= 8 - 8 = 0.$

$\because D = 0,$

$\therefore$ the roots are real and equal.

(ii) $D = 1 + 264$

$= 265.$

$\because D > 0,$

$\therefore$ the roots are real and unequal.

(iii) $D = 49 - 40 = 9.$

$\because D > 0,$

$\therefore$ the roots are real and unequal.

(iv) $D = 9 - 16 = -7.$

$\because D < 0,$

$\therefore$ the roots are non-real.

Example 2. For what real values of k will the equations

(i) $2x^2 + 3x + 1 - 2k = 0,$ (ii) $k(x + 1) = x(2 - kx),$

have real and equal roots?

Solution. (i) The discriminant $D = 9 - 8(1 - 2k)$

$= 1 - 16k.$

$\therefore$ the roots are real and equal if and only if

$1 - 16k = 0, k \in R,$

or $k = \frac{1}{16}.$

(ii) $k(x + 1) = x(2 - kx)$

$\leftrightarrow kx^2 + (k - 2)x + k = 0.$

$\therefore D = (k - 2)^2 - 4k^2$

$= -3k^2 - 4k + 4.$

$\therefore$ the roots are real and equal if and only if

$3k^2 + 4k - 4 = 0, k \in R,$

or $(3k - 2)(k + 2) = 0$

that is, $k = \frac{2}{3}$ or $k = -2.$

Example 3. If $k \in R$, $k \neq 0$, show, without solving the equation, that the roots of the equation

$$x^2 + (k + 2)x + 2k^2 + k + 1 = 0$$

are non-real numbers.

Solution. The discriminant $D = (k + 2)^2 - 4(2k^2 + k + 1)$

$= k^2 + 4k + 4 - 8k^2 - 4k - 4$

$= -7k^2.$

$\because$ D < 0 if $k \in R$, $k \neq 0,$

$\therefore$ the equation has non-real roots if $k \in R$, $k \neq 0.$

Solve the following problems and compare your solutions with those on page 442.

1. (i) Determine the character of the roots of the equation

$$2x(x + 1) = x - 4.$$

(ii) Confirm the result by finding the roots using the general quadratic formula.

2. Determine the real values of k for which the roots of the equation

$$2x(x + 1) = x - k$$

are (i) equal, (ii) real and unequal, (iii) non-real.

Exercise 10-1

(B)

Determine the character of the roots of each of the following equations:

1. $12x^2 - x - 6 = 0$

2. $3x^2 - 9x + 2 = 0$

3. $3y^2 - 4y + 10 = 0$

4. $z^2 - 81 = 0$

5. $z^2 - 8z + 16 = 0$

6. $4y^2 + 7y + 3 = 0$

7. $4y^2 + 7y + 4 = 0$

8. $\sqrt{5}z^2 - 3z - \sqrt{5} = 0$

9. $x^2 + 4x + 2 = 0$

10. $5x^2 - 4x + 5 = 0$

11. $9z^2 + 10z + 3 = 0$

12. $\sqrt{3}x^2 - 2x - \sqrt{3} = 0$

13. $121x^2 - 26x + \sqrt{2} = 0$

14. $\sqrt{7}x^2 - 3\sqrt{2}x + \frac{\pi}{2} = 0$

15. $z^2 + \pi z + \sqrt{5} = 0$

Determine the real values of k for which each of the following equations has real and equal roots:

16. $3x^2 - 4x - (3 + k) = 0$

17. $(k-1)x^2+5x+(k+1)=0$

18. $x^2-2x(1+3k)+7(3+2k)=0$

19. $x^2 - 15 - k(2x - 8) = 0$

(C)

Determine the real values of k for which each of the following equations has real and equal roots:

20. $3x^2 + kx + 2x = 1 - k$

21. $kx^2 + 4kx = x - 4k$

22. $(kx + 1)(x - 2) + k = 0$

23. $\dfrac{3x + 2k}{3x + 6} = x + k$

24. For what values of $k \in R$ does the equation of problem 16 have (i) real and unequal roots, (ii) non-real roots?

25. For what values of $k \in R$ do the equations of problems 17, 19, 20, and 18 respectively have (i) real and unequal roots, (ii) non-real roots?

Use the quadratic formula to solve the following equations for $x \in C$:

26. $x^2 + 2x + 2 = 0$ **27.** $4x^2 - 12x + 25 = 0$

28. $3x^2 + 3x + 7 = 0$ **29.** $\pi x^2 + 2x + 5 = 0$

10-2 The sum and product of the roots of the general quadratic equation.

Discovery Exercise 10-2

Write solutions for the following problems and compare them with those on page 443.

1. Complete the following table.

EQUATION	a	b	c	ROOTS	SUM OF ROOTS	PRODUCT OF ROOTS
$x^2 - 3x + 2 = 0$	1	-3	2	1, 2	3	2
$x^2 + \frac{1}{4}x, - \frac{1}{8} = 0$						
$x^2 - 7x + 12 = 0$						
$x^2 - 4x - 3 = 0$						
$2x^2 + 7x + 3 = 0$						
$3x^2 - 4x - 4 = 0$						
$5x^2 + 19x - 4 = 0$						

2. (i) By examining the "PRODUCT OF ROOTS" column of the table suggest an expression for the product of the roots of the equation

$$ax^2 + bx + c = 0,\ a \neq 0,$$

in terms of some or all of the coefficients a, b, c.

(ii) By examining the "SUM OF ROOTS" column of the table suggest an expression for the sum of the roots of the equation

$$ax^2 + bx + c = 0,\ a \neq 0,$$

in terms of some or all of the coefficients a, b, c.

3. Prove the results of 2 by finding (i) $x_1 + x_2$ (ii) $x_1 x_2$

where $x_1 = \dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$, $x_2 = \dfrac{-b - \sqrt{b^2 - 4ac}}{2a}$.

The solutions of these problems lead to the following theorem.

THEOREM: *If a, b, c are real, $a \neq 0$, and x_1, x_2 are the roots of the quadratic equation*

$$ax^2 + bx + c = 0,$$

then $x_1 + x_2 = -\dfrac{b}{a}$, $x_1 x_2 = \dfrac{c}{a}$.

Example 1. Find, without solving the equation, the sum and product of the roots of the quadratic equation $x^2 - 9x - 4 = 0$.

Solution. Sum of roots $= -\left(\frac{-9}{1}\right) = 9.$

Product of roots $= \frac{-4}{1} = -4.$

Example 2. If $\frac{3}{4}$ is one root of the equation $16x^2 - 8x - 3 = 0$, find the other root.

Solution. Represent the second root by x_1,

then $\frac{3}{4} + x_1 = \frac{8}{16}$

or $x_1 = -\frac{1}{4}.$

Solve the following problems and compare your solutions with those on page 443.

1. Find the sum and product of the roots of each of the quadratic equations:
   (i) $x^2 + 3x + 1 = 0$ (ii) $\sqrt{2}z^2 - 5z + \sqrt{2} = 0$
   (iii) $3r^2 - 7r + 3 = 0.$
2. Show that the roots of the quadratic equation
   $$ax^2 + bx + a = 0, (a, b \in R, a \neq 0)$$
   are reciprocals.
3. If one root of the equation $x^2 - 64x + c = 0$ is 35, determine c.

Exercise 10-3

(A)

Find, without solving the equation, the sum and product of the roots of each of the following quadratic equations:

1. $4z^2 + 28z + 53 = 0$
2. $7x^2 - 20x - 35 = 0$
3. $y^2 + \sqrt{2}y - 1 = 0$
4. $11z^2 - 121z - 40 = 0$
5. $2x^2 + \pi x - 2 = 0$
6. $3x^2 - 5 = 0$

(B)

7. (i) Show that not both of $\frac{1}{3}$ and -2 can be roots of the equation $3x^2 + 2x - 1 = 0$ by means other than verifying by substitution in the equation.
   (ii) Given that $\frac{1}{3}$ is a root, find the other root.

8. If the roots of $ax^2 - 11x + 2 = 0$ are reciprocals, determine a.

9. If each root of $7z^2 + bz - 5 = 0$ is the negative of the other, determine b.

10. If one root of the equation $4z^2 - 12z + c = 0$ is twice the other root, determine c.

11. If one root of the equation $x^2 + bx - 5 = 0$ is $\frac{1}{2}$, determine b.

12. Find the sum and product of the roots of the equation $(2x - 3)(3x - 2) = (x + 1)(2x - 1)$.

13. In which of the following is at least one of the given numbers not a root of the equation?

(i) $x^2 + 3x - 28 = 0$; $-4, 7$

(ii) $x^2 - 10x + 24 = 0$; $4, 6$

(iii) $2x^2 + 5x + 2 = 0$; $2, \frac{1}{2}$

14. Determine the condition that the quadratic equation

$$ax^2 + bx + c = 0, a \neq 0$$

has (i) reciprocal roots;

(ii) roots which are the negative of each other;

(iii) one root zero.

(C)

15. Show that there is no real number k for which the equation $(kx + 1)(x - 2) = k$ will have equal roots. Find the real value of k for which one root is equal to zero. Find the real value of k for which the two roots are equal in absolute value, but opposite in sign.

16. If the sum of the roots of $ax^2 - 10x + 5a = 0$, $a \neq 0$, $a \in R$, equals twice their product, find a.

17. If the sum of the roots of $5x^2 + bx + 10b = 0$, $b \neq 0$, $b \in R$, plus their product is 10, find b.

18. Find the sum and the product of the roots of

$$(x - a)^2 + (x - b)^2 = (x - c)^2, \text{ if } a, b, c \in R.$$

19. By inspection, a is one root of the equation

$$(x - b)(x - c) = (a - b)(a - c).$$

Find the other root.

20. If each root of the equation

$$(x^2 - bx)\ (m + 1) = (m - 1)\ (ax - c)$$

is the negative of the other, determine an expression for m in terms of a, b, and c.

21. Determine the relation among a, b, c if one root of $ax^2 + bx + c = 0$ is double the other.

10·3 Formation of quadratic equations given the roots.

If x_1 and x_2 are the roots of the quadratic equation

$$ax^2 + bx + c = 0,\ a, b, c \in R,\ a \neq 0,$$

then

THE QUADRATIC EQUATION	THE QUADRATIC EXPRESSION
$ax^2 + bx + c = 0$	$ax^2 + bx + c$
$\leftrightarrow\ x^2 + \frac{b}{a}x + \frac{c}{a} = 0$	$= a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right)$
$\leftrightarrow\ x^2 - \left(-\frac{b}{a}\right)x + \frac{c}{a} = 0$	$= a\left[x^2 - \left(-\frac{b}{a}\right)x + \frac{c}{a}\right]$
$\leftrightarrow\ x^2 - (x_1 + x_2)x + x_1x_2 = 0$	$= a[x^2 - (x_1 + x_2)x + x_1x_2]$
$\leftrightarrow\ (x - x_1)(x - x_2) = 0$	$= a(x - x_1)(x - x_2)$
and hence if x_1 and x_2 are the roots of a quadratic equation, then the equation is equivalent to $(x - x_1)(x - x_2) = 0$.	and hence the factors of the quadratic expression $ax^2 + bx + c,\ a \neq 0$ are $a,\ x - x_1,\ x - x_2,$ where x_1 and x_2 are the roots of $ax^2 + bx + c = 0,\ a \neq 0$.

Also, if x_1 and x_2 are any real (or non-real) numbers such that

$$x_1 + x_2 = -\frac{b}{a}, \quad x_1x_2 = \frac{c}{a},\ a \neq 0,$$

then x_1, x_2 are the roots of the quadratic equation

$$ax^2 + bx + c = 0,\ a \neq 0.$$

Example 1. Verify that $-2 - \sqrt{5}$ and $-2 + \sqrt{5}$ are the roots of the equation $x^2 + 4x - 1 = 0$.

Solution. Let $x_1 = -2 - \sqrt{5}$ and $x_2 = -2 + \sqrt{5}$.

Then $x_1 + x_2 = (-2 - \sqrt{5}) + (-2 + \sqrt{5})$

$= -4,$

and $x_1x_2 = (-2 - \sqrt{5})(-2 + \sqrt{5})$

$= (-2)^2 - (\sqrt{5})^2$

$= 4 - 5$

$= -1.$

For the equation $x^2 + 4x - 1 = 0$, the sum of the roots is -4 and the product of the roots is -1.

$\therefore\ -2 - \sqrt{5}$ and $-2 + \sqrt{5}$ are the roots of the equation.

Example 2. Form a quadratic equation which has as roots

(i) 5 and -3 (ii) 4 and $\frac{1}{2}$ (iii) $\sqrt{2}+1$ and $\sqrt{2}-1$.

Solution. (i) An equation with roots 5 and -3 is:

METHOD 1	METHOD 2
$x^2-(5-3)x+5(-3)=0$	$(x-5)(x+3)=0$
or $x^2-2x-15=0$.	or $x^2-2x-15=0$.

Either of these methods may be used to form a quadratic equation given the roots. Method 1 is usually more convenient if the roots involve radicals.

(ii) An equation with roots 4 and $\frac{1}{2}$ is

$$(x-4)\left(x-\frac{1}{2}\right)=0$$

$$\text{or} \qquad x^2-\frac{9}{2}x+2=0$$

$$\text{or} \qquad 2x^2-9x+4=0.$$

(iii) An equation with roots $\sqrt{2}+1$, $\sqrt{2}-1$ is

$$x^2-(\sqrt{2}+1+\sqrt{2}-1)x+(\sqrt{2}+1)(\sqrt{2}-1)=0$$

$$\text{or} \qquad x^2-2\sqrt{2}x+1=0.$$

Example 3. Form an equation which has the roots 1, -1, and 2.

Solution. An equation with roots 1, -1, and 2 may be formed using method 2 of example 2. Such an equation is

$$(x-1)(x+1)(x-2)=0$$

$$\text{or} \qquad (x^2-1)(x-2)=0$$

$$\text{or} \qquad x^3-2x^2-x+2=0.$$

Example 4. Factor the expression $9x^2+36x-9$.

Solution.

$$9x^2+36x-9=0$$

$$\leftrightarrow \quad x^2+4x-1=0.$$

$$\therefore \quad x_1=\frac{-4+\sqrt{16+4}}{2} \text{ and } x_2=\frac{-4-\sqrt{16+4}}{2}$$

$$\text{or} \quad x_1=-2+\sqrt{5} \quad \text{and} \quad x_2=-2-\sqrt{5}.$$

$$\therefore \quad 9x^2+36x-9$$

$$=9(x^2+4x-1)$$

$$=9[x-(-2+\sqrt{5})][x-(-2-\sqrt{5})].$$

Exercise 10–4

(A)

For each of the following, form a quadratic equation which has the given numbers as roots:

1. $1, -1$ **2.** $1, 2$ **3.** $-3, 0$ **4.** $2, -3$
5. $-2, -3$ **6.** $4, 5$ **7.** $-4, \frac{1}{5}$ **8.** $-\frac{1}{4}, -\frac{1}{5}$
9. $2, \frac{1}{2}$ **10.** $\sqrt{2}, -\sqrt{2}$

(B)

For each of the following, form a quadratic equation which has the given numbers as roots:

11. $1\frac{1}{2}, -2\frac{1}{3}$ **12.** $1-\sqrt{2}, 1+\sqrt{2}$ **13.** $-7, \pi$
14. $-2-\sqrt{3}, -2+\sqrt{3}$ **15.** $\sqrt{3} \pm \frac{1}{2}$ **16.** $\frac{1}{4}(5 \pm \sqrt{11})$
17. $a \pm b$ **18.** $m + n, mn$

For each of the following, form a cubic equation which has the given numbers as roots:

19. $1, 2, 3$ **20.** $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$
21. $3-\sqrt{2}, 3+\sqrt{2}, 5$ **22.** $a, b, a+b$

23. Show that 1.25 and 4.16 are the roots of $100x^2 - 541x + 520 = 0$.

24. For which of the following are the given numbers roots of the equation?

(i) $x^2 - 6x + 5 = 0$; $2, 3$
(ii) $4x^2 - 12x + 7 = 0$; $\frac{1}{2}(3 \pm \sqrt{2})$
(iii) $x^2 - 2\sqrt{5}x - 2 = 0$; $\sqrt{3} \pm \sqrt{5}$
(iv) $x^2 - 2\pi x + \pi^2 = 3$; $\pi \pm \sqrt{3}$

Factor each of the following quadratic expressions by first finding the roots of the corresponding quadratic equation:

25. $x^2 - 6x + 6$ **26.** $y^2 + 17y - 110$ **27.** $x^2 - 9x + 19$
28. $6x^2 + 19x - 85$ **29.** $3x^2 - 12x + 1$ **30.** $1800y^2 - 5y - 1$

(C)

31. If $a, b \in R$, $a \neq \pm b$, form a quadratic equation with roots

$$\frac{a+b}{a-b}, \quad \frac{a-b}{a+b}.$$

32. Construct a quadratic equation for which the sum of the roots is 7, and the difference of the squares of the roots is 14.

10·4 The formation of quadratic equations whose roots are related to the roots of a given equation (supplementary).

Example 1. Without solving the equation $x^2 - 4x - 5 = 0$,

(i) find an equation whose roots are greater than the roots of the given equation by 2;

(ii) find an equation whose roots are reciprocals of the roots of the given equation.

Solution.

(i) METHOD 1.

Represent the roots of the given equation by x_1, x_2; then

$$x_1 + x_2 = 4 \quad \text{and} \quad x_1x_2 = -5.$$

Represent the roots of the required equation by y_1, y_2; then

$$y_1 = x_1 + 2 \quad \text{and} \quad y_2 = x_2 + 2.$$

$$\therefore \quad y_1 + y_2 = x_1 + 2 + x_2 + 2 \quad \text{and} \quad y_1y_2 = (x_1 + 2)(x_2 + 2)$$
$$= x_1 + x_2 + 4 \qquad\qquad = x_1x_2 + 2(x_1 + x_2) + 4$$
$$= 4 + 4 = 8. \qquad\qquad = -5 + 2(4) + 4 = 7.$$

A quadratic equation with roots y_1, y_2 is

$$x^2 - (y_1 + y_2)x + y_1y_2 = 0$$

or

$$x^2 - 8x + 7 = 0.$$

METHOD 2.

If $y = x + 2$, then $x = y - 2$.

For $x = y - 2$,

$$x^2 - 4x - 5 = 0 \qquad (1)$$
$$\leftrightarrow \quad (y - 2)^2 - 4(y - 2) - 5 = 0$$
$$\leftrightarrow \quad y^2 - 4y + 4 - 4y + 8 - 5 = 0$$
$$\leftrightarrow \quad y^2 - 8y + 7 = 0. \qquad (2)$$

$\therefore$ if x is any root of (1), $y = x + 2$ is a root of (2) and conversely.

$\therefore$ the roots of equation (2) are each greater by 2 than the roots of (1).

(ii) METHOD 1.

Represent the roots of the given equation by x_1, x_2; then

$$x_1 + x_2 = 4 \text{ and } x_1x_2 = -5.$$

Represent the roots of the required equation by y_1, y_2 ; then

$$y_1 = \frac{1}{x_1} \quad \text{and} \quad y_2 = \frac{1}{x_2}.$$

$$\therefore \quad y_1 + y_2 = \frac{1}{x_1} + \frac{1}{x_2} \quad \text{and} \quad y_1y_2 = \frac{1}{x_1} \times \frac{1}{x_2}$$
$$= \frac{x_2 + x_1}{x_1\, x_2} \qquad\qquad = \frac{1}{x_1x_2}$$
$$= -\frac{4}{5}. \qquad\qquad = -\frac{1}{5}.$$

A quadratic equation with the roots y_1, y_2 is

$$x^2 - (y_1 + y_2)x + y_1y_2 = 0$$

$$\text{or} \quad x^2 - \left(-\frac{4}{5}\right)x + \left(-\frac{1}{5}\right) = 0$$

$$\text{or} \quad 5x^2 + 4x - 1 = 0.$$

METHOD 2.

If $y = \frac{1}{x}, (x \neq 0)$, then $x = \frac{1}{y}$.

For $x = \frac{1}{y}$, $\quad x^2 - 4x - 5 = 0 \qquad (1)$

$$\leftrightarrow \left(\frac{1}{y}\right)^2 - 4\left(\frac{1}{y}\right) - 5 = 0$$

$$\leftrightarrow \quad 1 - 4y - 5y^2 = 0$$

$$\leftrightarrow \quad 5y^2 + 4y - 1 = 0. \qquad (2)$$

Since 0 is not a root of equation (1), it follows that if x is any root of equation (1), then $y = \frac{1}{x}$ is a root of equation (2), and conversely.

$\therefore$ the roots of equation (2) are the reciprocals of the roots of equation (1).

Example 2. Without solving the equation $5x^2 - 3x - 4 = 0$, find an equation whose roots are the squares of the roots of the given equation.

Solution.

METHOD 1.

Represent the roots of the given equation by x_1, x_2 ; then

$$x_1 + x_2 = \frac{3}{5} \quad \text{and} \quad x_1, x_2 = -\frac{4}{5}.$$

Represent the roots of the required equation by y_1, y_2, then

$$y_1 = x_1^2 \quad \text{and} \quad y_2 = x_2^2.$$

$$\therefore \quad y_1 + y_2 = x_1^2 + x_2^2 \qquad \text{and} \qquad y_1y_2 = x_1^2x_2^2$$

$$= (x_1 + x_2)^2 - 2x_1x_2 \qquad = (x_1x_2)^2$$

$$= \left(\frac{3}{5}\right)^2 - 2\left(-\frac{4}{5}\right) \qquad = \left(-\frac{4}{5}\right)^2$$

$$= \frac{49}{25}. \qquad = \frac{16}{25}.$$

A quadratic equation with roots y_1, y_2 is

$$x^2 - (y_1 + y_2)x + y_1y_2 = 0$$

$$\text{or} \quad x^2 - \frac{49}{25}x + \frac{16}{25} = 0$$

$$\text{or} \quad 25x^2 - 49x + 16 = 0.$$

METHOD 2.

If $y = x^2$, then $x = \sqrt{y}$ or $x = -\sqrt{y}$.

For $x = \pm\sqrt{y}$, $\quad 5x^2 - 3x - 4 = 0 \qquad (1)$

$\leftrightarrow 5(\sqrt{y})^2 - 3\sqrt{y} - 4 = 0$ or $5(-\sqrt{y})^2 - 3(-\sqrt{y}) - 4 = 0$

$\leftrightarrow \quad 5y - 4 - 3\sqrt{y} = 0$ or $\quad 5y - 4 + 3\sqrt{y} = 0$

$\leftrightarrow \quad (5y - 4 - 3\sqrt{y})(5y - 4 + 3\sqrt{y}) = 0$

$\leftrightarrow \quad (5y - 4)^2 - (3\sqrt{y})^2 = 0$

$\leftrightarrow \quad 25y^2 - 40y + 16 - 9y = 0$

$\leftrightarrow \quad 25y^2 - 49y + 16 = 0. \qquad (2)$

$\therefore$ if x is any root of equation (1), $y = x^2$ is a root of equation (2), and conversely.

$\therefore$ the roots of equation (2) are the squares of the roots of equation (1).

Exercise 10-5

(B)

1. Find a quadratic equation whose roots are twice those of $2x^2 + 7x - 9 = 0$.
2. Find a quadratic equation whose roots are 5 less than those of $x^2 + 10x + 1 = 0$.
3. Find a quadratic equation whose roots are the reciprocals of the roots of $3x^2 - 5x + 2 = 0$.
4. Find a quadratic equation whose roots are the reciprocals of the roots of $2x^2 + 7x - 9 = 0$.
5. Find a quadratic equation whose roots are the squares of the roots of $3x^2 - 5x + 2 = 0$.
6. Find a quadratic equation whose roots are the squares of the roots of $2x^2 + 5x - 7 = 0$.
7. Find a quadratic equation whose roots are the reciprocals of the roots of $\pi x^2 + \sqrt{2}x - 3 = 0$.
8. Find a quadratic equation whose roots are the squares of the roots of $x^2 + 2\pi x - \sqrt{2} = 0$.

(C)

If the roots of the equation $x^2 + px + q = 0$, $p, q \in R$, are x_1, x_2, find a quadratic equation whose roots are:

9. $\frac{1}{2}x_1$, $\frac{1}{2}x_2$
10. $\frac{1}{x_1}$, $\frac{1}{x_2}$
11. x_1^2, x_2^2

12. $x_1^2 + x_2^2, -2x_1x_2$ **13.** $x_1 + 2x_2, x_2 + 2x_1$ **14.** $x_1(x_2+1), x_2(x_1+1)$

15. If the roots of $ax^2 + bx + c = 0$ are n and kn, show that $kb^2 = (1 + k)^2ac$.

16. If m, n are the roots of $x^2 + px + q = 0$, and if $m + k$, $n + k$ are the roots of $x^2 + Px + Q = 0$, show that $p^2 - 4q = P^2 - 4Q$.

***10·5* The maximum number of roots of a given equation (supplementary).** Any equation of the form

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_{n-1}x + a_n = 0, \qquad (1)$$

where (i) n is a positive integer,

(ii) $a_0, a_1, \ldots, a_n$ are real numbers,

and (iii) $a_0 \neq 0$,

is called a *polynomial equation of degree n*.

Equations of degree 1, 2, 3, 4, and 5 are also called linear, quadratic, cubic, quartic (or biquadratic), and quintic equations, respectively.

The linear equation

$$a_0x + a_1 = 0, \quad a_0 \neq 0$$

has exactly one root given by the formula

$$x = -\frac{a_1}{a_0}.$$

The quadratic equation

$$a_0x^2 + a_1x + a_2 = 0, \quad a_0 \neq 0$$

has exactly two roots (possibly equal), given by the quadratic formula

$$x = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_0a_2}}{2a_0}.$$

These roots are non-real if $a_1^2 - 4a_0a_2 < 0$.

In certain cases, the real roots of cubic, quartic, and quintic equations may be determined. Some of the methods which may be used are illustrated in the following examples.

Example 1. Find all the real roots of each of the cubic equations

(i) $x^3 - x^2 + 2x - 2 = 0$ (ii) $x^3 + 6x^2 + 6x - 4 = 0$.

Solution. (i) For $x \in R$, $x^3 - x^2 + 2x - 2 = 0$

$$\leftrightarrow \quad x^2(x - 1) + 2(x - 1) = 0$$

$$\leftrightarrow \quad (x^2 + 2)(x - 1) = 0$$

$$\leftrightarrow \quad x^2 + 2 = 0 \quad \text{or} \quad x - 1 = 0$$

$$\leftrightarrow \quad x = 1.$$

$\therefore$ $x^3 - x^2 + 2x - 2 = 0$ has one real root, 1.

(ii) If $x = -2$, then $x^3 + 6x^2 + 6x - 4 = 0$.

$\therefore$ $x + 2$ is a factor of the left side of the equation.

By division, the second factor is $x^2 + 4x - 2$.

For $x \in R$, $x^3 + 6x^2 + 6x - 4 = 0$

$\leftrightarrow (x + 2)(x^2 + 4x - 2) = 0$

$\leftrightarrow x + 2 = 0$ or $x^2 + 4x - 2 = 0$

$\leftrightarrow x = -2$ or $x = -2 + \sqrt{6}$ or $x = -2 - \sqrt{6}$.

$\therefore$ $x^3 + 6x^2 + 6x - 4 = 0$ has three real roots,

$-2,\ -2 + \sqrt{6},\ -2 - \sqrt{6}$.

Discovery Exercise 10-6

Write solutions for the following problems and compare them with those on page 444.

Find all the real roots of each of the following equations.

1. $x^3 - 4x^2 - 7x + 10 = 0$

2. $2x^3 - 9x^2 - 8x + 15 = 0$

3. $x^3 + 3x^2 + 3x + 2 = 0$

4. $x^4 + x^2 - 2 = 0$

5. $x^4 + x^3 - 7x^2 - x + 6 = 0$

6. $x^5 - x^4 + 2x^3 - 2x^2 + 3x - 3 = 0$

7. $x^5 - 2x^4 - 8x^3 - x^2 + 2x + 8 = 0$

8. $x^5 - 3x^4 - 3x^3 + 12x^2 - 4x = 0$

9. From the above examples make a conjecture concerning the maximum number of real roots that

(i) a cubic equation may have;

(ii) a quartic equation may have;

(iii) a quintic equation may have.

These problems suggest, and it may be proven that:

A polynomial equation of degree n has at most n real roots.

10·6 **The roots of polynomial equations (supplementary).** We have concluded that a polynomial equation of degree n has at most n real roots. The following two questions are of interest:

(a) Does every polynomial equation have at least one root?

(b) Is there an algebraic formula (such as those for the roots of the linear and quadratic equations) for the roots of polynomial equations if $n \geqq 3$?

The answer to question (a) is "no" if only *real* roots are meant, since equations such as $x^2 + 1 = 0$, $x^4 + 1 = 0$, etc., have no real root. On the other hand, the German mathematician, *Carl Friedrich Gauss* (1777-1855), while still a young man of 21, proved that every polynomial equation (with real or complex coefficients) has at least one root in the *complex* number system C. This important theorem is now called the Fundamental Theorem of Algebra. It may also be shown that *every polynomial equation of degree n has exactly n roots in C.*

The answer to question (b) depends on whether $n = 3$ or 4, or $n \geqq 5$. In the years 1535-1545 the Italian mathematicians *Tartaglia, Cardano,* and *Ferrari* obtained algebraic formulas for the roots of the general cubic and the general quartic equations. It was not until almost three hundred years later that the Norwegian mathematician *Niels Henrik Abel* (1802-1829), and the French mathematician *Evariste Galois* (1811-1832), proved that there can be no algebraic formulas for the roots of the general polynomial equation for $n \geqq 5$.

Practice Exercise 10-7

(B)

Determine the character of the roots of each of the following equations:

1. $3x^2 - 6x + 2 = 0$

2. $4y^2 + 5y + 3 = 0$

3. $2x^2 - 2\sqrt{6}x + 3 = 0$

4. $\pi y^2 + 4y + 1 = 0$

5. $a^2x^2 + 2\sqrt{3}ax + 3 = 0,\ a \in R,\ a \neq 0$

6. $2y^2 + 3y + \pi = 0$

Determine the real values of k for which each of the following equations has real and equal roots:

7. $5x^2 - 4x - (5 + k) = 0$

8. $(k + 2)x^2 + 3x + (k + 3) = 0$

9. $x^2 - x(2 + 3k) + 7 = 0$

10. $x^2 + 3 - k(2x - 2) = 0$

11. $(k - 1)x^2 + 2x + (k + 1) = 0$

12. $(k + 2)x^2 + 5kx - 2 = 0$

Practice Exercise 10-8

(B)

Find, without solving the equation, the sum and the product of the roots of each of the following quadratic equations:

1. $2x^2 - 7x + 3 = 0$

2. $2z^2 - 4 + 3z = 0$

3. $3\pi - 2x^2 + x = 0$

4. $7x^2 - \pi x = \sqrt{2} + x^2$

5. $8x(x + 1) = 8x + 3$

6. $(3x + 7)\pi = 3(\pi x^2 + 1)$

7. If the roots of $(k + 2)x^2 + 3kx + 7 = 0$ are reciprocals, determine k.
8. If one root of the equation $3y^2 + 2y + 9b = 0$ is three times the other root, determine b.
9. If one root of the equation $2x^2 + 3cx + 2 = 0$ is $\frac{1}{3}$, determine c.
10. If one root of the equation $3x^2 + 7x + 7k = 3\pi$ is 0, determine k.
11. If the roots of the equation $4x^2 + (\sqrt{2}b - \pi) + 7 = 0$ are the negative of each other, determine b.
12. Is there a value of k, $k \in R$, for which the equation $(7 - k)x^2 + 2x + 3 - k = 0$ has reciprocal roots?

Practice Exercise 10-9

(B)

For each of the following, form an equation which has the given numbers as roots.

1. $2-\sqrt{3}, 2+\sqrt{3}$
2. $-3, 2\pi$
3. $-1-\sqrt{5}, -1+\sqrt{5}$
4. $\sqrt{7} \pm \frac{1}{3}$
5. $\frac{1}{3}(3 \pm \sqrt{7})$
6. $1, 3, 5$
7. $\frac{1}{3}, 1, \frac{1}{4}$
8. $3-\sqrt{5}, 3+\sqrt{5}, 7$
9. $-1-\sqrt{7}, -1+\sqrt{7}, 5$

Review Exercise10-10

(B)

Determine the character of the roots of each of the following equations:

1. $2x^2 - 12x + 18 = 0$
2. $3x^2 + 5x - 11 = 0$
3. $3\sqrt{2}x^2 - 6x + 5\sqrt{2} = 0$
4. $4x^2 - x - \sqrt{2} = 0$

Determine the real values of k for which each of the following equations has real and equal roots:

5. $kx^2 - 2x + 5 = 0$
6. $x^2 + 3x = k - 5$
7. $(k + 1)x^2 + (2k - 3)x + (k - 1) = 0$

Find, without solving the equation, the sum and product of the roots of each of the following quadratic equations:

8. $3x^2 - 9x + 7 = 0$
9. $4x^2 = 7$
10. $3x^2 + 7x + 3 = 0$
11. $2x^2 + 4x - 9 = 0$
12. If the roots of $2x^2 - 11x + k = 0$ are reciprocals, determine k.
13. If each root of $5y^2 + ky - 7 = 0$ is the negative of the other, determine k.
14. If one root of the equation $5x^2 + (k - 2)x - 7 = 0$ is $\frac{1}{5}$, determine k.

For each of the following, form a quadratic equation which has the given numbers as roots:

15. $\sqrt{5} + \frac{1}{3}, \sqrt{5} - \frac{1}{3}$ **16.** $\sqrt{5} - 3, \sqrt{3} + 2$

Factor each of the following quadratic expressions by first finding the roots of the corresponding quadratic equations:

17. $x^2 - 8x + 13 = 0$ **18.** $5x^2 - 12x + 2 = 0$

(C)

19. Determine the real values of k for which $kx^2 + 5kx = x - 6k + 2$ has (i) real and equal roots; (ii) real and unequal roots.

20. Find a quadratic equation whose roots are 5 greater than the roots of $2x^2 - 5x + 2 = 0$.

21. Find the real roots of $x^3 + 3x^2 - 5x - 10 = 0$.

Chapter XI

THE CIRCLE

11·1 Basic definitions.

(i) *Circle.*

A circle is the set of all points of a given plane, which are at a given distance from a given point of the plane.

The given point is the *centre* of the circle and the given distance is the *length of the radius* of the circle.

A *radius* of a circle is a line segment with one end point the centre and the other a point of the circle.

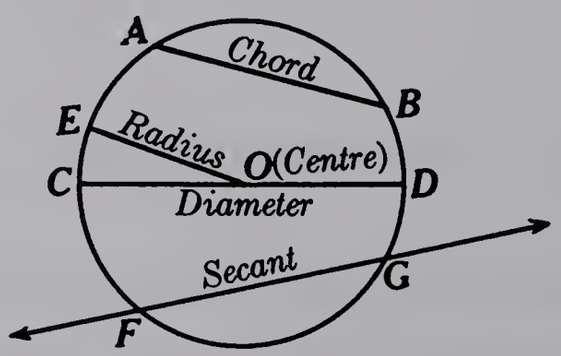


Fig. 11-1

It is customary to use the word *radius* to represent either the line segment or the length of the line segment. The context determines the meaning that is applicable.

(ii) *Interior and Exterior.*

The *interior* of a circle is the set of all points in the plane of the circle, such that the distance of each point from the centre is less than the radius.

The *exterior* of a circle is the set of all points in the plane of the circle, such that the distance of each point from the centre is greater than the radius.

(iii) *Disc or circular region.*

The union of a circle and its interior is called a *disc* or a *circular region.*

(iv) *Chord.*

A *chord* is a line segment whose end points are distinct points of the circle.

(v) *Diameter.*

A *diameter* is a chord on the centre.

(vi) *Secant.*

A line determined by any two distinct points of a circle.

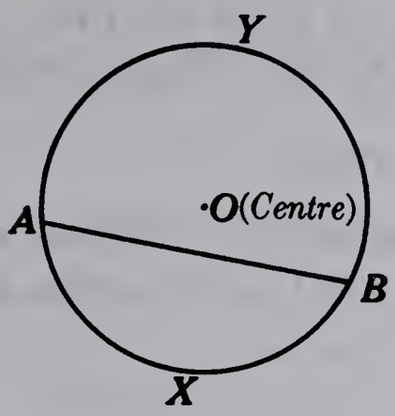


Fig. 11-2

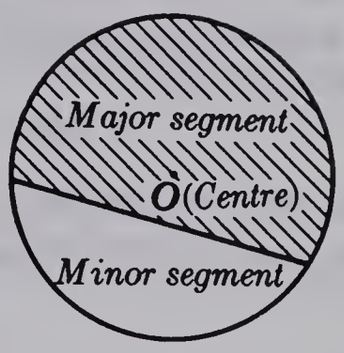


Fig. 11-3

(vii) *Arcs.*

The end points of a chord, which is not a diameter, separate the points of a circle into two subsets, each of which, including the end points, is called an *arc* of the circle.

A *major arc* is the arc which is on the same side of the chord as the centre. A *minor arc* is the arc on the opposite side of the chord to the centre. In *Fig. 11-2* AYB is a major arc, AXB is a minor arc.

(viii) *Semicircle.*

The end points of a diameter separate the points of a circle into two subsets, each of which, including the end points, is called a *semicircle.*

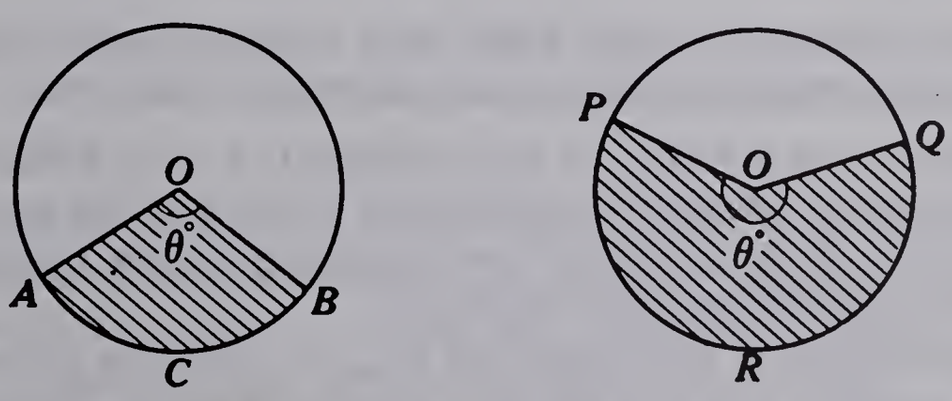
(ix) *Segment.*

A *segment* of a circle is the region bounded by a chord and the arc determined by its end points.

A *major segment*, *Fig. 11-3*, is a segment which contains the centre. A *minor segment* does not contain the centre.

(x) *Sector.*

A *sector* of a circle is the region bounded by an arc and the radii from its end points.



(*a*) (*b*)

Fig. 11-4

The angle determined by the radii of a sector is called the *sector angle.*

In *Fig. 11-4* (*a*) sector $OACB$ is determined by minor arc ACB. The sector angle is determined by rays on OA and OB, and its degree measure, θ, is the same as that of $\angle AOB$.

In *Fig. 11-4* (*b*) sector $OPRQ$ is determined by major arc PRQ. The sector angle is determined by rays on OP and OQ, and its degree measure, θ, is defined to be

$$360 - (\text{degree measure } \angle POQ).$$

When we are concerned with this degree measure of $\angle POQ$ we speak of *reflex angle POQ.* Thus the sector angle in *Fig. 11-4* (*b*) is reflex angle POQ.

11•2 Chord property theorems.

Write a proof for each of the following chord property theorems; compare your proofs for 1, 4, 6 *with those on page* 445.

1. The centre of a given circle lies on the right bisector of any chord of the circle.

2. If a line is on the centre of a circle and on the midpoint of a chord, then the line is perpendicular to the chord.

3. If a line on the centre of a circle is perpendicular to a chord, then it bisects the chord.

4. If chords of a circle are equidistant (perpendicular distances) from the centre of the circle, then they are equal in length.

5. If chords of a circle are equal, then they are equidistant from the centre of the circle.

6. Of two chords of a circle, the length of the one nearer to the centre is greater than the length of the other. (Hint: apply the Pythagorean Theorem.)

7. Of two chords of a circle which have different lengths, the chord having the greater length is nearer the centre of the circle.

8. A *locus* is a set of points. A point belongs to a locus if and only if it satisfies the conditions for membership in the set. To prove that a certain set of points is the locus satisfying certain conditions two converse propositions must be proved.

   For example, to prove that the locus of centres of circles on two given points A and B is the right bisector of the line segment AB, it is necessary and sufficient to prove:

(i) any point of the right bisector of AB is equidistant from A and from B;

(ii) any point equidistant from A and from B is a point of the right bisector of AB.

When (i) and (ii) have been proved it may be concluded:
"the locus of centres of circles on A and B is the right bisector of AB".
Prove this theorem.

9. Any three non-collinear points are points of a circle.

10. DEFINITION. The *circumcircle* of a triangle is the circle on the three vertices of the triangle.

The centre of this circle is called the *circumcentre* of the triangle.

Prove the centre of the circumcircle of a triangle is the point common to the right bisectors of two of the sides of the triangle.

***11·3* Relation between sector angle and sector arc.** In *Fig. 11-5* sector angle AOB = sector angle COD. Since OA, OB, OC, and OD are equal radii and $\angle AOB = \angle COD$, we can match the point sets of $\angle AOB$ and $\angle COD$ so that A is matched with C, and B with D.

Intuitively we see that there is a one-to-one matching of the points of arc AB with the points of arc CD which preserves distances between corresponding points. Thus arc AB is congruent to arc CD and arc AB = arc CD. This leads to the following arc length postulates:

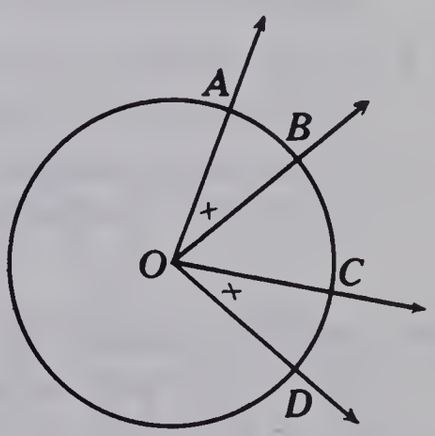


Fig. 11-5

1. *In the same circle or in equal circles (circles having equal radii), equal sector angles have equal sector arcs and equal arcs have equal sector angles.*

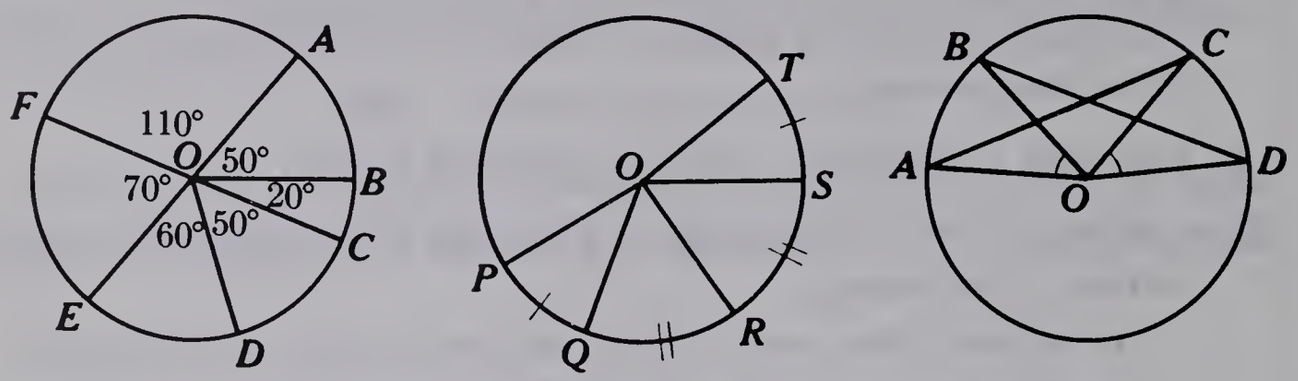
2. *In the same circle or in equal circles, if two sector angles are unequal they have unequal sector arcs, the greater sector angle having the greater sector arc; if two arcs are unequal, they have unequal sector angles, the greater arc having the greater sector angle.*

The length of an arc of a circle is discussed further in Section 11·13.

Exercise 11-1

(A)

1. In the diagram at the left below name pairs of equal arcs.



2. In the diagram, centre above, name pairs of equal angles.
3. In the diagram, right above, name pairs of equal arcs. Justify your answer.

(B)

4. In the same circle or in equal circles, if two chords are equal, they determine equal arcs.
5. In the same circle or in equal circles, if two arcs are equal, they determine equal chords.
6. A diameter perpendicular to a chord of a circle bisects both arcs determined by the chord.

11•4 **Inscribed angles.** In *Fig. 11-6 (a)*, $\angle ABC$ is inscribed in the circle or in the major segment ABC. Arc AXC or chord AC is said to subtend $\angle ABC$ at the circle. In *Fig. 11-6 (b)* $\angle DEF$ is inscribed in a semicircle. In *Fig 11-6 (c)* $\angle QPR$ is inscribed in the circle or in the minor segment

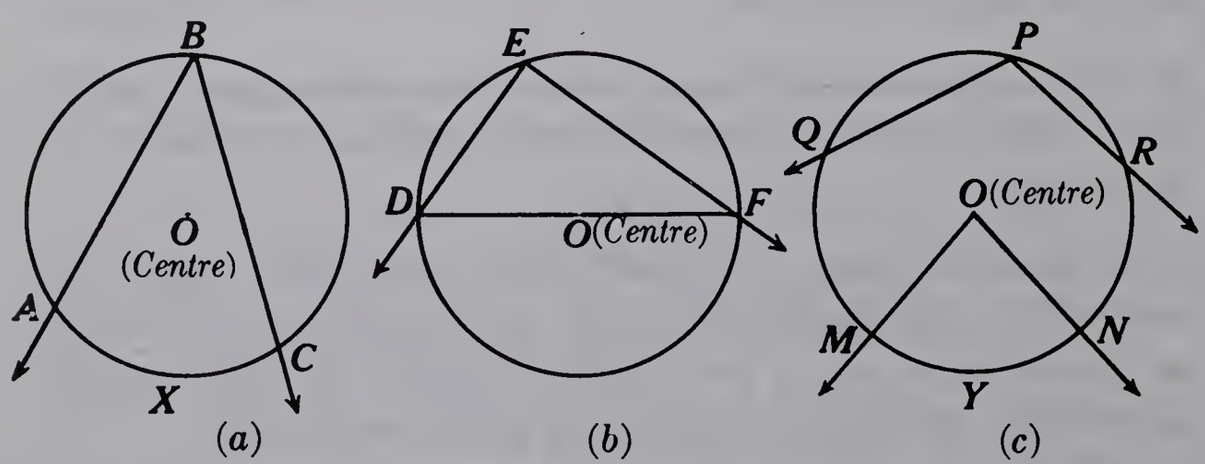


Fig. 11-6

QPR; $\angle QPR$ is subtended by arc $QMNR$ or by chord QR. Also sector angle MON may be described as the angle at the centre subtended by arc MYN (or chord MN).

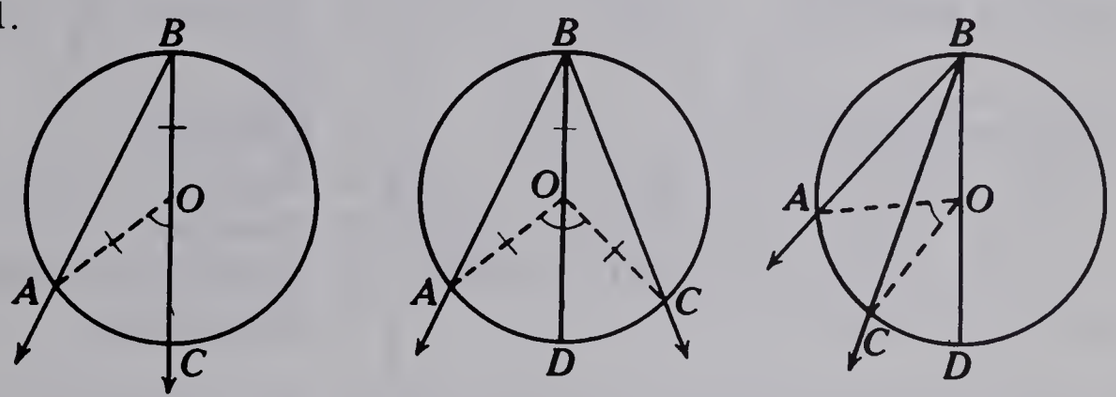
11•5 **Basic inscribed angle theorem.**

THEOREM

Inscribed Angle Theorem

An angle inscribed in a circle is one-half the sector angle subtended by the same arc.

CASE 1.



(*a*) (*b*) (*c*)

CASE 2.

CASE 3.

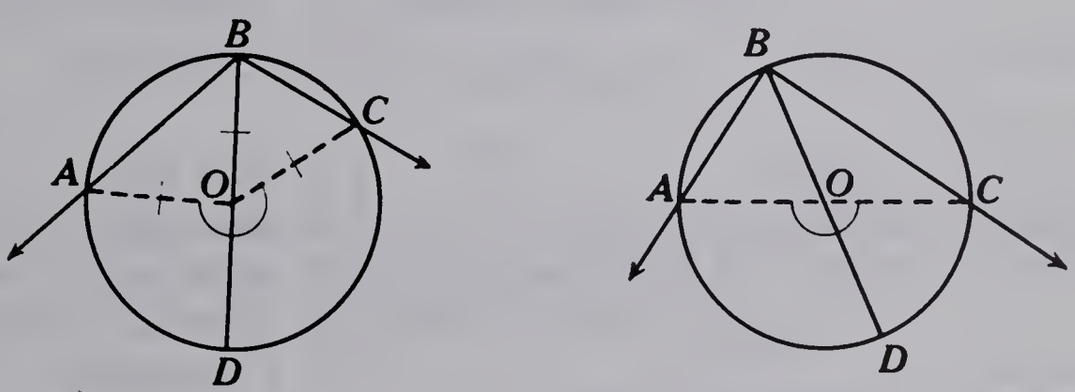


Fig. 11-7

Hypothesis: ABC is an inscribed angle of a circle with centre O and sector angle AOC subtended by the same arc, *Fig. 11-7.*

CASE 1. $\angle ABC$ *subtended by a minor arc:*

(a) O lies on the chord BA or the chord BC, say on BC.

(b) O lies on the same side of the chord BC as A.

(c) O lies on the opposite side of the chord BC to A.

CASE 2. $\angle ABC$ *subtended by a major arc.*

CASE 3. $\angle ABC$ *subtended by a semicircle.*

Conclusion: $\angle ABC = \frac{1}{2} \angle AOC$.

Proof:

STATEMENTS	AUTHORITIES
CASE 1.	
(*a*)	
1. $OB = OA$	1. Definition
2. $\angle OAB = \angle OBA$	2. Isosceles Triangle Th.
3. $\angle AOC = \angle OBA + \angle OAB$	3. Exterior Angle Th.
4. $\angle AOC = \angle OBA + \angle OBA$	4. Replacement
5. $\angle AOC = 2\ \angle OBA$	5. Addition
6. $\angle OBA = \frac{1}{2}\ \angle AOC$	6. Division
(*b*)	
1. $\angle ABD = \frac{1}{2}\ \angle AOD$	1. Case 1(*a*)
2. $\angle CBD = \frac{1}{2}\ \angle COD$	2. Case 1(*a*)
3. $\angle ABD + \angle CBD = \frac{1}{2}\ \angle AOD + \frac{1}{2}\ \angle COD$	3. Addition
4. $= \frac{1}{2}(\angle AOD + \angle COD)$	4. Distributive property
5. $\angle ABC = \frac{1}{2}\ \angle AOC$	5. Completion
(*c*)	
1. $\angle ABD = \frac{1}{2}\ \angle AOD$	1. Case 1(*a*)
2. $\angle CBD = \frac{1}{2}\ \angle COD$	2. Case 1(*a*)
3. $\angle ABD - \angle CBD = \frac{1}{2}\ \angle AOD - \frac{1}{2}\ \angle COD$	3. Subtraction
4. $= \frac{1}{2}(\angle AOD - \angle COD)$	4. Distributive property
5. $\angle ABC = \frac{1}{2}\ \angle AOC$	5. Completion
CASES 2 AND 3.	
1. $\angle ABD = \frac{1}{2}\ \angle AOD$	1. Case 1(*a*)
2. $\angle CBD = \frac{1}{2}\ \angle COD$	2. Case 1(*a*)
3. $\angle ABD + \angle CBD = \frac{1}{2}\ \angle AOD + \frac{1}{2}\ \angle COD$	3. Addition
4. $= \frac{1}{2}(\angle AOD + \angle COD)$	4. Distributive property
5. CASE 2. $\angle ABC = \frac{1}{2}$ (reflex angle AOC)	5. Completion
CASE 3. $\angle ABC = \frac{1}{2}$ (straight angle AOC)	

Corollary 1. Angles inscribed in a circle that are subtended by the same arc are equal.

Corollary 2. An angle inscribed in a semicircle is a right angle.

DEFINITION: An *inscribed quadrilateral* or *cyclic quadrilateral* is a quadrilateral having all its vertices on a circle.

Corollary 3. The opposite angles of an inscribed quadrilateral are supplementary.

Corollary 4. An exterior angle at a vertex of an inscribed quadrilateral is equal to the interior angle at the opposite vertex.

In each of the following O is the centre of the given circle; find x and y, and compare your solutions with those on page 446.

1. **2.** **3.**

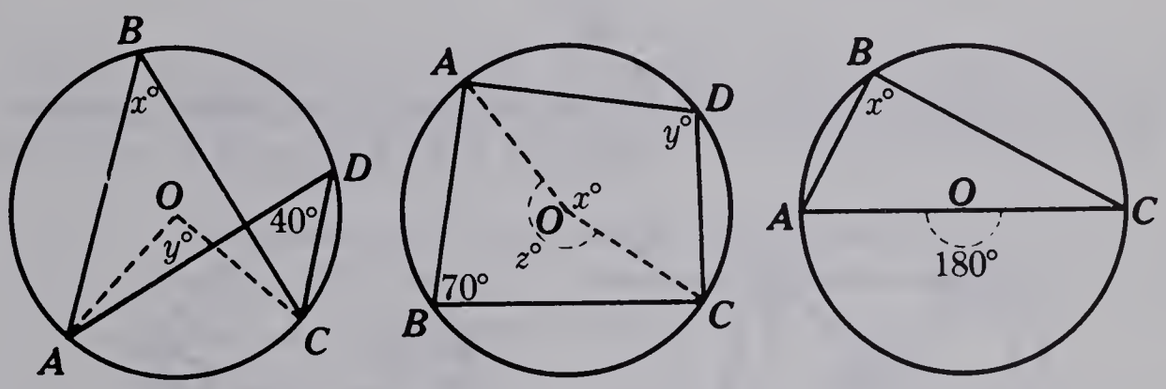


Fig. 11-8

Write proofs for each of the preceding corollaries; compare them with those on page 446.

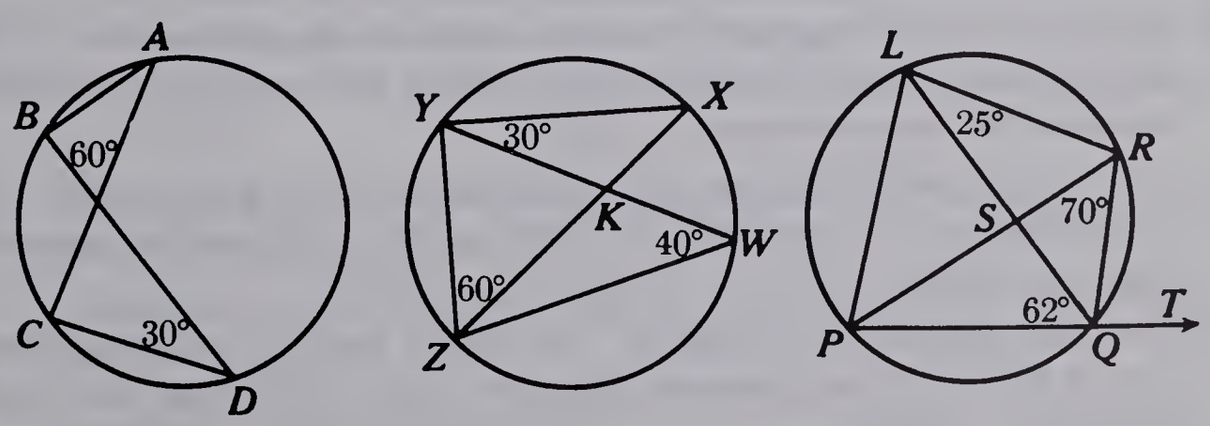
Exercise 11-2

Numerical Exercise

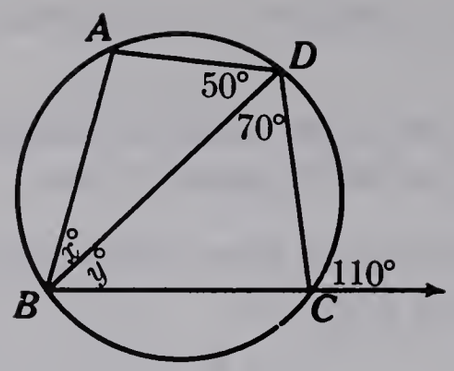
(A)

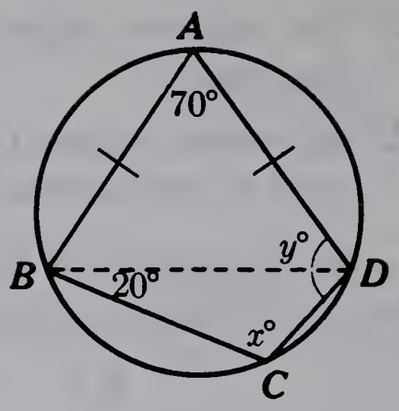
1. For each of the following, state the measurement of each of the angles whose measurement is not indicated:

(i) (ii) (iii)



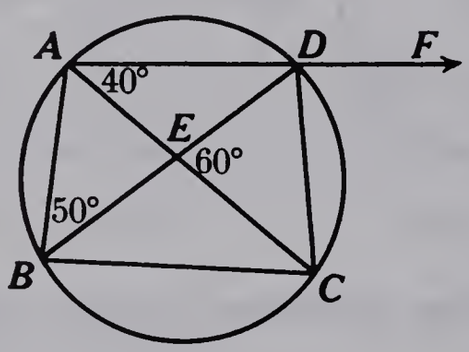
2. For each of the diagrams at the top of page 294 calculate the values of x and y:

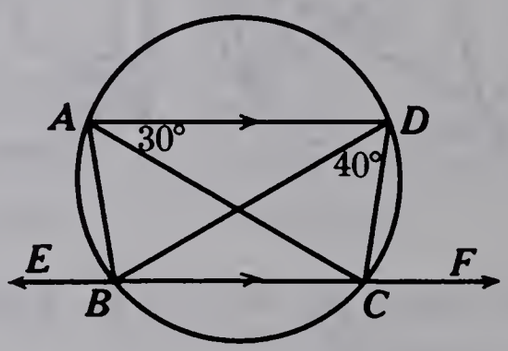




(B)

3. In the diagram at the left below, calculate the degree measure of (i) $\angle AED$ (ii) $\angle CBD$ (iii) $\angle ADC$ (iv) $\angle BAE$ (v) $\angle BCD$ (vi) $\angle CDF$.





4. In the diagram at the right above, chord AD is parallel to chord BC, $\angle DAC = 30°$, $\angle BDC = 40°$. Calculate the degree measure of (i) $\angle ADB$ (ii) $\angle ACD$ (iii) $\angle ABE$ (iv) $\angle DCF$.

For each of the following, draw a representative sketch, mark on all given data and complete the required calculation.

5. AOB is a diameter of a circle, centre O. P is any point of the circle. AE and BE bisect $\angle PAB$ and $\angle PBA$ respectively. Calculate the measurement of $\angle AEB$.

6. A, B, C, D are four points of a circle taken in the given order. COD is the diameter and O is the centre of the circle. If $\angle AOD = 40°$, calculate the measurement of $\angle ABC$.

7. A, B, C, D are four points of a circle taken in the given order. CD is the diameter of the circle. If $\angle BAC = 40°$, calculate the measurement of $\angle BCD$.

8. Two line segments AB and AC are drawn from A. Circles drawn with AB and AC as diameters intersect on D. Calculate the degree measure of $\angle BDC$.

9. $ABCD$ is a cyclic quadrilateral and the diagonals intersect on E. If $\angle DAC = 20°$ and $\angle ACB = 50°$, find the degree measure of $\angle AEB$.

10. A, B, C, D are four points of a circle taken in the given order. If $\angle DAC = 25°$, $\angle ADB = 33°$, and $\angle ABD = 41°$, find the degree measure of each of the four angles of the quadrilateral.

Exercise 11-3

(B)

1. ABC and ADE are two rays on a point A of the exterior of a circle. The rays intersect the circle on B, C and D, E respectively. Prove that $\triangle ADC$ is equiangular to $\triangle ABE$.

2. A, B, C, D are four points of a circle such that $AD \parallel BC$. If AC and BD intersect on E, a point of the interior of the circle, prove that triangles EBC and EAD are isosceles.

3. Prove that angles inscribed in a circle that are subtended by equal chords are equal.

4. A, B, C, D are four points of a circle taken in the given order. AD and BC intersect on E, a point of the exterior of the circle. Prove that $\angle DCF = \angle BAD$.

5. $ABCD$ is a cyclic quadrilateral. $\angle A = 2\angle C$, and $\angle B = 3\angle D$. Find the number of degrees in each angle of the quadrilateral.

6. If $\angle AXB$, $\angle BYC$, and $\angle CZA$ are angles inscribed in the segments of the circumcircle of $\triangle ABC$ which are exterior to the triangle, prove that $\angle AXB + \angle BYC + \angle CZA = 360°$.

7. If a quadrilateral is inscribed in a circle, find the sum of the inscribed angles in the four segments exterior to the quadrilateral.

8. Two circles of unequal radii intersect on X and Y. AXB is any line on X intersecting the circles on A and B respectively. Prove that the degree measure of $\angle AYB$ remains the same regardless of the position of line AXB.

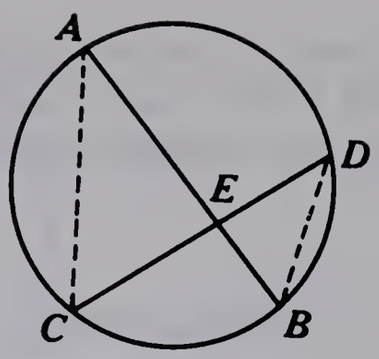
9. *If two chords of a circle intersect, the product of the segments of one is equal to the product of the segments of the other.*

Examine the following analysis of this theorem and then write a proof of the theorem.

Hypothesis: Chords AB and CD intersect on E.

Conclusion: $AE \cdot EB = CE \cdot ED$.

Analysis:



I CAN PROVE	IF I CAN PROVE
1. $AE \cdot EB = EC \cdot DE$	1. $\frac{AE}{EC} = \frac{DE}{EB}$
2. $\frac{AE}{EC} = \frac{DE}{EB}$	2. $\triangle AEC \sim \triangle DEB$
3. $\triangle AEC \sim \triangle DEB$	3. $\angle A = \angle D$ $\angle C = \angle B$

4. $\angle A = \angle D$ (Inscribed Angle Theorem)
   $\angle C = \angle B$ (Inscribed Angle Theorem)

10. AB and CD are two chords of a circle intersecting on X. If $AX = 4$ cm., $XD = 6$ cm., $CX = 3$ cm., calculate the length of AB to one-tenth centimetre.

11. AB and CD are any two chords of a circle. EF, a third chord, intersects AB on G and CD on H so that $EG = GH = HF$. Prove that $AG \cdot GB = HD \cdot CH$.

12. Two circles intersect on P and Q. CD is a chord of one circle on any point of the common chord of the circles. EF is a chord of the other circle on the same point of the common chord. Prove $CA \cdot AD = EA \cdot AF$.

(C)

13. $\triangle ABC$ is an equilateral triangle inscribed in a circle. P is any point of the minor arc AC. D is a point of the chord BP such that $BD = PC$. Prove (i) $\triangle ADB \cong \triangle APC$,

(ii) $\triangle ADP$ is equilateral,

(iii) $BP = AP + PC$.

11·6 The tangent at a point of a curve.

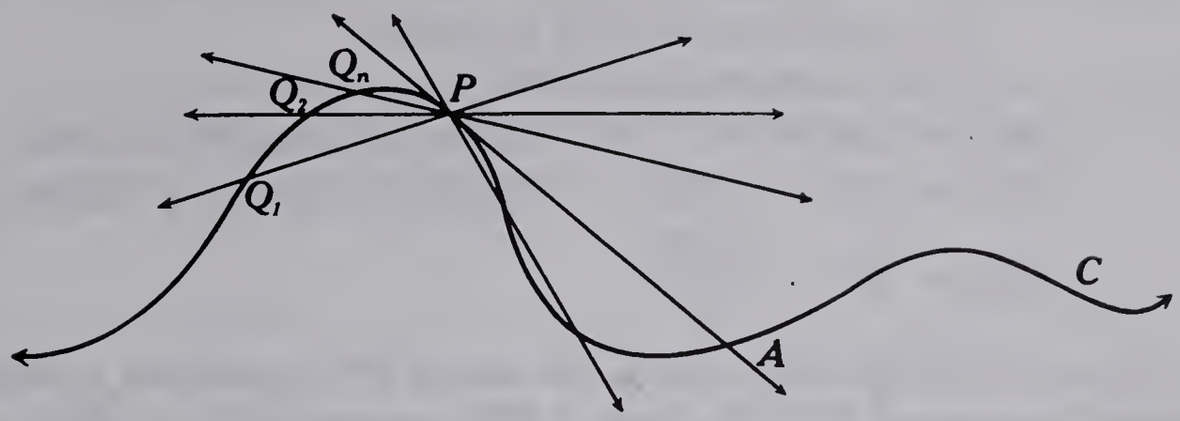


Fig. 11-9

In *Fig. 11-9* P is a point of the curve C. If Q is any other point of C, then the line determined by the two points P and Q is called a *secant* of the curve. In *Fig.11-9* lines PQ_1, PQ_2, . . ., PQ_n, . . . are members of the family of secants of the curve C, all of which lie on the point P.

The line PA which "*touches*" the curve C at the point P is said to be the *tangent (line) at the point* P of the curve C.

We may think of the position of the tangent line PA as being the *limiting position* of any family of secants PQ_1, PQ_2, . . ., PQ_n, . . . where Q_1, Q_2, . . ., Q_n, . . . are points of the curve in that order, and taken closer and closer to the point P. The line in the limiting position of the line PQ_n as Q_n approaches P is called the tangent line at P of the curve.

The point P is called the *point of contact* of the tangent. The tangent at a point P may intersect the curve at some other point. In *Fig.11-9* the tangent PA at P intersects the curve at A.

In *Fig. 11-10* the curve C is a *closed curve*, the tangent at P intersects the curve at A and B.

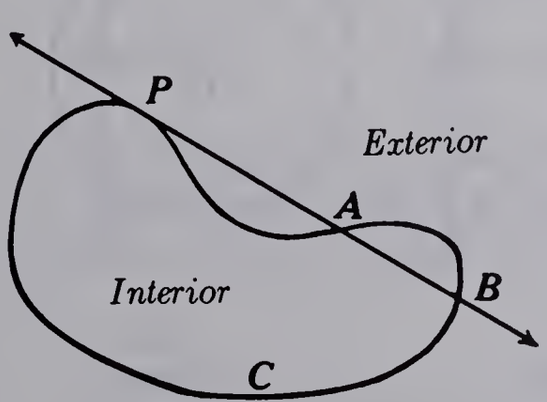


Fig. 11-10

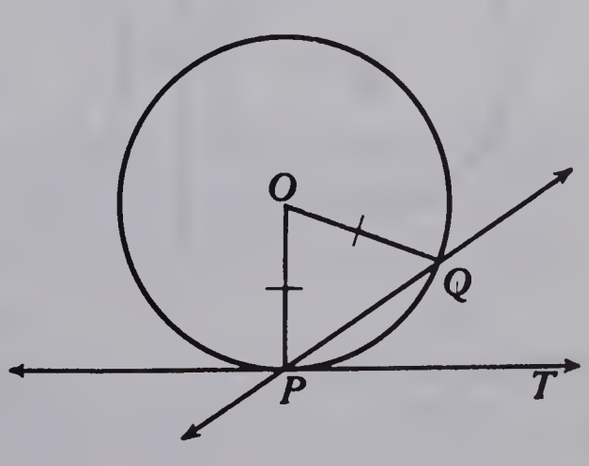


Fig. 11-11

11•7 The tangent at a point of a circle.

In *Fig. 11-11* PT is the tangent at P of the circle with centre O;
OP and PQ are radii of the circle;
PQ is a secant of the circle.

$\angle OPQ = \angle OQP$ (Isosceles Triangle Theorem)

$2\angle OPQ = 180° - \angle POQ$ (Triangle Angle Sum Theorem)

$$\therefore \quad \angle OPQ = 90° - \frac{\angle POQ}{2}.$$

It seems intuitively clear that as the secant PQ approaches a limiting position (the tangent PT at P), then $\angle POQ$ approaches 0°, in which case, $PT \perp OP$ at the point of contact P.

This implies that the tangent line at a point P on a circle is a line at a distance from the centre of the circle equal to the radius of the circle.

We will assume this to be true and state the following basic postulate concerning the tangent at a point of a circle.

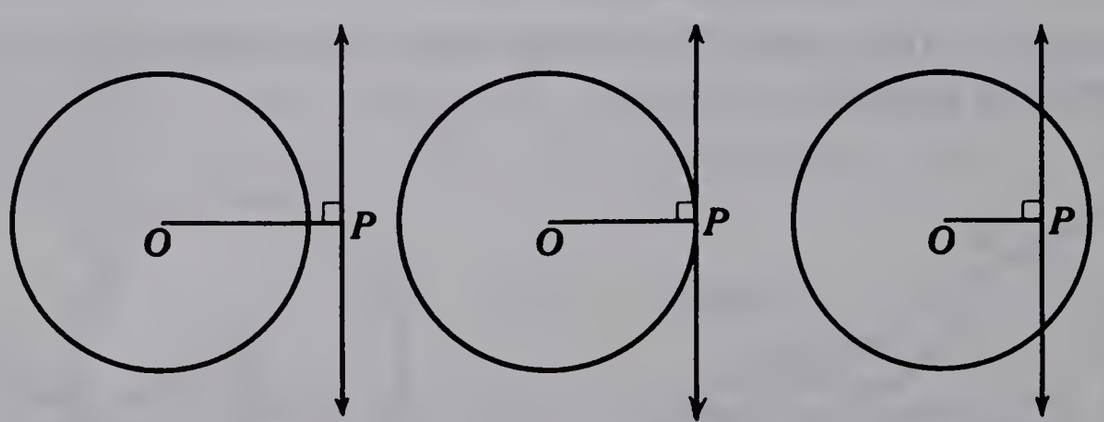
A line is a tangent to a circle if and only if the line is perpendicular to a radius at the point of contact.

Immediate consequents of this postulate are:

(i) *a line is tangent to a circle if and only if the (perpendicular) distance of the line from the centre of the circle is equal to the radius of the circle;*

(ii) *a line perpendicular to a tangent to a circle at the point of contact lies on the centre of the circle.*

The three ways in which a line and a circle may be related are illustrated in *Fig. 11-12*.



(*a*) exterior to circle: $OP > r$

(*b*) Tangent: $OP = r$

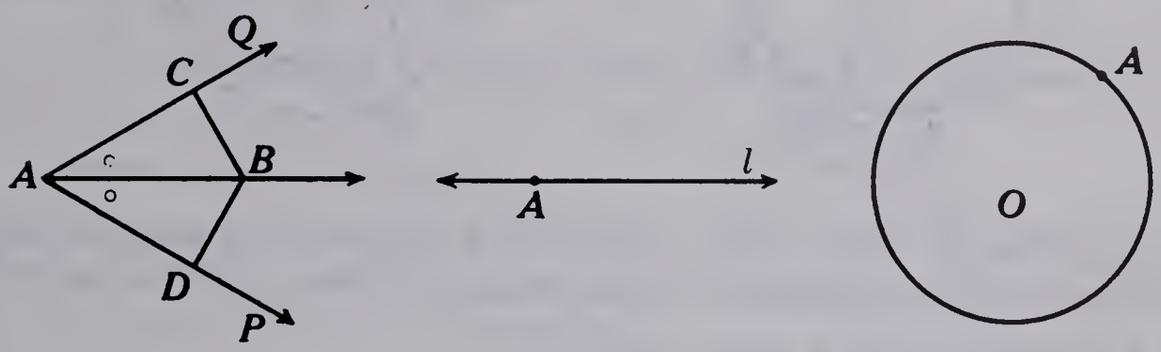
(*c*) Secant: $OP < r$

Fig. 11-12

Exercise 11-4

(A)

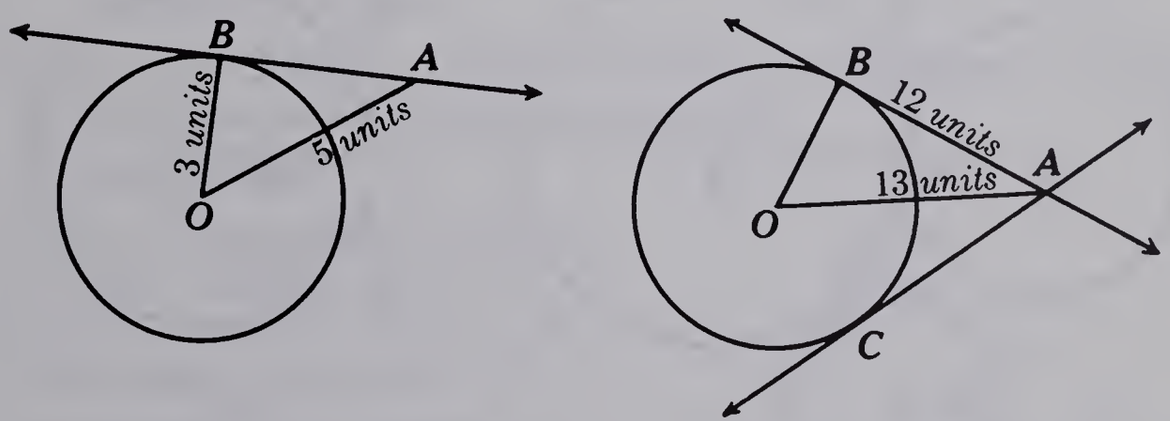
1. In the diagram at the left below, what additional conditions must be given in order to ensure that a circle with centre B and radius BC is tangent to the rays AQ and AP at C and D respectively?



2. In the centre diagram above, describe a method of constructing any circle to touch the line l at A.

3. In the diagram at the right above, A is a point of the circle with centre O. Describe a method of constructing a tangent to the circle at point A.

4. In the diagram at the left below, AB is a segment of the tangent at B to the circle, centre O. If the length of the radius is 3 units and the length of OA is 5 units, find the length of AB.

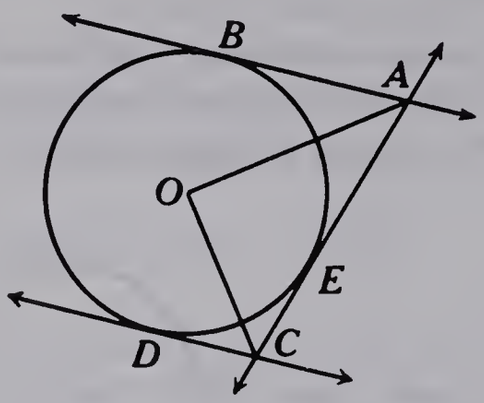


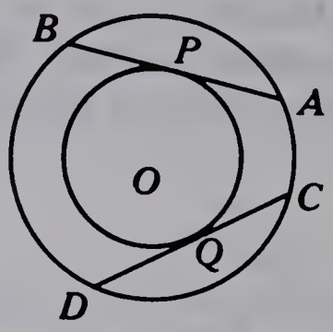
5. In the diagram at the right above, AB and AC are tangent segments to the circle; the length of AB is 12 units, and the length of OA is 13 units. Find the length of the radius OB and the tangent segment AC.

(B)

6. AOB is a diameter of a circle, centre O. Prove that the tangents to the circle at A and B are parallel.

7. In the diagram at the left below, AB, AEC, and CD are tangents to the circle at B, E, and D respectively. If $AB \parallel CD$, prove $\angle AOC = 90°$.





8. In the diagram at the right, chords APB and DQC are tangent to the inner of the two concentric circles, centre O. Prove $AB = DC$.

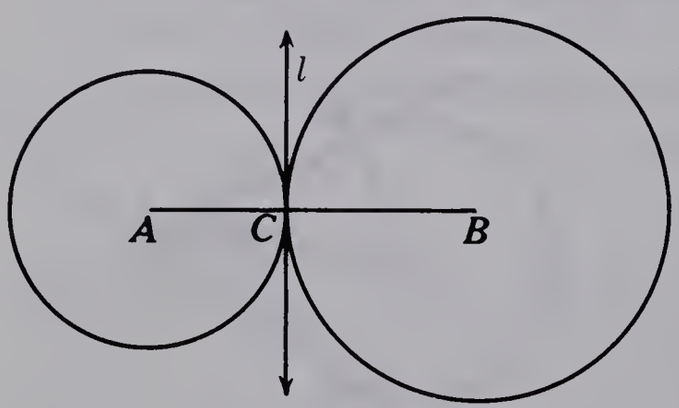
9. AB is any chord of a circle, centre O.

(i) Describe a method of constructing a circle with centre O to touch the chord AB.

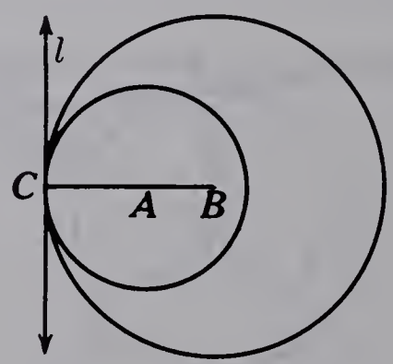
(ii) Prove that this circle is tangent to any other chord equal in length to AB.

10. Prove that all points of a tangent at a point of a circle, other than the point of contact, are points of the exterior of the circle.

(C)



(*a*)



(*b*)

11. Two circles with centres A and B respectively are tangent to each other at C. l is the *common tangent* to each circle at C. In diagram (a) the circles touch *externally;* in diagram (b) the circles touch *internally*.

(i) For the circles in (a), prove that A, C, and B are collinear and hence the *line of centres* is on the point of contact of the circles, and the distance between the centres of two circles which touch externally is the sum of the radii.

(ii) For the circles in (*b*), page 307, prove that $C, A,$ and B are collinear and hence the line of centres is on the point of contact of the circles, and the distance between the centres of two circles which touch internally is the absolute value of the difference of their radii.

12. State and prove the converse theorems for (i) and (ii) of question 11.

11•8 Tangent property theorems.

Write a proof for each of the following; compare your proofs of 1, 3, *and* 5 *with those on page* 447.

a. *Tangent segment properties.*

In *Fig. 11-13* P is a point of the exterior of a circle, centre O. A and B are the points of contact of tangents to the circle on P.

The line segments PA and PB are referred to as the *tangent segments* to the circle from the point P.

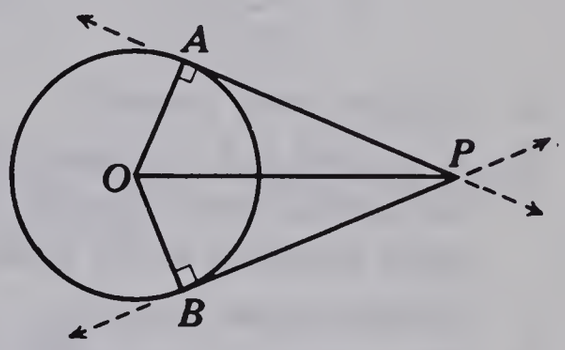


Fig. 11-13

1. Tangent segments to a circle from an external point:

(i) are equal in length;

(ii) make equal angles with the line segment from the point to the centre of the circle;

(iii) the angles at the centre which are subtended by the tangent segments are equal.

2. Describe a ruler and compasses method of contructing the tangents to a given circle on a given external point.

b. *Incircle property.*

A circle is the *inscribed circle* or *incircle* of a triangle, *Fig. 11-14* if and only if the sides of the triangle are each tangent to the circle.

3. The centre of the incircle of a triangle is the point of intersection of the bisectors of two of the angles of the triangle.

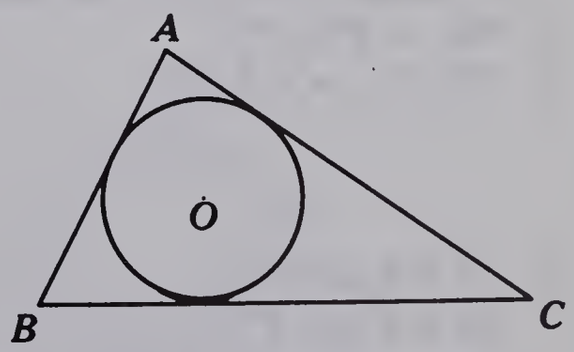


Fig. 11-14

c. *Escribed circle property.*

A circle is an *escribed circle* of a triangle, *Fig. 11-15*, if and only if the circle is tangent to one side and to the lines formed when the other two sides are extended.

4. The centre of an escribed circle of a triangle is the point of intersection of the bisectors of the exterior angles at two of the vertices.

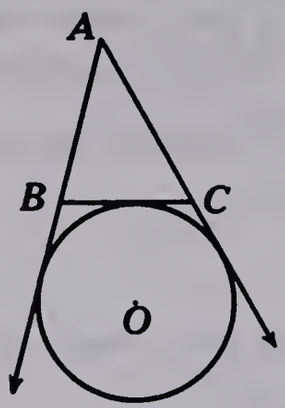


Fig. 11-15

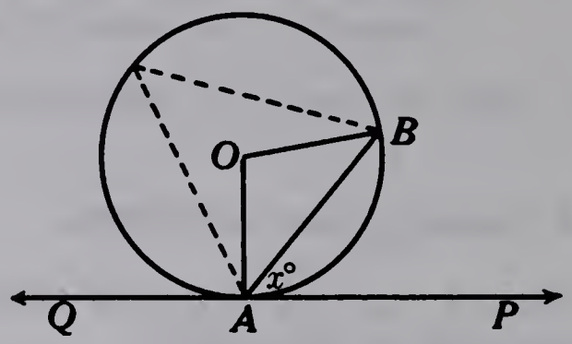


Fig. 11-16

d. *Tangent-chord property.*

5. If a chord and a tangent, *Fig. 11-16*, are on a point of a circle, each of the angles determined by the tangent and the chord is equal to the angle inscribed in the circle on the opposite side of the chord.

e. *Tangent-secant property.*

6. The square of the length of a tangent segment to a circle from an external point is equal to the product of the lengths of the *secant* segments from the point.

In *Fig. 11-17* the segments PC and PB are called the secant segments from P of any secant on P.

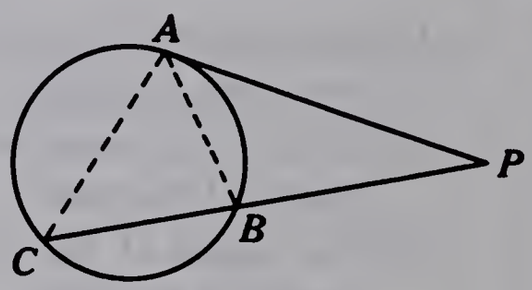


Fig. 11-17

Analysis:

I CAN PROVE	IF I CAN PROVE
1. $PA^2 = PB \cdot PC$	1. $PB : PA = PA : PC$
2. $PB : PA = PA : PC$	2. (i) PB and PA (ii) PA and PC are corresponding sides of two similar triangles.
3. (i) PB and PA (ii) PA and PC are corresponding sides of two similar triangles.	3. $\triangle PAB \sim \triangle PCA$
4. $\triangle PAB \sim \triangle PCA$	4. (i) $\angle P = \angle P$ (ii) $\angle PAB = \angle PCA$

5. (i) $\angle P = \angle P$ (Reflexive property)
   (ii) $\angle PAB = \angle PCA$ (Tangent-chord property).

Corollary:

If l_1 and l_2 are two secants of a circle with common point P, *Fig. 11-18*, and intersecting the circle at A, B, and C, D, respectively, then $PA \cdot PB = PC \cdot PD$.

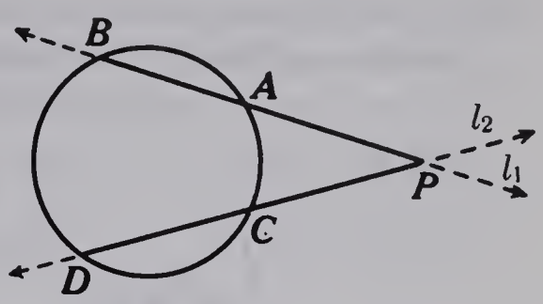
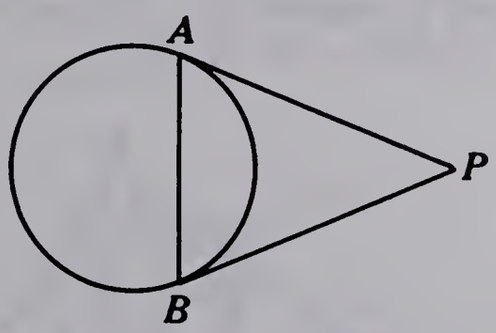


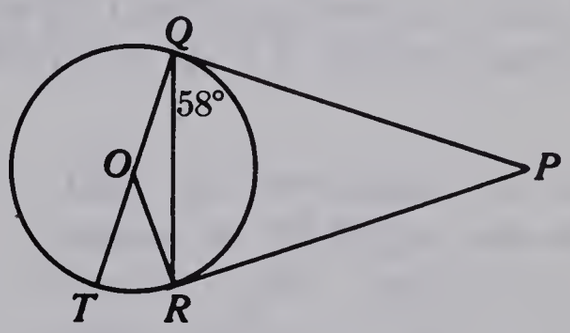
Fig. 11-18

Exercise 11-5

(A)

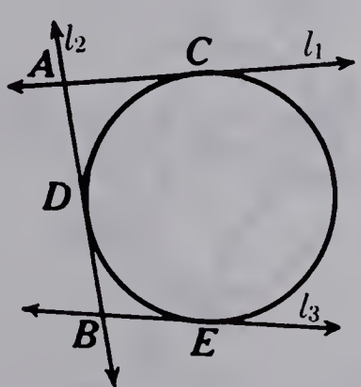
1. In the diagram at the left below, PA and PB are tangent segments from P to the circle. Prove $\angle PAB = \angle PBA$.

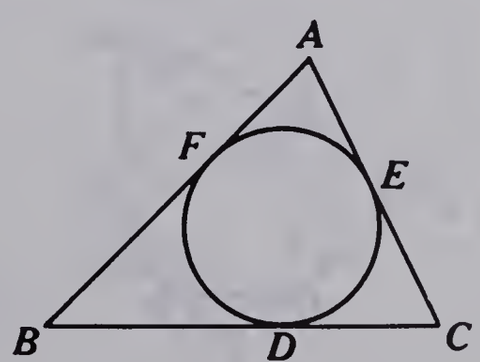




2. In the diagram at the right above, PQ and PR are tangent segments to the circle, centre O. $\angle PQR = 58°$. Find the number of degrees in $\angle TOR$.

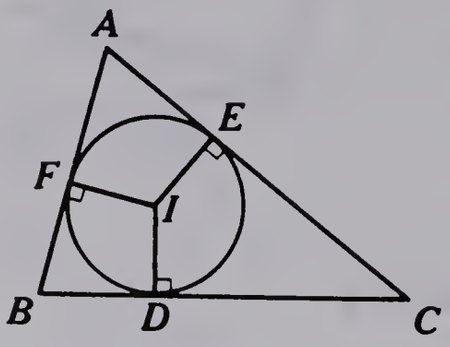
3. In the diagram at the left below, l_1, l_2, and l_3, are tangents to the circle, with centre O, at C, D, and E respectively. Prove $AB = AC + BE$.

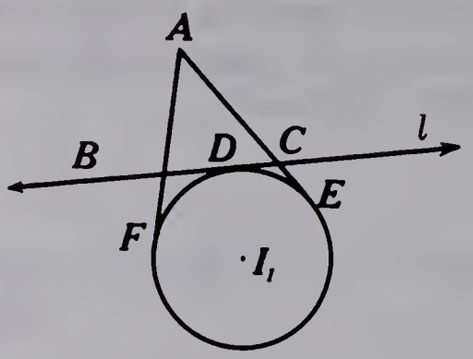




4. In the diagram at the right above, the inscribed circle of triangle ABC is tangent to the sides AB, BC, CA at F, D, and E respectively. Prove that $AF + BD + CE$ equals half the perimeter of the triangle.

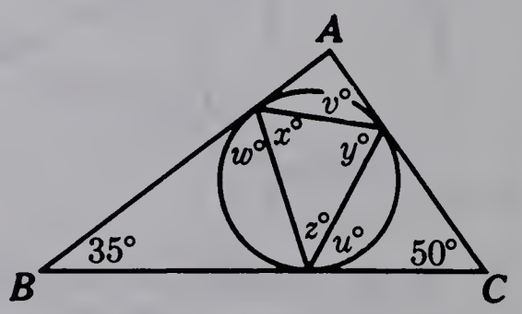
5. In the diagram at the left below, the inscribed circle is tangent to the sides at D, E, and F respectively. If $BC = a$ units, $CA = b$ units, $AB = c$ units, and $a + b + c = 2s$, prove $AF = s - a$, $BD = s - b$, and $DC = s - c$.



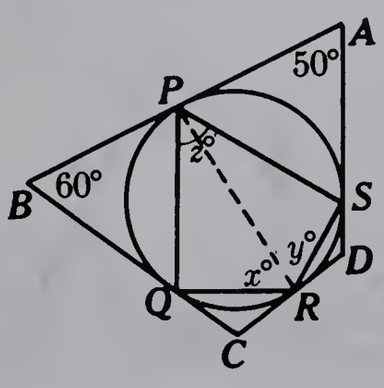


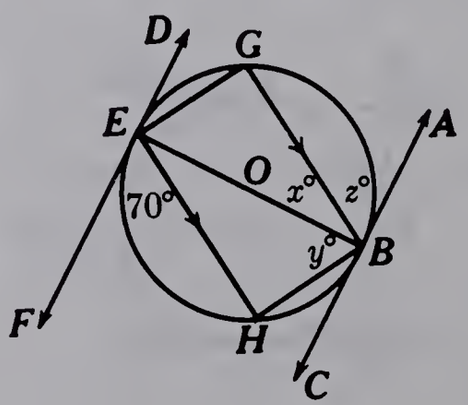
6. In the diagram at the right above, AF and AE are tangent segments from A to the circle with centre I_1. l is a tangent to the circle at D and intersecting AF and AE at B and C respectively. Prove that for any position of l, the perimeter of $\triangle ABC = AF + AE$.

7. In the diagram at the right, the sides of $\triangle ABC$ are tangent to the circle, $\angle B = 35°$ and $\angle C = 50°$. Find the values of x, y, z, w, v, u.



8. In the diagram at the left below, AB, BC, CD, and DA are tangent to the circle. $\angle B = 60°$, $\angle A = 50°$. Find the values of x, y, and z.



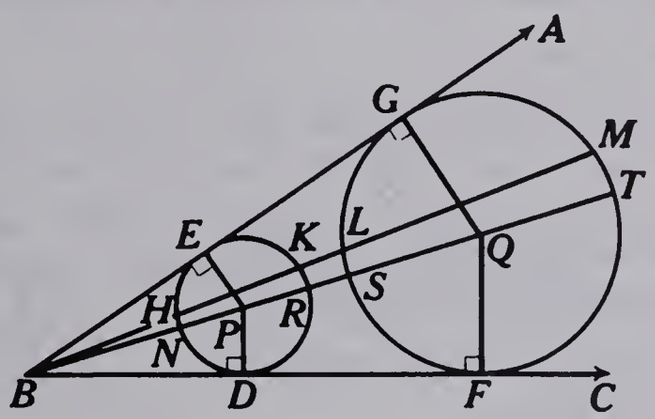


9. In the diagram at the right above, DF and AC are tangent to the circle, centre O, at the ends of a diameter EOB. $EH \parallel BG$ and $\angle FEH = 70°$. Find the values of x, y, and z.

10. If P and Q are the centres of the circles in the accompanying diagram, use the Tangent-secant Property Theorem or its corollary to obtain equivalent expressions for each of the following:

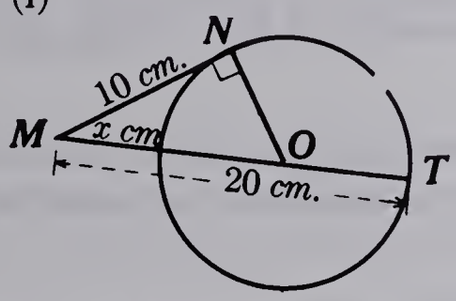
(i) BE^2 (ii) BG^2 (iii) BF^2 (iv) $BH \cdot BK$

(v) $BS \cdot BT$ (vi) $BR \cdot BN$ (vii) BD^2 (viii) $BM \cdot BL$

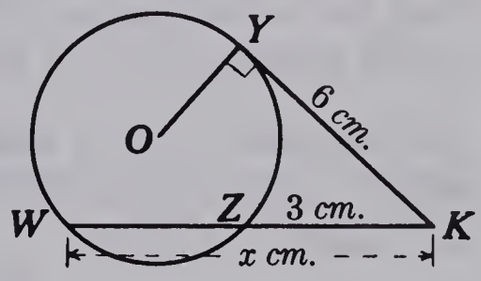


11. In each of the following calculate the value of x. (O is the centre of each circle.)

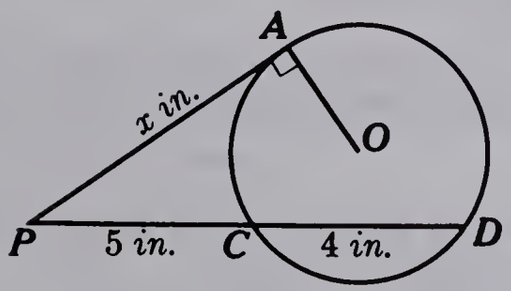
(i)



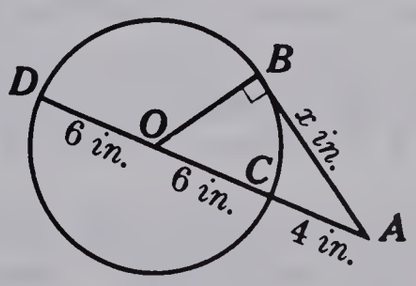
(ii)



(iii)



(iv)



(B)

12. If PA and PB are tangents to a circle, centre O, at A and B respectively and on an external point P, prove PO right bisects the chord of contact AB.

13. PA and PB are tangent segments from an external point P to a circle with centre O. AOC is a diameter. Prove $\angle CAB = \frac{1}{2}\angle P$.

14. AB is a diameter of a circle and AD and AC are chords such that $\angle CAD = \angle DAB$. Prove that the tangent at D is perpendicular to the line on AC.

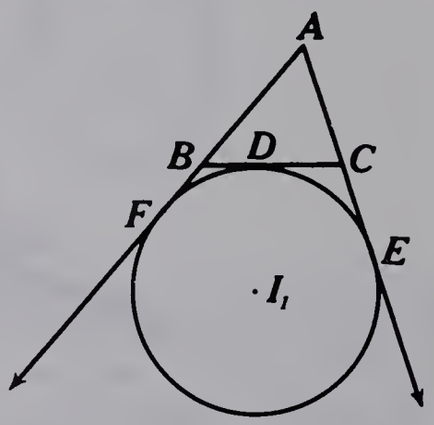
15. In triangle ABC, if $BC = a$ units, $CA = b$ units, and $AB = c$ units, then the perimeter is $(a + b + c)$ units. The inscribed circle is tangent to AB, BC, and CA, at F, D, and E, respectively.

If $a + b + c = 2s$, prove:

(i) $AF = s - a$ (ii) $BD = s - b$

(iii) $DC = s - c$.

16. In the diagram at the right, I_1 is the centre of the escribed circle opposite A.



If a, b, and c have the same meaning as in question 15, prove:

(i) $AF = AE = s$ (ii) $BD = BF = s - c$

(iii) $CD = CE = s - b$.

17. PQR is a triangle inscribed in a circle. $\angle P = 80°$ and $\angle Q = 40°$. Find the measurements of the three angles of the triangle determined by the tangents at P, Q, and R.

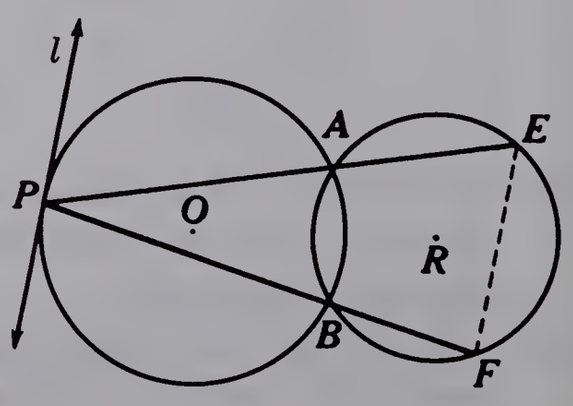
18. The inscribed circle of triangle ABC is tangent to the sides at D, E, and F. $\angle A = 45°$ and $\angle B = 75°$. Calculate the measurement of each angle of $\triangle DEF$.

19. A and B are two points of a circle. The tangents at A and B intersect on C. AD is a chord parallel to BC. $\angle CAB = 70°$. Find the measurement of each angle of $\triangle ABD$.

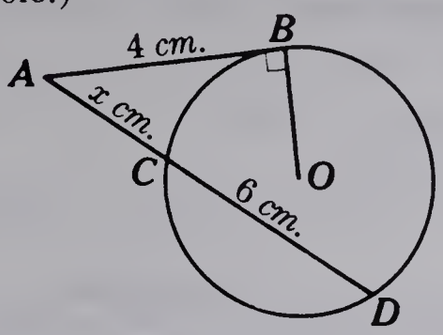
20. ABC is a tangent to a circle at B. DE is a chord of the circle parallel to ABC. Prove $\triangle DEB$ is isosceles.

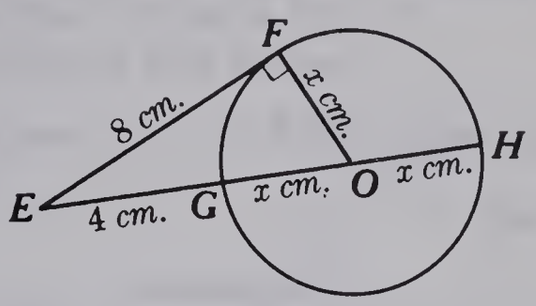
21. AB is a diameter of a circle and AC is a chord. AD is perpendicular to the tangent at C. Prove that AC bisects $\angle DAB$.

22. In the diagram at the right, line l is tangent to the circle, centre O, at P. The two circles intersect at A and B. PAE and PBF are line segments on the points A and B, terminated in the circumference of the second circle on E and F respectively. Prove $EF \parallel l$.



23. In the accompanying diagrams, calculate x. (O is the centre of each circle.)





24. If two circles intersect, the tangent segments to each circle from any point on the common chord extended are equal.

25. If two circles intersect, the line on their common chord bisects the common tangent segment of the two circles.

26. C is a point on a semicircle whose diameter is a line segment AB. E is in CB and D is in AB. $ED \perp AB$. Prove $AB \cdot BD = BC \cdot BE$.

27. AD, BE, and CF are the altitudes of $\triangle ABC$; and O is their point of intersection. Prove $AB \cdot AF = AD \cdot AO = AC \cdot AE$.

(C)

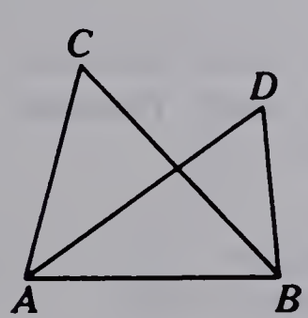
28. State and prove the converse theorem of the Tangent-chord Property Theorem.

11·9 **Concyclic points, cyclic quadrilaterals.** If the four vertices of a quadrilateral are on a circle, the quadrilateral is a *cyclic* quadrilateral. The four vertices are *concyclic* points.

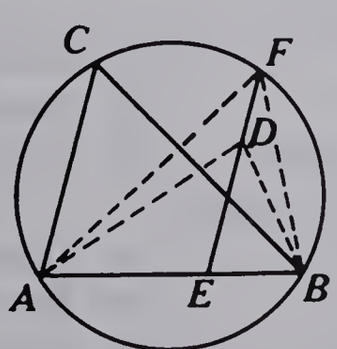
THEOREM

Basic Concyclic Point Theorem
(Converse of Corollary 1, page 292)

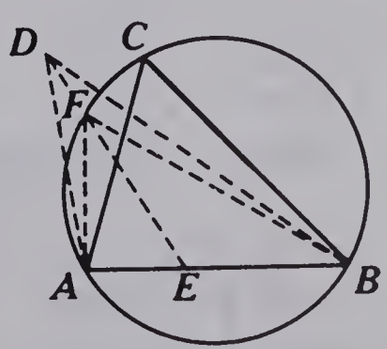
If a line segment subtends two equal angles at points on the same side of it, then the vertices of the angles and the end points of the line segment are concyclic points.



(a)



(b)



(c)

Fig. 11-19

Hypothesis: ACB and ADB are equal angles subtended by line segment AB at points C and D respectively, on the same side of the line on AB, *Fig. 11-19(a)*.

Conclusion: A, B, C, D are concyclic points.

Proof:

STATEMENTS	AUTHORITIES
1. Circle on A, B, and C does not lie on D, but; CASE 1. D is in the interior of the circle; or CASE 2. D is in the exterior of the circle.	1. Assumption of the negation of the conclusion; circle exists by problem 9 page 289.
CASE 1. *Fig. 11-19(b)*	
2. E is any point of AB, and the line on ED meets the circle on A, B and C at F.	2. Existence postulates
3. $\angle ADE > \angle AFE$	3. Exterior Angle Theorem
4. $\angle BDE > \angle BFE$	4. Exterior Angle Theorem
5. $\angle ADE + \angle BDE > \angle AFE + \angle BFE$	5. Addition
6. $\angle ADB > \angle AFB$	6. Completion
7. $\angle ADB = \angle ACB$	7. Hypothesis
8. $\angle ACB > \angle AFB$	8. Replacement
9. D is in the interior of the circle is false.	9. 8 contradicts the Inscribed Angles Theorem
CASE 2. *Fig 11-19 (c)*	
10. E is any point of AB, and the line on ED intersects the circle on A, B, and C at F.	10. Existence postulates
11. $\angle ADE < \angle AFE$	11. Exterior Angle Theorem
12. $\angle BDE < \angle BFE$	12. Exterior Angle Theorem
13. $\angle ADE + \angle BDE < \angle AFE + \angle BFE$	13. Addition
14. $\angle ADB < \angle AFB$	14. Completion
15. $\angle ADB = \angle ACB$	15. Hypothesis
16. $\angle ACB < \angle AFB$	16. Replacement
17. D is in the exterior of the circle is false.	17. 16 contradicts the Inscribed Angles Theorem
18. D is on the circle on A, B, and C.	18. Law of contradiction

Example. In $\triangle ABC$, $AB = AC$. Points X and Y of AB and AC respectively are such that $XB = YC$. Prove that quadrilateral $XBCY$ is cyclic.

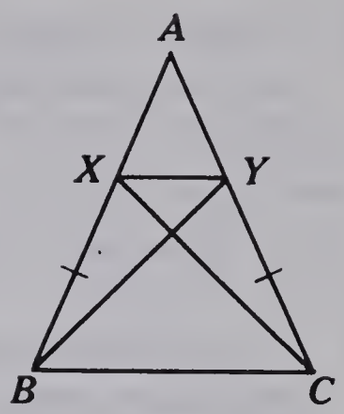


Fig. 11-20

Hypothesis: $\triangle ABC$, $AB = AC$, $XB = YC$.

Conclusion: $XBCY$ is a cyclic quadrilateral.

Analysis:

I CAN PROVE	IF I CAN PROVE
1. $XBCY$ is a cyclic quadrilateral	1. $\angle BXC = \angle BYC$
2. $\angle BXC = \angle BYC$	2. $\triangle BXC \cong \triangle CYB$
3. $\triangle BXC \cong \triangle CYB$	3. $\begin{cases} BX = CY \\ BC = CB \\ \angle XBC = \angle YCB \end{cases}$

4. $\begin{cases} BX = CY \text{ by hypothesis} \\ BC = CB \text{ by reflexive property} \\ \angle XBC = \angle YCB \text{ by Isosceles Triangle Theorem} \end{cases}$

Write a proof of this deduction.

Write a proof for each of the following theorems; use the suggestion in the given diagram and the Basic Concyclic Point Theorem in 1 and 2. Compare your proof with that on page 449.

1. **A circle on the hypotenuse of a right triangle as diameter, is on the vertex of the right angle.** (Converse of corollary 2 page 292).

Hypothesis: ABC is a right triangle. *Fig. 11-21* with a circle on AB as diameter.

Conclusion. Circle lies on C.
(Hint: select any point D on the circle and show $\angle ADB = \angle ACB$.)

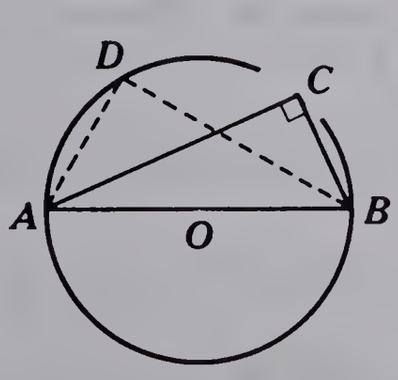


Fig. 11-21

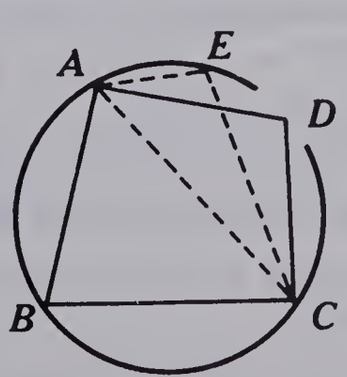


Fig. 11-22

2. **If a pair of interior, opposite angles of a quadrilateral are supplementary, the quadrilateral is cyclic.** (Converse of Corollary 3, page 293.)

Hypothesis: $ABCD$ is a quadrilateral, *Fig 11-22* in which $\angle B + \angle D = 180°$.

Conclusion: $ABCD$ is a cyclic quadrilateral.

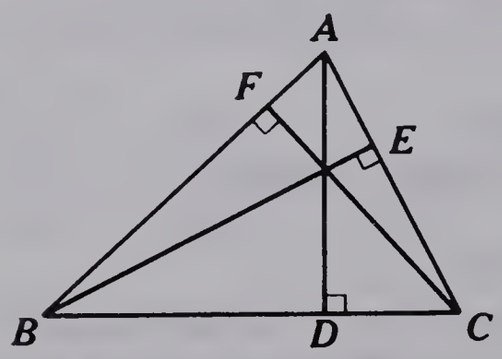
(Hint: select any point E on the circumcircle of $\triangle ABC$ and show $\angle AEC = \angle ADC$.)

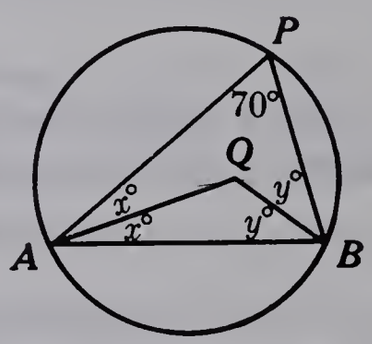
3. **If an exterior angle of a quadrilateral is equal to the interior angle at the opposite vertex, the quadrilateral is cyclic.** (Converse of Corollary 4, page 293.)

Exercise 11-6

(A)

1. In the diagram at the left below, $AD \perp BC$, $BE \perp AC$, $CF \perp AB$. Name three sets of four points which are concyclic.



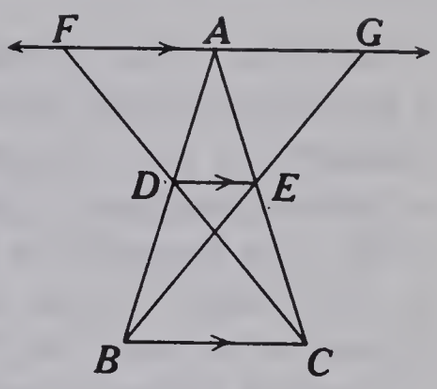


2. In the diagram at the right above, $\angle APB = 70°$, QA bisects $\angle A$, QB bisects $\angle B$. Prove: (i) $\angle AQB = 125°$; (ii) $\angle AQB$ is always $125°$ for any position of P on major arc APB.

(B)

3. Prove that the vertices of a rectangle are concyclic.
4. In $\triangle ABC$, BE and CF are perpendicular to AC and AB respectively with E on AC and F on AB. Prove $\angle EBF = \angle FCE$.
5. The altitudes AD, BE, CF of $\triangle ABC$ meet at O. Prove: (i) $\angle EBC = \angle EFC$; (ii) $\angle EBC = \angle DAC$; (iii) points A, F, O, E are concyclic points.
6. The mid-point of the hypotenuse of a right triangle is equidistant from the three vertices of the triangle.
7. If a chord of a circle subtends a right angle at the circle, the chord is a diameter.
8. ABC is an isosceles triangle in which $AB = AC$. A circle on AB as diameter intersects BC on D. Prove $BD = DC$.

9. In the diagram at the right, $AB = AC'$ and $FAG \parallel DE \parallel BC$. Prove that F, D, E, and G are concyclic points.



10. In trapezoid $ABCD$, $AD \parallel BC$, $AB = CD$. Prove that A, B, C, D are concyclic.

11. $ABCD$ is a parallelogram. A circle on A and B intersects AD and BC or their extensions at E and F respectively. Prove that E, F, C, D are concyclic points.

12. Show that the altitudes of a triangle are concurrent.

DEFINITION. *The point of intersection of the altitudes of a triangle is called the orthocentre of the triangle.*

11•10 Perimeter and area of a regular polygon.

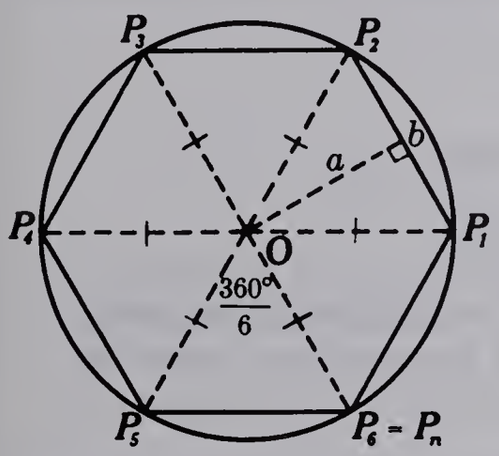


Fig. 11-23

The convex polygon $P_1P_2 \ldots P_6$ in *Fig. 11-23* is a regular polygon (hexagon) inscribed in the circle with centre O.

DEFINITION: *A convex polygon is regular if all its sides are equal and all its angles are equal.*

A polygon is inscribed in a circle if all its vertices lie on the circle.

A regular polygon, (n-gon), $P_1P_2 \ldots P_n$ inscribed in a circle, centre O, has

$$\triangle OP_1P_2 \cong \triangle OP_2P_3 \cong \triangle OP_3P_4 \cong \ldots \cong \triangle OP_{n-1}P_n \cong \triangle OP_nP_1;$$

each triangle has the same base length, b units;
each triangle has the same altitude, a units;
each triangle has the same area, $\frac{1}{2}ab$ square units.

$\therefore$ the *perimeter* of the polygon, p_n units, is given by the formula

$$p_n = nb,$$

and the *area* of the polygon, A_n square units, is given by the formula

$$A_n = n \times \tfrac{1}{2}ab,$$

$$\text{or} \quad A_n = \frac{nab}{2},$$

which may be expressed: $$A_n = \frac{p_n a}{2}.$$

11•11 The circumference of a circle.

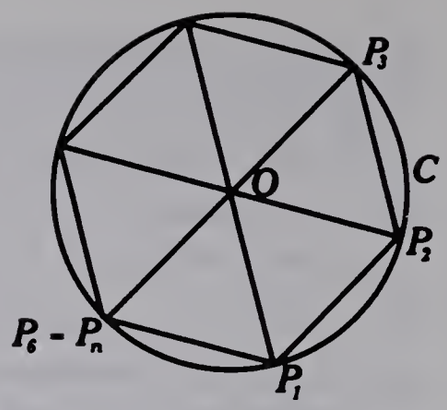


Fig. 11-24

We know what is meant by the length of a line segment and hence can determine the perimeter of any polygonal figure. To ascribe a measurement to a curved line such as a circle, it is necessary to define what we mean by the *"length"* of the curve or the *"circumference"* of a circle. In *Fig. 11-24* $P_1P_2 \ldots P_6$ is a regular polygon inscribed in the circle with centre O. Intuitively we feel that if we inscribe a polygon with a large number of sides in a circle, then the perimeter, p_n units, of the polygon should be a close approximation to the "circumference" of the circle. Indeed if we make n, the number of sides, great enough we ought to be able to make the difference between the perimeter, p_n units, of the polygon and the "circumference", c units, of the circle as small as we like. In other words: the perimeter of the polygon *"approaches"* the circumference of the circle.

or p_n approaches c

as the number of sides of the polygon is increased.

Therefore we will make the following definition:

DEFINITION: *The circumference of a circle is the limit of the perimeters of the inscribed regular polygons as the number of sides is increased indefinitely.*

Now consider the two circles in *Fig. 11-25* with centres O_1 and O_2 and radii r_1 and r_2, $r_1 \neq r_2$, respectively.

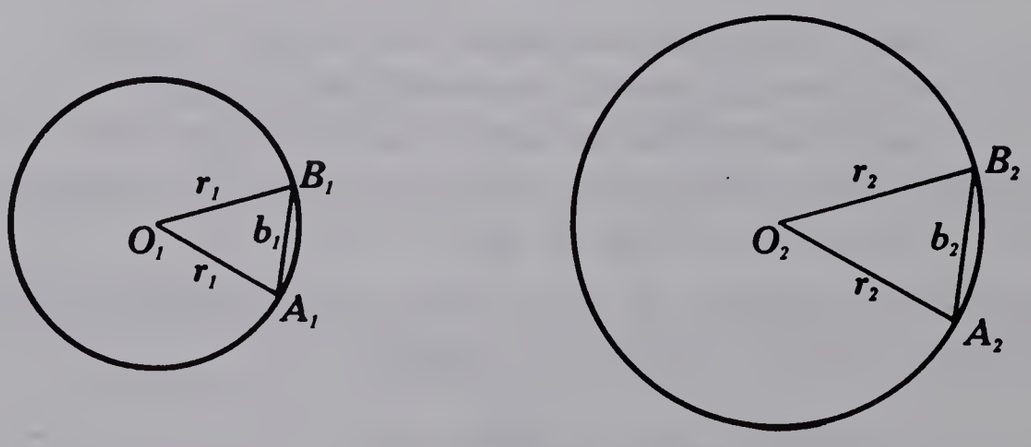


Fig. 11-25

If regular n-gons with the same number of sides are inscribed in each circle, then

$$\angle A_1O_1B_1 = \angle A_2O_2B_2 = \left(\frac{360}{n}\right)^\circ$$

and $O_1B_1 : O_2B_2 = O_1A_1 : O_2A_2 = r_1 : r_2$.

$\therefore$ $\triangle A_1O_1B_1 \sim \triangle A_2O_2B_2$ (sas Similarity Theorem).

$\therefore$ $$\frac{b_1}{b_2} = \frac{r_1}{r_2},$$

or $$\frac{b_1}{r_1} = \frac{b_2}{r_2},$$

and $$\frac{p_1}{r_1} = \frac{p_2}{r_2} \qquad (\because\ p_1 = nb_1 \text{ and } p_2 = nb_2).$$

If c_1 units and c_2 units are the circumferences of the circles respectively, then

$$\frac{c_1}{r_1} = \frac{c_2}{r_2} \qquad (\because\ p_1 \text{ approaches } c_1 \text{ and } p_2 \text{ approaches } c_2 \text{ as the number of sides is increased.})$$

or $$\frac{c_1}{2r_1} = \frac{c_2}{2r_2},$$

from which we may conclude that the ratio of the circumference of a circle to its diameter is a constant.

The number $\frac{c}{2r}$, which is the same for all circles, is represented by the symbol π.

Thus $$\frac{c}{2r} = \pi \quad \text{or} \quad c = 2\pi r.$$

Some rational approximations to the irrational number π are:

$$3.14, \quad \tfrac{22}{7}, \quad 3.1416, \quad 3.14159265358979.$$

***11•12* Area of a circle.** A circular region or disc is the union of a circle and its interior. We know what is meant by the area of any polygonal region. By an argument similar to that which led to the definition of "circumference of a circle" in Section 11•11, we may conclude that if we make n (the number of sides of a regular polygon inscribed in a circle, *(Fig11-26)* great enough, the difference between the area, A_n square units, of the polygonal region (referred to as the area of the polygon) and the area, A square units, of the circular region (referred to as the area of the circle) will be as small as we like. In other words:

The area of the polygon "approaches" the area of the circle,

or A_n approaches A,

as the number of sides of the polygon is increased.

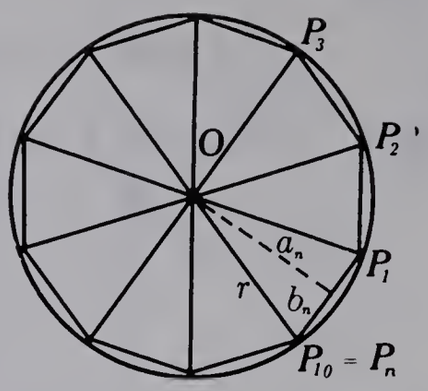


Fig. 11-26

Therefore we will make the following definition:

DEFINITION: *The area of a circle is the limit of the areas of the inscribed regular polygons as the number of sides is increased indefinitely.*

In general (*Fig 11-26*) $a_n < r$,

but a_n approaches r (as n is increased).

Also p_n approaches c (as n is increased).

Whence $\frac{1}{2}a_n p_n$ approaches $\frac{1}{2}rc$.

and $\therefore$ A_n approaches $\frac{1}{2}rc$.

But by definition A_n approaches A.

$\therefore$ we conclude $A = \frac{1}{2}rc$.

$A = \frac{1}{2}r \cdot 2\pi r$ ($\because$ $c = 2\pi r$).

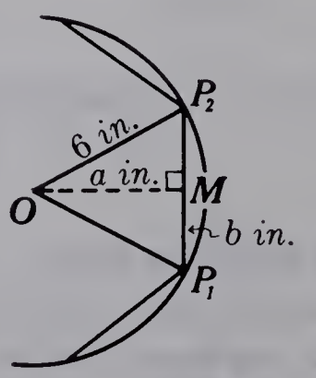
$A = \pi r^2$.

Thus the area of a circle of radius r is πr^2.

Exercise 11-7

(B)

1. A regular polygon $P_1P_2 \ldots P_6$ is inscribed in a circle, centre O, of radius 6 inches.



(i) Find $\angle P_1OP_2$ and $\angle MOP_2$.

(ii) Find the altitude, a in., of $\triangle P_1OP_2$ ($a = 6 \cos \angle MOP_2$).

(iii) Find the length of the base, b inches, of $\triangle P_1OP_2$.

$\left(\frac{b}{2} = 6 \sin \angle MOP_2\right)$.

(iv) Find the perimeter, p inches, of the polygon.

(v) Find the circumference, c inches, of the circle.

(vi) Find the difference $(c - p)$ inches.

(vii) Find the area, A_n square inches, of the polygon.

(viii) Find the area, A square inches, of the circle.

(ix) Find the difference $(A - A_n)$ square inches.

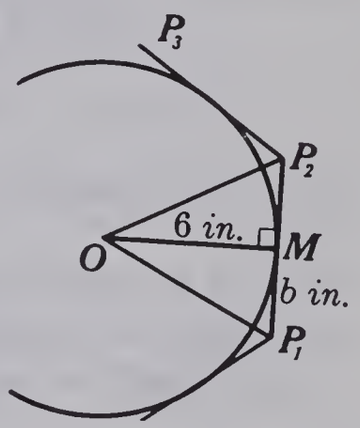
(x) Repeat the above calculations for a polygon with 12 sides.

(xi) Compare the difference found in each case for items (vi) and (ix).

(xii) Calculate $\angle P_1P_2P_3$ for each polygon. (Recall the sum of the interior angles of a polygon is $(2n - 4)$ right angles.)

2. A regular polygon $P_1P_2 \ldots P_6$ is circumscribed about a circle, centre O, with radius 6 inches. Using whatever information is available from question 1;

(i) find the length of the base, b inches, of $\triangle P_1OP_2$ (recall $\frac{b}{2} = 6 \tan \angle MOP_2$);



(ii) find the perimeter, p inches, of the polygon;

(iii) find the difference $(p - c)$ inches;

(iv) find the area, A_n square inches, of the polygon;

(v) find the difference $(A_n - A)$ square inches;

(vi) repeat the above calculations for a polygon with 12 sides;

(vii) compare the differences found in each case for items (iii) and (v);

(viii) draw a conclusion concerning the circumference and area of a circle with respect to the limit of the perimeter and of the area of a circumscribed polygon respectively, as the number of sides is increased without bound.

***11·13* Length of an arc of a circle.** By an argument similar to that of Section 11·11, we may agree to define the length of an arc AB of a circle, *Fig 11-27* as the limit of

$$AP_1 + P_1P_2 + \ldots + P_{n-1}B$$

as n, the number of sides of the inscribed regular polygonal line is increased indefinitely.

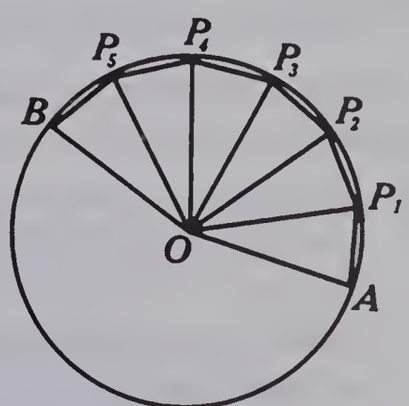
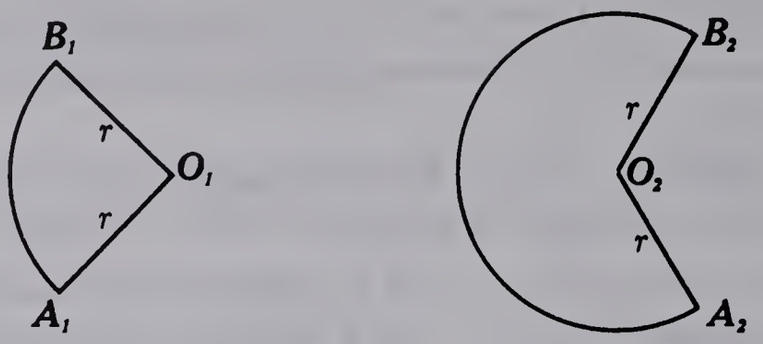


Fig. 11-27

However, since each arc is associated with a sector and is related to the sector angle of the sector, it is more convenient to begin by considering the circumference of a circle as an arc with sector angle 360°. Then it may be proved:

if two arcs, Fig. 11-28 have equal radii, then their lengths are proportional to their sector angles.



$$\frac{\text{length of arc } A_1B_1}{\text{sector angle } A_1O_1B_1} = \frac{\text{length of arc } A_2B_2}{\text{sector angle } A_2O_2B_2}$$

Fig. 11-28

Since this theorem is difficult to prove it is accepted here as a postulate. Thus,

if (i) c units is the circumference of a circle of radius r units,

and (ii) $m°$ is the measurement of the sector angle of a sector of this circle having an arc length l units,

then $$\frac{l}{m} = \frac{c}{360},$$

or $$\frac{l}{m} = \frac{2\pi r}{360},$$

whence $$l = \frac{\pi m r}{180}.$$

11·14 Area of a sector of a circle.

Write a discussion leading to the following conclusions; compare your discussion with that on page 450.

1. The area of a sector (A_s square units) is half the product of its radius and the length of its arc.
   (Hint: use the idea suggested in *Fig 11-29* and an argument similar to that of Section 11·12 in which the area of a circle is developed.)
2. The area of a sector of radius r units and sector angle $m°$ is $\dfrac{m\pi r^2}{360}$ square units.
   (Hint: use the result of question 1 and Section 11·13.)

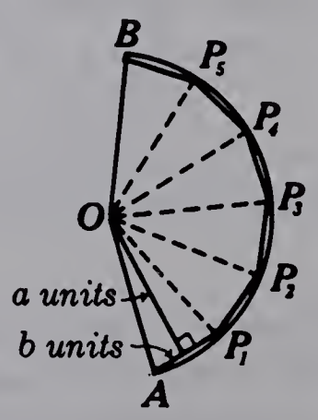


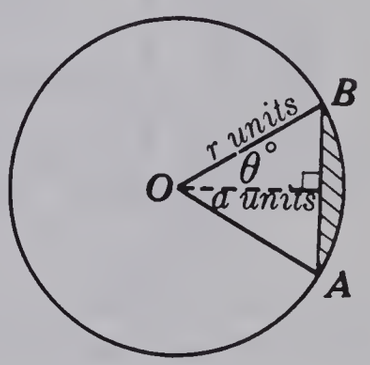
Fig. 11-29

Exercise 11-8

(B)

(Unless otherwise stated, use $\pi = 3.14$ *and round-off answers to the nearest tenth of a unit.)*

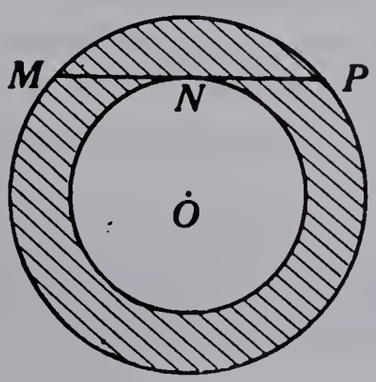
1. The radius of a circle is 20 inches. Find the lengths of the arcs whose sector angles are (i) 60° (ii) 90° (iii) 36° (iv) 135°.
2. The radius of a circle is 3 inches. Find the areas of the sectors whose sector angles are (i) 30° (ii) 72° (iii) 120° (iv) 150°.
3. The radius of a circle is 12 cm. Find the area of the sectors whose arcs have lengths (i) 10 cm. (ii) 12 cm. (iii) 25 cm. (iv) 60 cm.
4. If the length of an arc whose sector angle is 60° is 2 cm., find (i) the radius of the arc and (ii) the length of the chord of the arc.
5. Find the area, to nearest tenth of a square inch, of each of the following segments of circles:

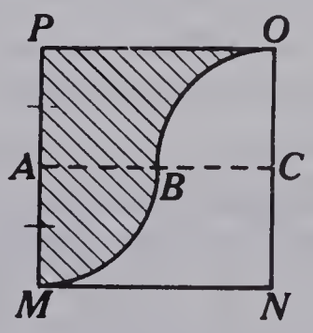


   (i) $\angle AOB = 20°$, radius = 12 inches;
   (ii) $\angle AOB = 52°$, radius = 20 inches;
   (iii) $\angle AOB = 72°$, radius = 15 inches.
   (Recall: $a = r\cos\theta°$.)

6. (i) Find the area of the annulus in the diagram at the left below if the inner and outer diameters have lengths 2 inches and 4 inches respectively.

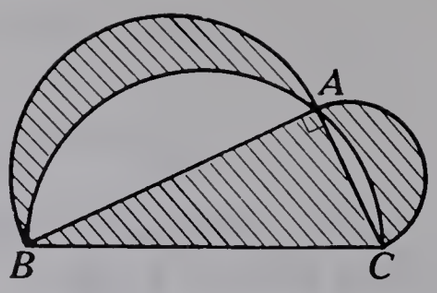
   (iii) In the diagram at the left below, MP is a chord of the larger circle and is tangent at N to the smaller circle. Prove that the area of the annulus is $\pi(NP)^2$.



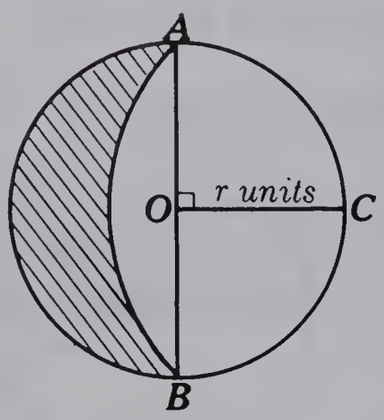


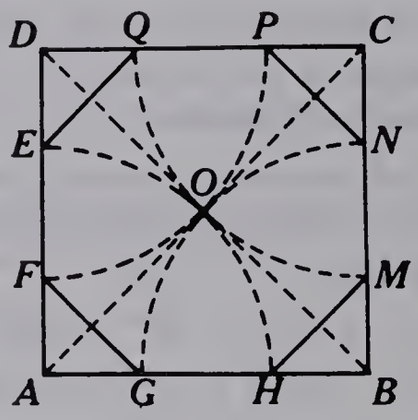
7. In the diagram at the right above, $MNOP$ is a square, A and C are the midpoints of PM and ON respectively. OB and BM are circular arcs with centres C and A respectively. If the side of the square is $2a$ units, find the area of the shaded portion.

8. Semicircles are constructed on the sides of a right triangle as shown in the diagram at the right. Prove that the sum of the areas of the shaded crescents equals the area of triangle ABC.



9. AB is the diameter of a circle and OC is a radius perpendicular to AB. Prove that the shaded figure (crescent) in the diagram at the left below has an area equal to r^2 square units. (Centres of the arcs of the crescent are O and C respectively.)





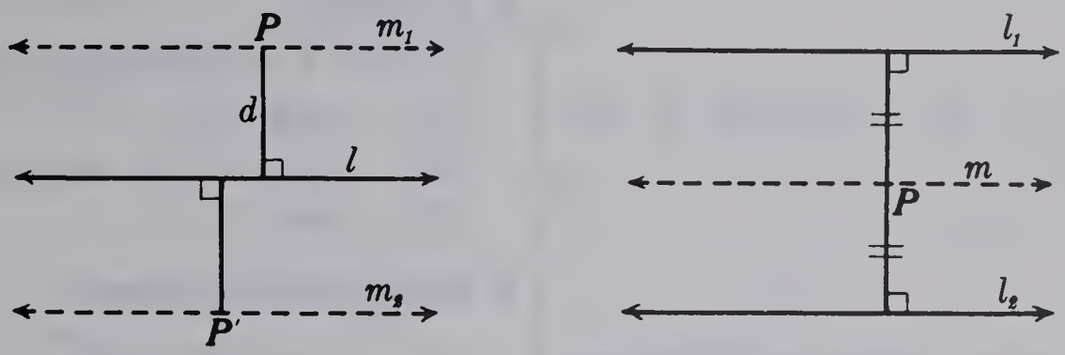
10. The diagram at the right above illustrates a method which may be used for cutting a regular octagonal region from a square region. Study the diagram, write a description of the method, and prove $EFGHMNPQ$ is a regular octagon.

***11•15* Construction problems involving circles (supplementary).** We have seen that three points determine a circle. This suggests that to determine any particular circle three conditions are necessary (and sufficient). These three conditions are equivalent to the position of the centre of the circle and the length of the radius. The position of the centre is equivalent to two conditions, since to determine a point two conditions are needed; in analytic geometry these correspond to the x-coordinate and the y-coordinate; in synthetic geometry these correspond to the intersection of two lines or arcs, for example, the centre of a circle is determined by the right bisectors of two chords of a circle.

To construct a circle with ruler and compasses it is necessary to know three independent facts or conditions concerning the circle or the construction cannot be made. Three conditions on a circle may be expressed in many ways and it is necessary to reduce these three conditions to a determination of the centre and radius.

The following fundamental loci and definitions are recalled:

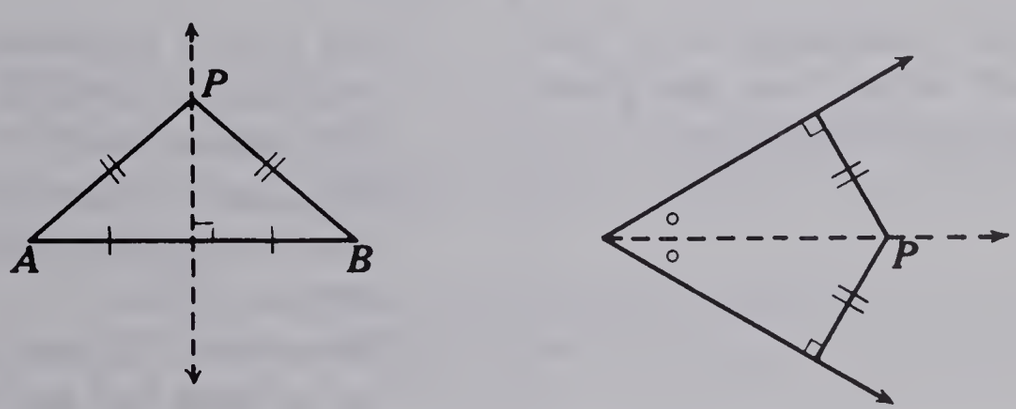
1. A circle is the set of points each of which is a given distance (radius) from a fixed point (centre).
2. The locus or set of points, such that each point is a distance d units from a given line l, is the pair of straight lines m_1 and m_2 parallel to l and a distance d units on either side of it (*Fig.11-30*(*a*).)



(*a*) (*b*)

Fig. 11-30

3. The locus or set of points, such that each point is equally distant from two parallel lines l_1 and l_2, is the line m which is parallel to l_1 and l_2 and is midway between them. (*Fig.11-30*(*b*).)
4. The locus or set of points, such that each point is equally distant from two points A and B, is the right bisector l of the line segment of which A and B are the end points. (*Fig.11-31* (*a*).)



(*a*) (*b*)

Fig. 11-31

5. The locus or set of points, such that the points are equally distant from the sides of an angle, is the bisecting ray of the angle (*Fig11-31*.*b*)).
6. The locus of the centres of circles tangent to a given line l at a given point A of the line is the line m, such that $m \perp l$ and m is on A.
7. The locus of the centres of circles which lie on two given points A and B is the right bisector m of the line segment AB.

The following examples illustrate a method of analysis for construction problems.

Example 1. Construct a circle on three given distinct points.

Solution.

Step 1.

Set down the conditions of the problem.

Step 2.

Mark on the representative diagram the given conditions and pertinent facts known or deduced.

Step 3.

Discuss the determination of the centre and radius from the given conditions and deduced facts.

Also discuss the possible number of solutions and the conditions for these.

Analysis.

1. *Conditions:*

A circle is required such that

(i) point A
(ii) point B
(iii) point C

} lie on the circle.

2. *Representative diagram.*

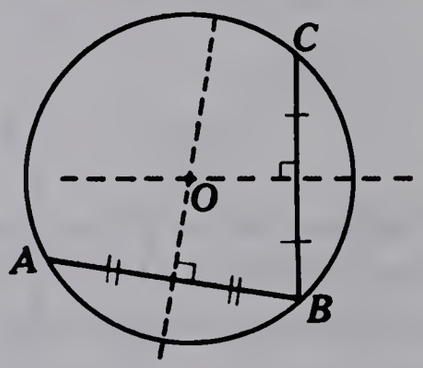


Fig. 11-32

3. *Discussion.*

(a) The locus of centres to satisfy conditions (i) and (ii) is the right bisector of AB.

(b) The locus of centres to satisfy conditions (ii) and (iii) is the bisector of BC.

(c) The centre O is the intersection of these two loci.

(d) The radius is OA or OB or OC.

(e) The solution is unique if A, B, and C are not in the same straight line. If A, B, and C are collinear, then the right bisectors are parallel and no circle can be constructed.

The circle(s) can now be constructed by constructing the required loci or the parts of them necessary to determine the required points or lines.

Example 2. Construct a circle with a given radius tangent to two intersecting straight lines.

Analysis.

1. *Conditions.*
   A circle (centre O) is required such that:
   (i) radius = r units;
   (ii) tangent to l_1 } intersecting lines.
   (iii) tangent to l_2 }
2. *Representative diagram.*

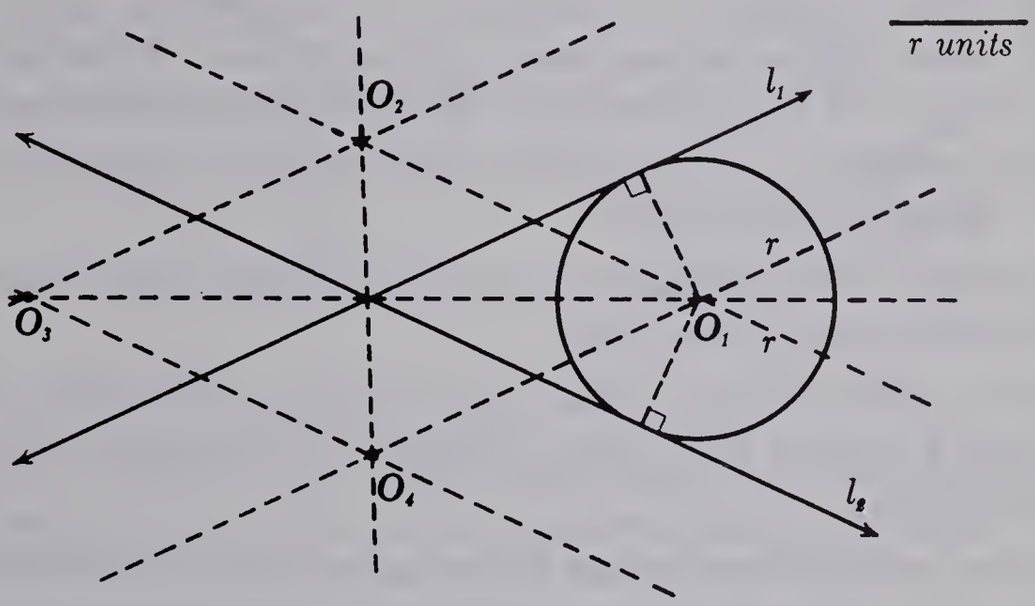


Fig. 11-33

3. *Discussion.*
   It is sufficient to determine O, since r is given.
   (a) The locus of centres of circles satisfying conditions (i) and (ii) is the pair of lines parallel to l_1 and a distance r units from l_1 on either side of l_1.
   (b) The locus of centres of circles satisfying conditions (i) and (iii) is the pair of lines parallel to l_2 and a distance r units from l_2 on either side of l_2.
   (c) The centre(s) of the required circle(s) is the intersection of these two loci.
   (d) It should be noted that the locus of centres of circles satisfying conditions (ii) and (iii) is the bisector of the angles determined by the two lines. The intersections of the three loci will be the same points.
   (e) There are four circles which satisfy the given conditions.

The circles may now be constructed by constructing the required loci or the parts of them necessary to determine the required points.

Exercise 11-9

(B)

In each of the following, for which a construction is required, make an analysis of the problem and the ruler-compasses construction.

1. Construct a circle to touch two parallel straight lines and a given transversal of these lines.

2. Construct a circle to touch a given line on a given point of the line and to lie on a given point not on the line.

3. (i) What is the locus of centres P of circles which touch a given circle externally and have a given radius? Illustrate such a locus and four of the family of circles satisfying the conditions.

   (ii) Construct a circle of given radius to touch a given line and a given circle externally.

4. Construct a circle tangent to a given line and to touch a given circle at a given point of the circle.

5. Draw a circle of given radius with its centre on one given circle and touching another given circle. Consider both internal and external possibilities.

6. If two parallel diameters are drawn in two circles which touch each other externally, the point of contact and one end of each diameter are collinear. Prove that the same is true if the circles touch internally.

7. Construct a circle to touch externally a given circle and a given straight line at a given point. (Hint: apply the results of question 6.) Consider the case in which the circles touch internally.

8. (i) State the relationship between the tangent segment PA to a circle and the secant segments PC and PD, if the secant PCD intersects the circle at C and D.

   (ii) Show how to construct a line segment which is a mean proportional to two given line segments.

   (iii) Construct a circle tangent to a given line and on two given points not on the line.

9. Construct a circle tangent to two intersecting lines and on a given point not on either line.

10. Draw four circles tangent to three lines which intersect in pairs.

11. Draw a circle to touch two given circles externally and to be tangent to a given line.

11•16 **Loci and intersections of loci in three-dimensional space (supplementary).**

1. *Lines and planes.*

(i) The locus of points in three-space equally distant from two fixed points, A and B, (*Fig.11-34*) is a plane, p, which is the perpendicular bisector of the line segment joining the two points. Only a portion of the plane is shown; it is unlimited in extent.

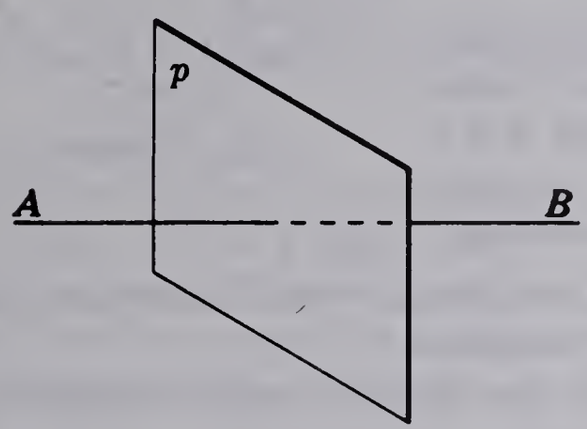


Fig. 11-34

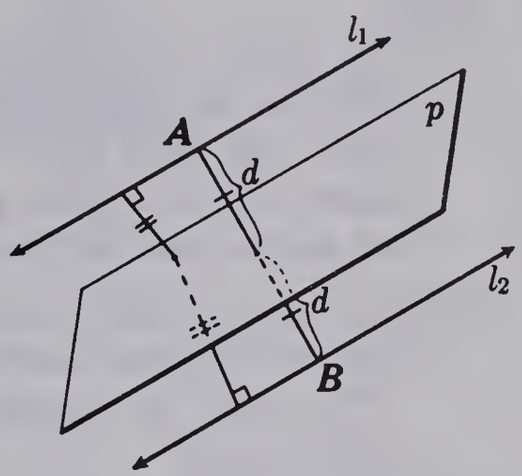


Fig. 11-35

(ii) The locus of points in three-space equidistant from two parallel lines, l_1 and l_2, (*Fig.11-35*) is a plane which is the perpendicular bisector of a line segment AB, determined by the lines and a perpendicular to the lines.

(iii) The locus of points at a given distance, d units, from a given plane, q, (*Fig.11-36*) is a pair of parallel planes, p and r, one on each side of the given plane, parallel to it and at the given distance from it.

(iv) The locus of points equidistant from two parallel planes, p and r, (*Fig.11-36*) is a plane q which is parallel to them and midway between them.

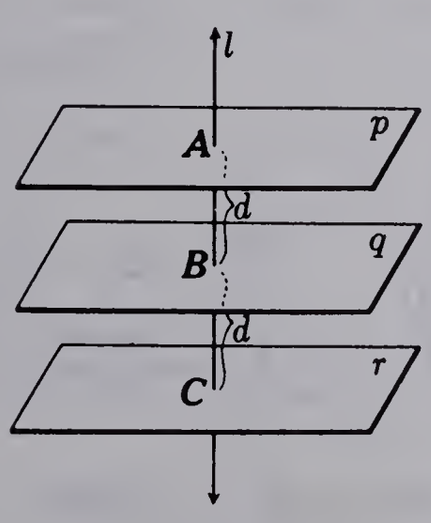


Fig. 11-36

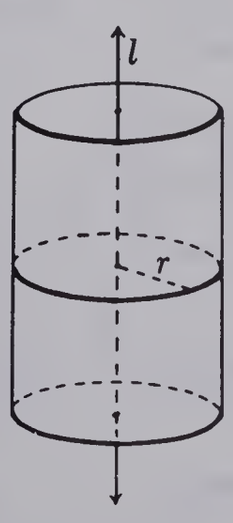


Fig. 11-37

2. *Circular cylindrical surface.*

The locus of points in three-space at a given distance, r units, (*Fig.11-37*) from a given line, l, is a circular cylindrical surface with the line as axis and the radius equal to the given distance.

3. *Sphere.*

(i) The locus of points in three-space which are at a given distance, r units, (*Fig.11-38*) from a given point A is a sphere with the given point as centre and radius equal to the given distance.

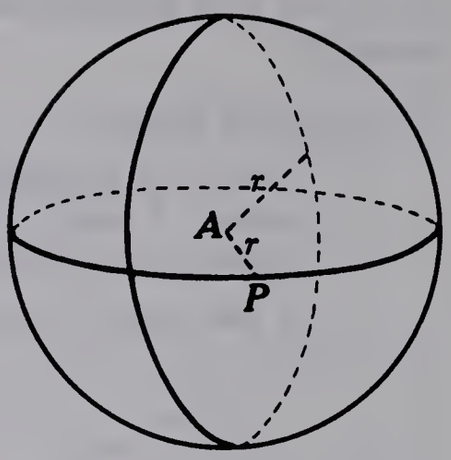


Fig. 11-38

(ii) The locus of points such that the distance from the centre is less than r units is the *interior* of the sphere.

The union of a sphere and its interior is called a *globe* or a *spherical solid.*

(iii) The locus of points such that the distance from the centre is greater than r units is the exterior of the sphere.

It should be noted that the locus of (i) is a space-separating locus, while those of parts (ii) and (iii) are space-filling loci.

4. *Intersection of sphere and plane.*

The locus of points in three-space a distance r units from a given point O and equidistant from given points A and B is illustrated in *Fig. 11-39*.

The points satisfying the first condition are the points of a sphere with centre O and radius r units. The points satisfying the second condition are points of a plane, p, which is the perpendicular bisector of AB.

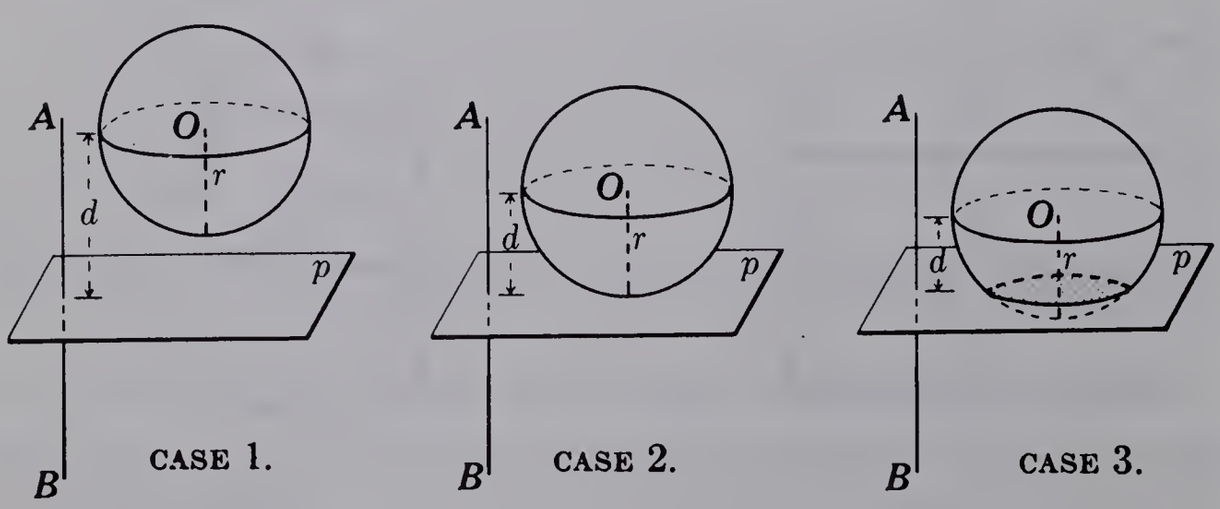


Fig. 11-39

If d units is the distance from the centre O of the sphere to the plane, the following three cases arise:

CASE 1. If $d > r$, the intersection set of the sphere and the plane is the null set.

CASE 2. If $d = r$, the intersection set consists of one point, C, the point of tangency.

CASE 3. If $d < r$, the intersection set is a circle.

5. *Right-circular cone.*

In *Fig 11-40* (*a*) a line AB intersects a fixed line XY at a fixed point V at an angle AVX whose measurement is $\theta°$. For any given θ the locus of points determined by all such lines AB is a *double-napped right-circular cone.*

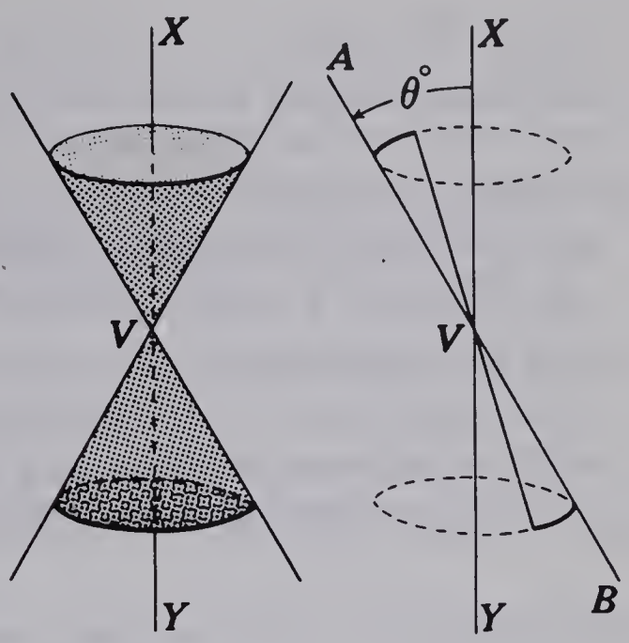


Fig. 11-40

The line AB is said to *generate* the cone, *Fig 11-40* (*b*); the fixed point is called the *vertex;* each half of the cone is called a *nappe* of the cone; the fixed line XY is the *axis* of the cone; AB is an element of the cone.

6. *Conic sections.*

The intersection of a cone and a plane define four important loci. Each of these loci illustrated in *Fig 11-41* and *11-42* is called a conic section.

CASE 1. *The circle.*

The locus defined by the intersection of a cone and a plane perpendicular to the axis of the cone is a *circle*, *Fig 11-41* (*a*).

A single point, the vertex, is obtained as a limiting case, *Fig 11-41* (*b*).

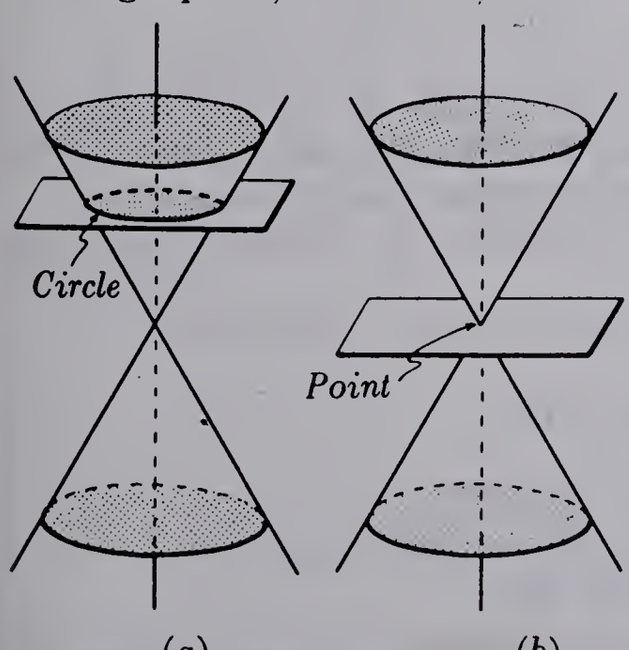


Fig. 11-41

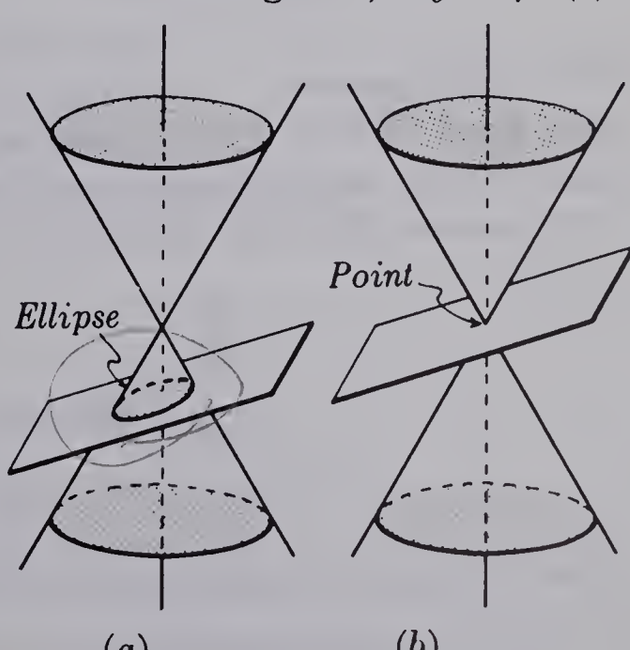


Fig. 11-42

CASE 2. *The ellipse.*

The locus defined by the intersection of a cone and a plane which intersects only one nappe of the cone and is not parallel to an element of the cone is an *ellipse, Fig.11-42* (*a*).

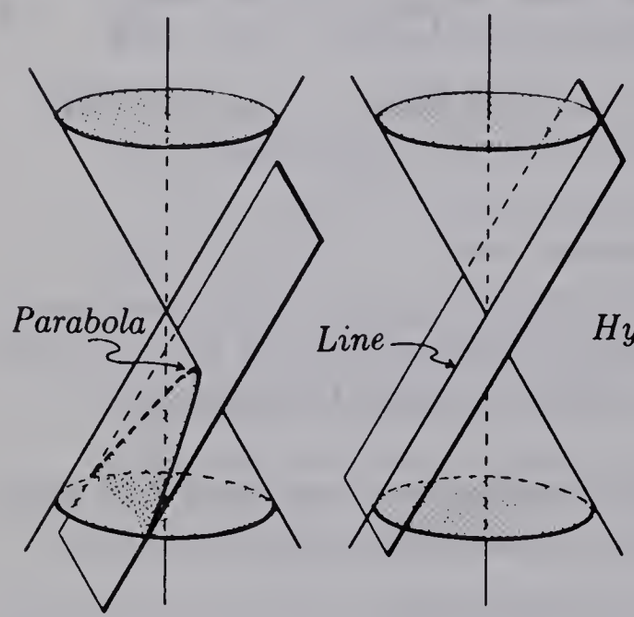
A single point, the vertex, is obtained as a limiting case, *Fig.11-42* (*b*).

A circle is also a special case of an ellipse.

CASE 3. *The parabola.*

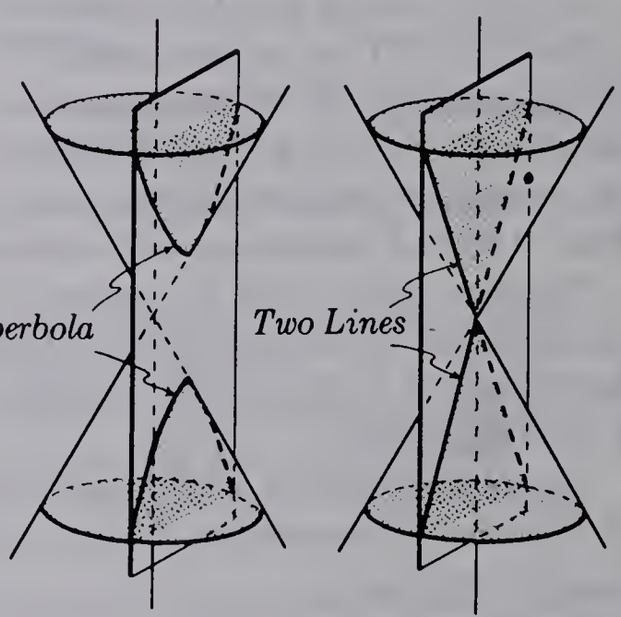
The locus defined by the intersection of one nappe of a cone and a plane which is parallel to an element of the cone is a *parabola, Fig 11-43*(*a*).

A line, (an element) is obtained as a limiting case, *Fig.11-43* (*b*).



(*a*) (*b*)

Fig. 11-43



(*a*) (*b*)

Fig. 11-44

CASE 4. *The hyperbola.*

The locus defined by the intersection of a cone and a plane which intersects both nappes of the cone is a *hyperbola, Fig 11-44*(*a*).

Two intersecting lines (elements) are obtained as a limiting case, *Fig. 11-44* (*b*).

Exercise 11-10

(B)

Draw diagrams to illustrate the following loci:

1. The locus of points a distance d_1 units from a plane p and a distance d_2 units from a plane q which is parallel to plane p.

Draw diagrams to illustrate the following loci:

2. The locus defined by the intersection of two parallel planes and a third plane.
3. The locus of points in three-dimensional space equidistant from two given points A and B and at a distance d units from the line on AB.
4. The locus of points equidistant from the faces of a dihedral angle.
5. The locus defined by the intersection of a dihedral angle and a plane perpendicular to its edge.
6. The locus of points at a distance $\frac{1}{2}$ in. from a sphere of radius 1 in.
7. The locus defined by the intersection of a sphere with a plane through its centre.
8. The locus of points in three-space 1 in. from point A and equidistant from points A and B where $AB = 1\frac{1}{2}$ in.
9. The locus of points in three-space at which a line segment subtends a right angle.
10. The locus of points of a sphere between two parallel planes perpendicular to a diameter of the sphere and distance apart less than the diameter of the sphere.
11. The section of one nappe of a cone bounded by its surface and the intersections of two parallel planes perpendicular to its axis is called a *frustum* of a cone. Draw a diagram of a frustum of a cone.

11·17 The circle as the graph of a relation.

Example. For the relation

$$C = \{(x, y) \mid x^2 + y^2 = 25,\ x, y \in R\}:$$

(i) determine the intercepts of the graph;
(ii) determine the domain and range of C;
(iii) discuss the symmetry of the graph;
(iv) make a table of values and sketch the graph.

Solution.

(i) *x-intercepts.* Let $y = 0$ in $x^2 + y^2 = 25$.

$$\therefore \quad x^2 = 25.$$

$$\therefore \quad x = \pm 5.$$

$\therefore$ the x-intercepts are 5 and -5.

y-intercepts. Let $x = 0$ in $x^2 + y^2 = 25$.

$$\therefore \quad y^2 = 25.$$

$$\therefore \quad y = \pm 5.$$

$\therefore$ the y-intercepts are 5 and -5.

The four points with coordinates (5, 0), (− 5, 0), (0, 5), and (0, − 5) are on the graph (*Fig 11-45*).

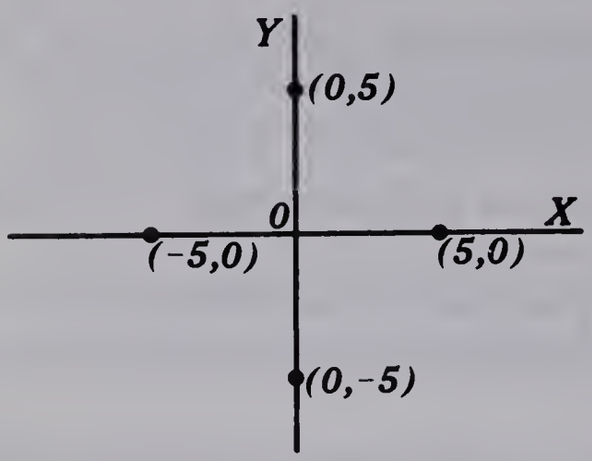


Fig. 11-45

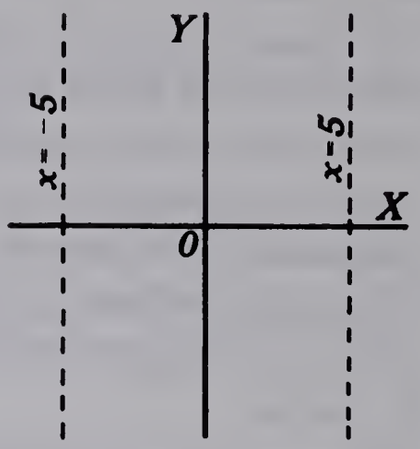


Fig. 11-46

(ii) *Domain.*

$$x^2 + y^2 = 25$$
$$\leftrightarrow \quad y^2 = 25 - x^2$$
$$\leftrightarrow \quad y = \pm\sqrt{25 - x^2}.$$
$$\therefore\ y \in R \leftrightarrow 25 - x^2 \geqq 0,\ x \in R$$
$$\leftrightarrow \quad x^2 \leqq 25$$
$$\leftrightarrow \quad |x| \leqq 5.$$

$\therefore$ the domain is $\{x \mid -5 \leqq x \leqq 5, x \in R\}$.

The graph must lie on or between the vertical broken lines in *Fig 11-46*.

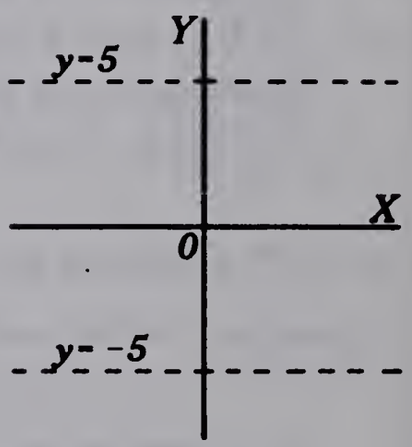


Fig. 11-47

Range.

$$x^2 + y^2 = 25$$
$$\leftrightarrow \quad x^2 = 25 - y^2$$
$$\leftrightarrow \quad x = \pm\sqrt{25 - y^2}.$$
$$\therefore\ x \in R \leftrightarrow 25 - y^2 \geqq 0,\ y \in R$$
$$\leftrightarrow \quad y^2 \leqq 25$$
$$\leftrightarrow \quad |y| \leqq 5.$$

$\therefore$ the range is $\{y \mid -5 \leqq y \leqq 5, y \in R\}$.

The graph must lie on or between the horizontal broken lines in *Fig 11-47*. The actual shape of the graph is not yet known. It might be as shown in Fig. *11-48*.

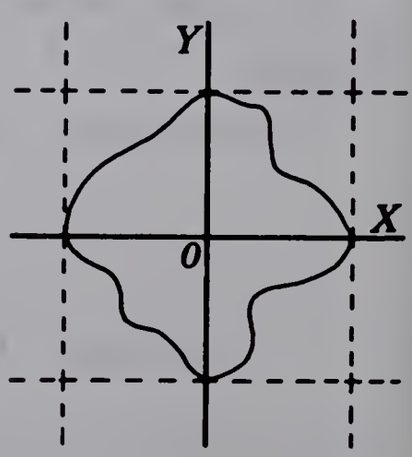


Fig. 11-48

(iii) *Symmetry.* If y is replaced by $-y$, then $x^2 + y^2 = 25$ is unchanged.
$\therefore$ the graph is symmetric with respect to the x-axis.
If x is replaced by $-x$, then $x^2 + y^2 = 25$ is unchanged.
$\therefore$ the graph is symmetric with respect to the y-axis.
If x and y are replaced by $-x$ and $-y$ respectively, then $x^2 + y^2 = 25$ is unchanged.
$\therefore$ the graph is symmetric with respect to the origin.

Although the actual shape of the graph is not yet known, the symmetry indicates that it might be as shown in *Fig. 11-49*.

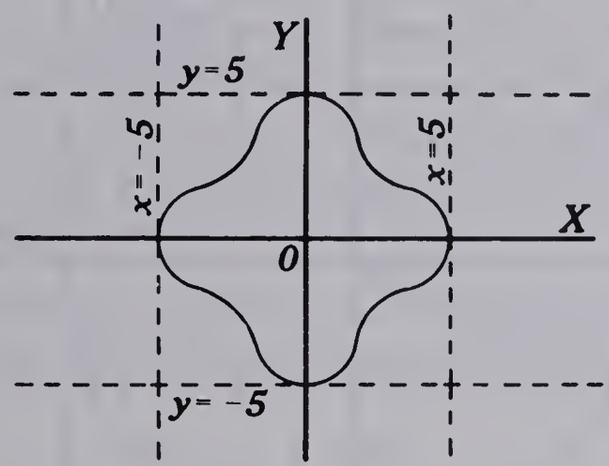


Fig. 11-49

Note: Symmetry may also be deduced from the equations:

$$y = \pm\sqrt{25 - x^2} \qquad (1)$$

and $$x = \pm\sqrt{25 - y^2}. \qquad (2)$$

From (1), for each x in the domain, there are two values of y equal in magnitude but opposite in sign. That is, for each point $P(x, y)$ on the graph there is a corresponding point $P'(x, -y)$ also on the graph. Therefore, the graph is symmetric with respect to the x-axis (*Fig. 11-50*).

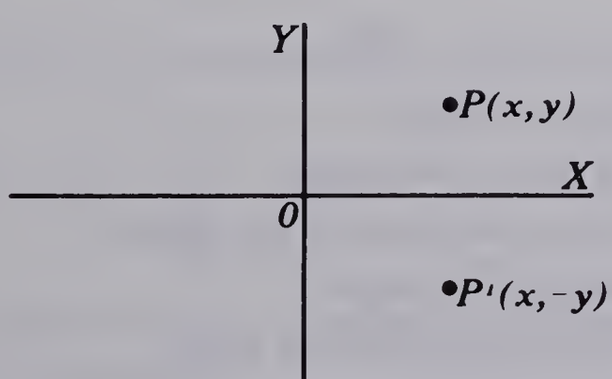


Fig. 11-50

Similarly symmetry with respect to the y-axis may be deduced using equation (2).

(iv) *Table of values.* It is sufficient to determine coordinates for points in the first quadrant. When these points are plotted, points in the other three quadrants may be determined by symmetry (*Fig.11-51*).

x	0	3	4	5
y	5	4	3	0

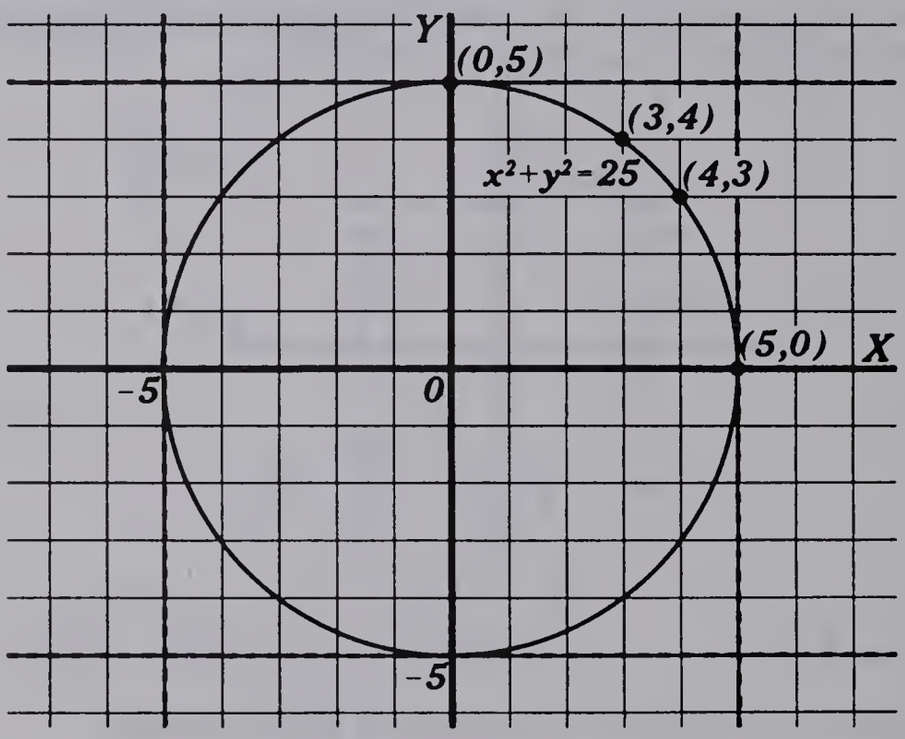


Fig. 11-51

The graph of C appears to be a circle with centre $O(0, 0)$ and radius 5.

Exercise 11-11

(B)

For each of the following relations:

(i) *determine the intercepts of the graph;*

(ii) *determine the domain and range of the relation;*

(iii) *discuss the symmetry of the graph;*

(iv) *make a table of values and sketch the graph.*

1. $A = \{(x, y) \mid x^2 + y^2 = 16, x, y \in R\}$
2. $B = \{(x, y) \mid x^2 + y^2 = 13, x, y \in R\}$
3. $C = \{(x, y) \mid x^2 + y^2 = 25, |x| > 2, x, y \in R\}$
4. $D = \{(x, y) \mid y = \sqrt{36 - x^2}, x, y \in R\}$

(C)

5. $E = \{(x, y) \mid x^2 + y^2 - 2y = 8, x, y \in R\}$
   (Hint: complete the square of the terms containing y.)
6. $F = \{(x, y) \mid x^2 + y^2 + 2x - 4 = 0, x, y \in R\}$.

***11·18* The standard form of the equation of the circle with centre $O(0, 0)$.**

In Section 11·17 the graph of the relation

$$C = \{(x, y) \mid x^2 + y^2 = 25, x, y \in R\}$$

was sketched after considering the intercepts, the domain, the range, and symmetry. The conjecture was made that the graph of this relation is a circle. From the definition of a circle it may be deduced that any quadratic relation with a defining equation of the form $x^2 + y^2 = r^2$ has as its graph a circle with centre $O(0, 0)$ and radius r.

Theorem. *The graph of a relation is a circle with centre $O(0, 0)$ and radius $r (r \in {}^+R)$ if and only if the defining sentence of the relation may be written in the form*

$$x^2 + y^2 = r^2, \quad x, y \in R.$$

Proof:

(i) If $P(x, y)$ represents any point on the circle with centre $O(0, 0)$ and radius r, $(r > 0)$, then

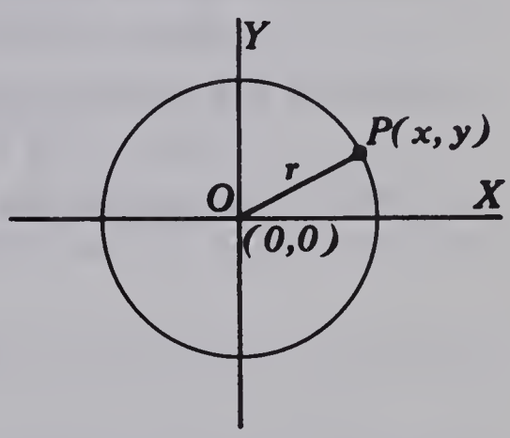


Fig. 11-52

$$PO = r \text{ (Definition)}$$

$$\therefore \sqrt{(x - 0)^2 + (y - 0)^2} = r,$$

or $$x^2 + y^2 = r^2.$$

(ii) Conversely, if $P(x, y)$ represents any point such that

$$x^2 + y^2 = r^2,$$

then $$\sqrt{(x - 0)^2 + (y - 0)^2} = |r| = r \qquad (\because r > 0)$$

or $$PO = r.$$

Thus $P(x, y)$ represents those points and only those points of the circle with centre $O(0, 0)$ and radius r.

$\therefore$ the defining sentence of the circle is

$$x^2 + y^2 = r^2, x, y \in R.$$

The equation $x^2 + y^2 = r^2$ is called the *standard form of the equation of the circle with centre $O(0, 0)$ and radius r.*

Example 1. Identify and sketch the graph of the relation

$A = \{(x, y) \mid x^2 + y^2 = 5, x, y \in R\}$.

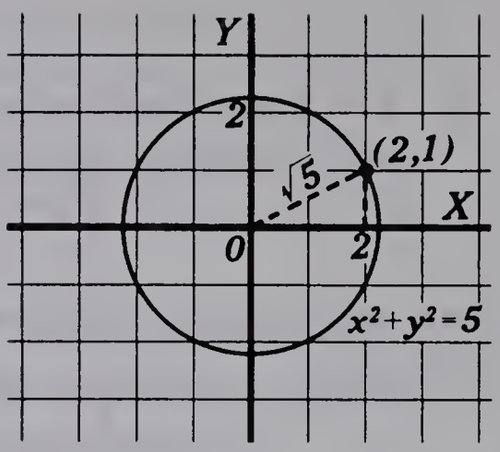


Fig. 11-53

Solution.

$$x^2 + y^2 = 5$$
$$\leftrightarrow \quad x^2 + y^2 = (\sqrt{5})^2.$$

$\therefore$ the graph of the relation is the circle with centre $O(0, 0)$ and radius $\sqrt{5}$ *(Fig. 11-53)*.

Example 2. Determine whether the points $A(1, 1)$, $B(-2, 7)$, $C(3, 4)$ are points of the interior, or of the exterior, or of the circle defined by $x^2 + y^2 = 25$.

Solution.

The centre of the circle is $O(0, 0)$, and the radius is 5.

$$AO = \sqrt{(1-0)^2 + (1-0)^2} = \sqrt{2} < 5.$$

$\therefore$ $A(1, 1)$ is a point of the interior of the circle.

$$BO = \sqrt{(-2-0)^2 + (7-0)^2} = \sqrt{53} > 5.$$

$\therefore$ $B(-2, 7)$ is a point of the exterior of the circle.

$$CO = \sqrt{(3-0)^2 + (4-0)^2} = \sqrt{25} = 5.$$

$\therefore$ $C(3, 4)$ is a point of the circle.

Exercise 11-12

(A)

1. State the standard form of the equation of each of the following circles:

(i) centre $O(0, 0)$, radius 3; (ii) centre $O(0, 0)$, radius $\sqrt{5}$;

(iii) centre $O(0, 0)$, radius $3\sqrt{3}$; (iv) centre $O(0, 0)$, radius $3\sqrt{5}$.

(B)

2. Write the standard form of the equation of each of the following circles:

(i) centre $O(0, 0)$, radius 9;

(ii) centre $O(0, 0)$, radius $5\sqrt{3}$;

(iii) centre $O(0, 0)$, and on $A(5, 0)$;

(iv) centre $O(0, 0)$, and on $B(0, -\sqrt{7})$;

(v) centre $O(0, 0)$, and on $C(-8, -6)$;

(vi) centre $O(0, 0)$, and on $D(\sqrt{3}, 1)$.

3. Write the standard form of the equation of each of the following circles:

(i) centre the origin, tangent to the line defined by $y - 3 = 0$;

(ii) centre the origin, tangent to the line defined by $x + 6 = 0$;

(iii) centre the origin, tangent to the line defined by $x + y = 7$;

(iv) centre the origin, tangent to the line defined by $x - y = 2$.

4. For each of the following points, determine whether it is a point of the interior, or of the exterior, or of the circle defined by $x^2 + y^2 = 25$:

(i) $A(0, 5)$ (ii) $B(1, 4.5)$ (iii) $C(-4, -3)$ (iv) $D(-4, 3)$

(v) $E(2, -\sqrt{21})$ (vi) $F(-2, -\sqrt{21})$ (vii) $G(3.5, 4)$

11·19 Inequalities.

Example 1. Draw the graphs of the following relations:

$$\text{(i)}\ A = \{(x, y) \mid x^2 + y^2 < 4,\ x, y \in R\}$$
$$\text{(ii)}\ B = \{(x, y) \mid x^2 + y^2 > 4,\ x, y \in R\}$$
$$\text{(iii)}\ C = \{(x, y) \mid x^2 + y^2 \leqq 4,\ x, y \in R\}.$$

Solution. The equation $x^2 + y^2 = 4$ defines the set of points $P(x, y)$ of the circle, centre $O(0, 0)$ and radius 2. Thus, for the circle, $OP = 2$.

(i) $P(x, y)$ is a point of the graph defined by $x^2 + y^2 < 4$ if and only if $OP < 2$.

Thus the graph is the set of points of the interior of the circle (*Fig.11-54*). (The broken line indicates that the circle is not part of the graph.)

(ii) $P(x, y)$ is a point of the graph defined by $x^2 + y^2 > 4$ if and only if $OP > 2$.

Thus the graph is the set of points of the exterior of the circle. (*Fig.11-55*).

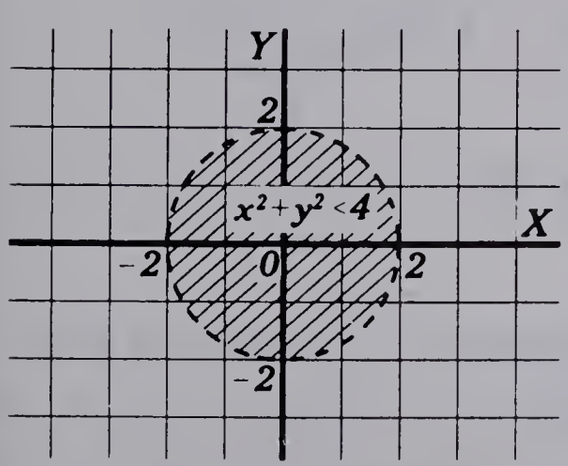


Fig. 11-54

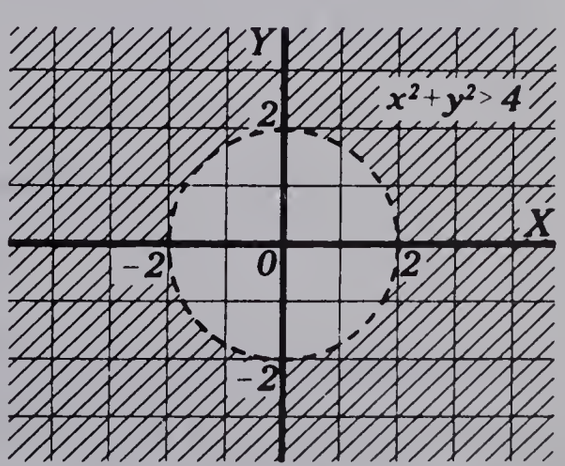


Fig. 11-55

(iii) $P(x, y)$ is a point of the graph defined by $x^2 + y^2 \leqq 4$ if and only if $OP \leqq 2$.

Thus the graph is the set of points of the interior of the circle and of the circle *(Fig.11-56)*.

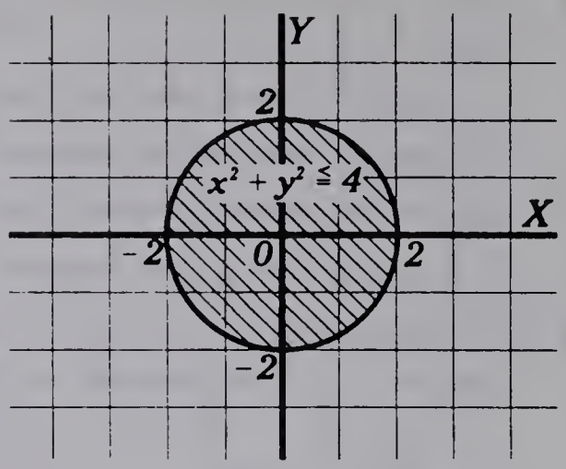


Fig. 11-56

Example 2. Use the set-builder notation to define a relation for which each of the following is a graph:

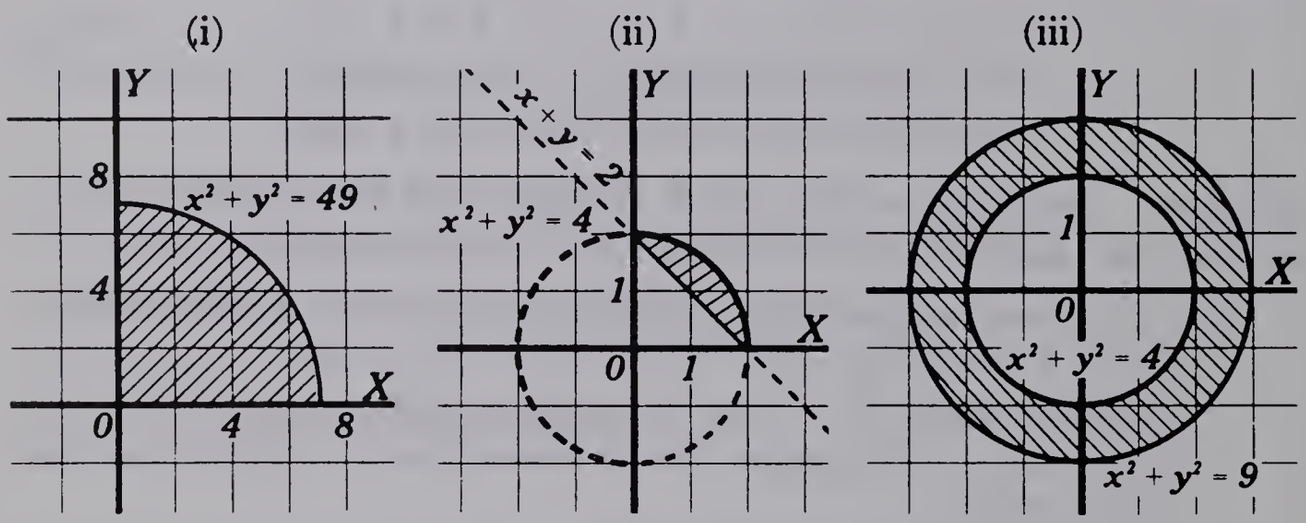


Fig. 11-57

Solution.

(i) $\{(x, y) \mid x^2 + y^2 \leqq 49, x \geqq 0, y \geqq 0, x, y \in R\}$

(ii) $\{(x, y) \mid x^2 + y^2 \leqq 4, x + y \geqq 2, x, y \in R\}$

(iii) $\{(x, y) \mid x^2 + y^2 \geqq 4, x^2 + y^2 \leqq 9, x, y \in R\}$

Exercise 11-13

(B)

Draw the graphs of the following relations:

1. $A = \{(x, y) \mid x^2 + y^2 > 16, x, y \in R\}$
2. $B = \{(x, y) \mid x^2 + y^2 \geqq 9, x, y \in R\}$
3. $C = \{(x, y) \mid x^2 + y^2 < 49, x > 0, y > 0, x, y \in R\}$

Draw the graphs of the following relations:

4. $D = \{(x, y) \mid x^2 + y^2 \leqq 16, x^2 + y^2 \geqq 9, x, y \in R\}$

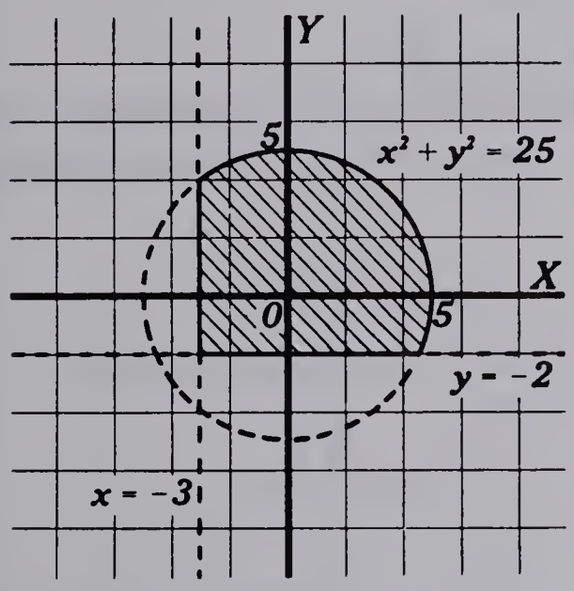
5. $E = \{(x, y) \mid x^2 + y^2 \leqq 9, x + y \geqq 2, x, y \in R\}$

6. $F = \{(x, y) \mid x^2 + y^2 \leqq 25, x + y \leqq -3, x, y \in R\}$

7. $G = \{(x, y) \mid x^2 + y^2 \leqq 4, y \geqq 1, x \geqq 0, x, y \in R\}$

Use the set-builder notation to define a relation for which each of the following is a graph:

8.



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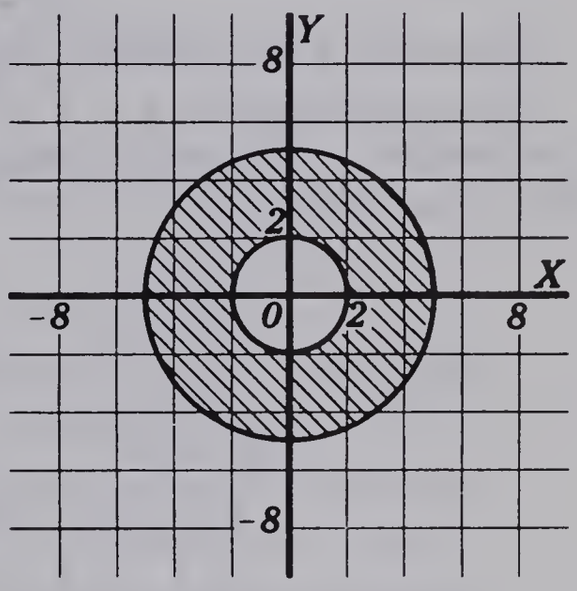

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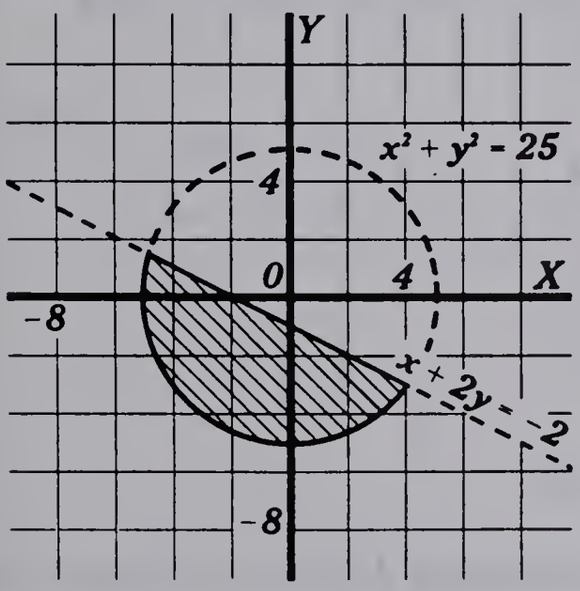

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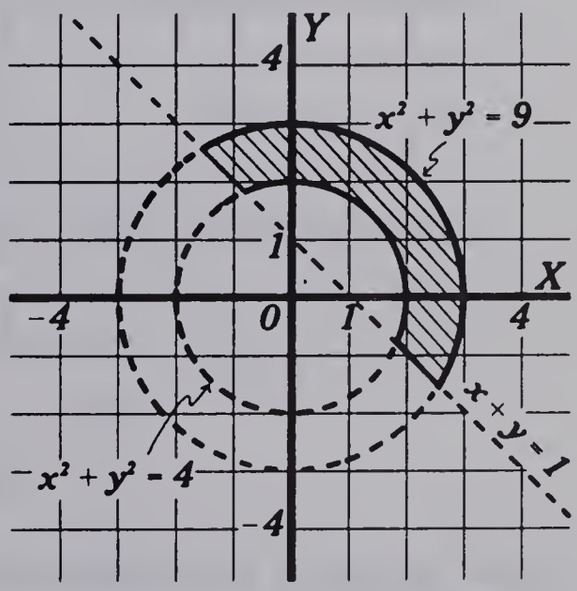


(C)

Draw the graphs of the following relations:

12. $H = \{(x, y) \mid x^2 + y^2 \leqq 16, y \geqq x^2, x, y \in R\}$

13. $J = \{(x, y) \mid x^2 + y^2 \leqq 9, \quad y \geqq -x^2 + 3, x, y \in R\}$

14. $K = \{(x, y) \mid x^2 + y^2 \leqq 16, y \geqq x^2 + 4, x, y \in R\}$

11•20 **The solution of systems of one linear and one quadratic equation in two variables.**

Example 1. Determine, algebraically, the ordered pairs of

$$S = \{(x, y) \mid x - y = 1,\ x^2 + y^2 = 25,\ x, y \in R\}.$$

Solution.

S is the solution set of the system of equations:

$$\begin{cases} x - y = 1 \\ x^2 + y^2 = 25. \end{cases}$$

The graph of this system is shown in *Fig. 11-58*,

The graph defined by $x - y = 1$ is a straight line with x-intercept 1 and y-intercept -1. The graph defined by $x^2 + y^2 = 25$ is a circle with centre $O(0, 0)$ and radius 5.

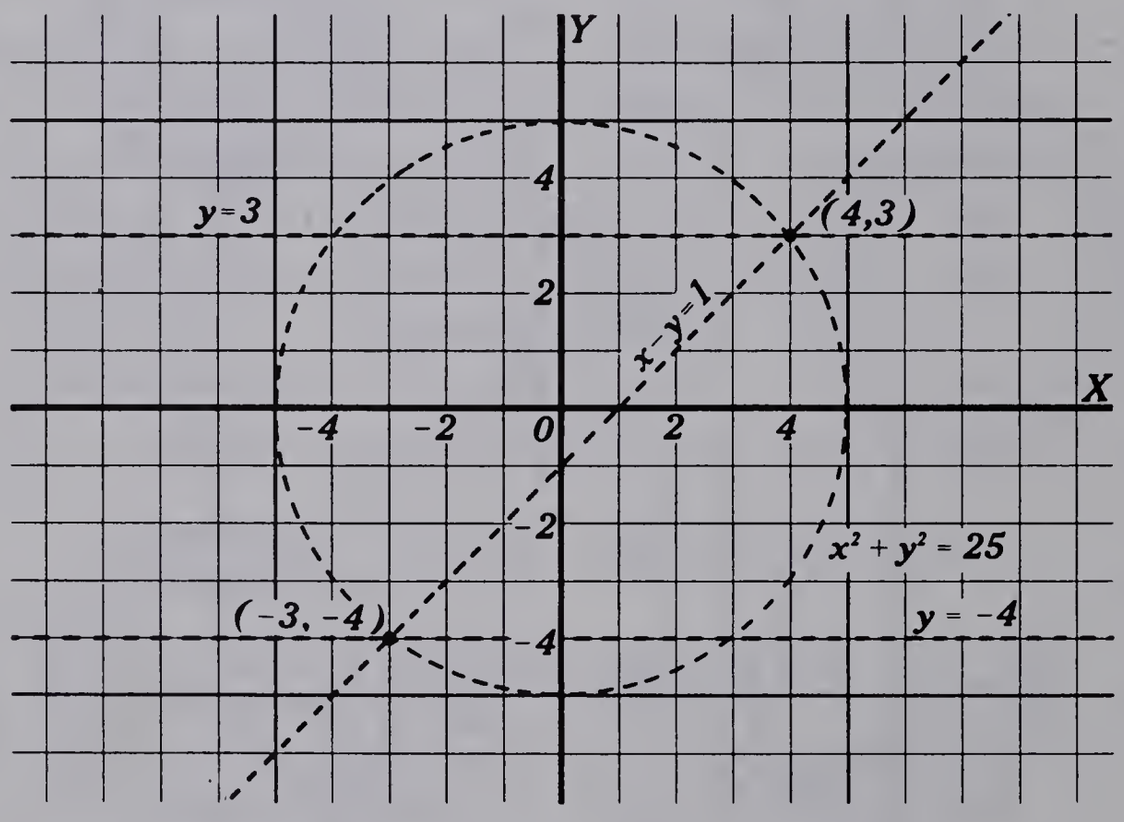


Fig. 11-58

From the graph we conclude that the original system is equivalent to the union of the two systems

$$\begin{cases} x - y = 1 \\ y = 3 \end{cases} \qquad \begin{cases} x - y = 1 \\ y = -4. \end{cases}$$

We may obtain these systems algebraically by the method of elimination of a variable by substitution. The algebraic solution is written in the following form:

$$\begin{cases} x - y = 1 \\ x^2 + y^2 = 25 \end{cases}$$

$$\leftrightarrow \quad \begin{cases} x = 1 + y \\ x^2 + y^2 = 25 \end{cases}$$

$$\leftrightarrow \quad \begin{cases} x = 1 + y \\ (1 + y)^2 + y^2 = 25 \end{cases}$$

$$\leftrightarrow \quad \begin{cases} x = 1 + y \\ 2y^2 + 2y - 24 = 0 \end{cases}$$

$$\leftrightarrow \quad \begin{cases} x = 1 + y \\ (y + 4)(y - 3) = 0 \end{cases}$$

$$\leftrightarrow \quad \begin{cases} x = 1 + y \\ y + 4 = 0 \text{ or } y - 3 = 0 \end{cases}$$

$$\leftrightarrow \quad \begin{cases} x = 1 + y \\ y = -4 \end{cases} \quad \text{or} \quad \begin{cases} x = 1 + y \\ y = 3 \end{cases}$$

$$\leftrightarrow \quad \begin{cases} x = -3 \\ y = -4 \end{cases} \quad \text{or} \quad \begin{cases} x = 4 \\ y = 3 \end{cases}.$$

$$\therefore \ S = \{(-3, -4), (4, 3)\}.$$

Example 2. Solve the system

$$\begin{cases} 2x - y = 1 \\ x^2 + y^2 = 25 \end{cases}$$

Solution.

$$\leftrightarrow \quad \begin{cases} 2x - y = 1 \\ x^2 + y^2 = 25 \end{cases}$$

$$\leftrightarrow \quad \begin{cases} y = 2x - 1 \\ x^2 + y^2 = 25 \end{cases}$$

$$\leftrightarrow \quad \begin{cases} y = 2x - 1 \\ x^2 + (2x - 1)^2 = 25 \end{cases}$$

$$\leftrightarrow \quad \begin{cases} y = 2x - 1 \\ 5x^2 - 4x - 24 = 0 \end{cases}$$

$$\leftrightarrow \quad \begin{cases} y = 2x - 1 \\ x = \dfrac{2 + 2\sqrt{31}}{5} \quad \text{or} \quad x = \dfrac{2 - 2\sqrt{31}}{5} \end{cases}$$

$$\leftrightarrow \quad \begin{cases} y = 2x - 1 \\ x = \dfrac{2 + 2\sqrt{31}}{5} \end{cases} \quad \text{or} \quad \begin{cases} y = 2x - 1 \\ x = \dfrac{2 - 2\sqrt{31}}{5} \end{cases}$$

$$\leftrightarrow \begin{cases} y = \dfrac{-1+4\sqrt{31}}{5} \\ x = \dfrac{2+2\sqrt{31}}{5} \end{cases} \quad \text{or} \quad \begin{cases} y = \dfrac{-1-4\sqrt{31}}{5} \\ x = \dfrac{2-2\sqrt{31}}{5} \end{cases}.$$

$\therefore$ the solution is $x = \dfrac{2+2\sqrt{31}}{5}$, $\quad y = \dfrac{-1+4\sqrt{31}}{5}$

or $x = \dfrac{2-2\sqrt{31}}{5}$, $\quad y = \dfrac{-1-4\sqrt{31}}{5}$.

Write a verification for this example and compare it with that on page 450.

Exercise 11-14

(B)

Determine algebraically the ordered pairs of the following relations; illustrate by a graph:

1. $A = \{(x, y) \mid y = x - 2,\ x^2 + y^2 = 100,\ x, y \in R\}$

2. $B = \{(x, y) \mid x + 2y = 5,\ x^2 + y^2 = 25,\ x, y \in R\}$

3. $C = \{(x, y) \mid x + y = 9,\ x^2 + y^2 = 41,\ x, y \in R\}$

4. $D = \{(x, y) \mid 2x + y = 2,\ x^2 + y^2 = 40,\ x, y \in R\}$

5. $E = \{(x, y) \mid 3x + 4y = 25,\ x^2 + y^2 = 25,\ x, y \in R\}$

6. $F = \{(x, y) \mid 2x - y = 5,\ x^2 + y^2 = 4,\ x, y \in R\}$

7. For each of the above problems, state whether the straight line: (i) is a secant of; (ii) is a tangent of; or (iii) does not intersect, the circle.

(C)

Determine algebraically the ordered pairs of the following relations; illustrate by a graph:

8. $G = \{(x, y) \mid 2x - y = -3,\ y = x^2,\ x, y \in R\}$

9. $H = \{(x, y) \mid 2x - y = 1,\ y = x^2,\ x, y \in R\}$

10. $K = \{(x, y) \mid 2x - y = 2,\ y = x^2,\ x, y \in R\}$

11. $L = \{(x, y) \mid 2x - y = 5,\ y = x^2 - 4,\ x, y \in R\}$

12. $M = \{(x, y) \mid 2x - y = 4,\ y = x^2 - 4,\ x, y \in R\}$

13. $N = \{(x, y) \mid 2x - y = 8,\ y = x^2 - 4,\ x, y \in R\}$

***11•21* Geometric relations proved by analytic methods.** Coordinate geometry enables us to solve problems in geometry by using algebraic methods.

Example 1. Prove that the centre of the circle defined by $x^2 + y^2 = 9$ is a point of the right bisector of the chord determined by $2x + y = 4$.

Solution.

The circle has centre $O(0, 0)$ and radius 3.

The secant determining the chord AB has slope -2, x-intercept 2, and y-intercept 4 (*Fig 11-59*).

CM, the right bisector of AB, is perpendicular to AB.

$\therefore$ the slope of CM is $-\dfrac{1}{-2}$ or $\dfrac{1}{2}$.

To find the coordinates of A and B:

$$\begin{cases} 2x + y = 4 \\ x^2 + y^2 = 9 \end{cases}$$

$$\leftrightarrow \begin{cases} y = 4 - 2x \\ x^2 + (4 - 2x)^2 = 9 \end{cases}$$

$$\leftrightarrow \begin{cases} y = 4 - 2x & (1) \\ 5x^2 - 16x + 7 = 0. & (2) \end{cases}$$

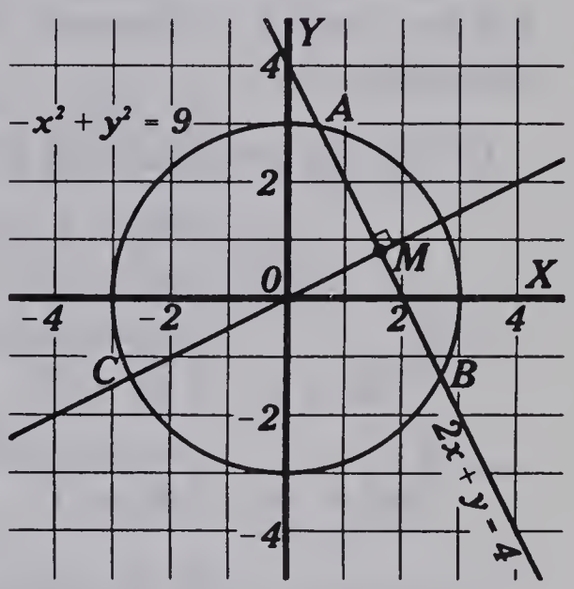


Fig. 11-59

If x_1 and x_2 are the roots of equation (2), then x_1 and x_2 are the abscissas of A and B.

The abscissa of M, the midpoint of AB, is $\dfrac{x_1 + x_2}{2}$.

But this is one-half the sum of the roots of equation (2).

$\therefore$ the abscissa of M is $\dfrac{1}{2}\left(\dfrac{16}{5}\right)$ or $\dfrac{8}{5}$.

Since M is in AB, the ordinate of M may be found by replacing x by $\dfrac{8}{5}$ in equation (1).

$\therefore$ the ordinate of M is $4 - 2\left(\dfrac{8}{5}\right)$ or $\dfrac{4}{5}$.

Thus CM has slope $\dfrac{1}{2}$ and is on the point $M\left(\dfrac{8}{5}, \dfrac{4}{5}\right)$

$\therefore$ the equation of CM is

$$y - \frac{4}{5} = \frac{1}{2}\left(x - \frac{8}{5}\right)$$

$$\leftrightarrow \quad 5x - 10y = 0.$$

$\therefore$ CM is on the origin.

$\therefore$ the centre of the circle is a point of the right bisector of the chord.

Example 2. Prove that the line on the centre of the circle defined by $x^2 + y^2 = 64$ and on the midpoint of the chord determined by $x - 2y = 4$ is perpendicular to the chord.

Solution.

The circle has centre $O(0, 0)$ and radius 8.

The secant determining the chord AB has slope $\frac{1}{2}$, x-intercept 4, and y-intercept -2.

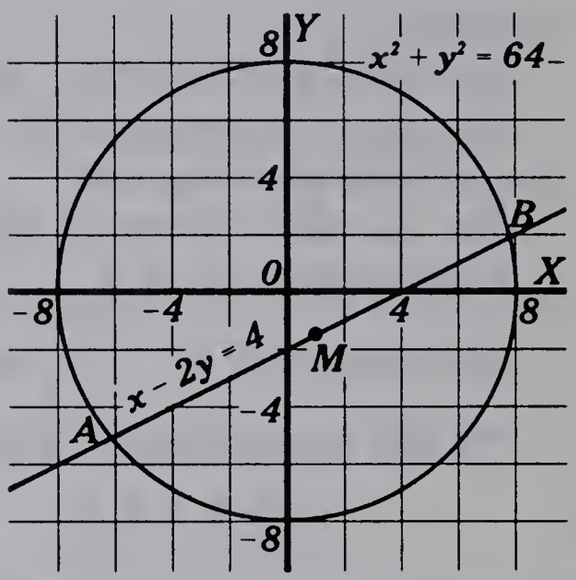


Fig. 11-60

To find the coordinates of A and B:

$$\begin{cases} x - 2y = 4 \\ x^2 + y^2 = 64 \end{cases}$$

$$\leftrightarrow \begin{cases} x = 2y + 4 \\ (2y + x)^2 + y^2 = 64 \end{cases}$$

$$\leftrightarrow \begin{cases} x = 2y + 4 & (1) \\ 5y^2 + 16y - 40 = 0 & (2) \end{cases}$$

The ordinate of M, the midpoint of AB, is

$$\frac{1}{2} \text{ (sum of the roots of equation (2))}$$

$$= \frac{1}{2}\left(-\frac{16}{5}\right)$$

$$= -\frac{8}{5}.$$

The abscissa of M is found by replacing y by $-\frac{8}{5}$ in (1).

$\therefore$ the abscissa of M is $\frac{4}{5}$.

Thus OM is on $O(0, 0)$ and $M\left(\frac{4}{5}, -\frac{8}{5}\right)$.

$\therefore$ the slope of OM is $-\dfrac{\frac{8}{5}}{\frac{4}{5}} = -2.$

$\because$ the slope of AB and OM are negative reciprocals,

$\therefore$ $OM \perp AB$.

Example 3. Determine the length of the tangent segment from the point $A(5, 4)$ to the circle defined by $x^2 + y^2 = 9$.

Solution. The circle has centre $O(0, 0)$ and radius 3.

AP is the tangent segment from A to the circle.

$$\begin{aligned} AP^2 &= AO^2 - OP^2 \\ &= (\sqrt{25 + 16})^2 - 3^2 \\ &= 41 - 9 = 32. \end{aligned}$$

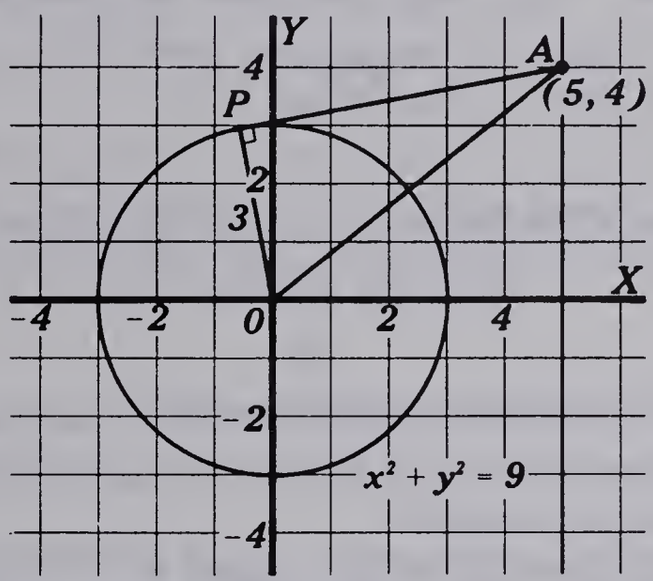


Fig. 11-61

$\therefore$ the length of the tangent segment from the point $A(5, 4)$ to the circle is $\sqrt{32} = 4\sqrt{2}$.

Exercise 11-15

(B)

1. Prove that the centre of the circle defined by $x^2 + y^2 = 25$ lies on the right bisector of the chord determined by $x + y = 5$.
2. Prove that the centre of the circle defined by $x^2 + y^2 = 16$ lies on the right bisector of the chord determined by $x - 3y = 4$.
3. Prove that the line drawn on the centre of the circle defined by $x^2 + y^2 = 25$ and on the midpoint of the chord determined by $x - 2y + 5 = 0$ is perpendicular to the chord.
4. Prove that the line drawn on the centre of the circle defined by $x^2 + y^2 = 13$ and on the midpoint of the chord determined by $3x - y = 6$ is perpendicular to the chord.
5. Determine the length of the tangent segment from the point $P(7, 5)$ to the circle defined by $x^2 + y^2 = 64$.

6. Determine the length of the tangent from the point $B(6, \sqrt{3})$ to the circle defined by $x^2 + y^2 = 15$.
7. Determine the length of the tangent from the point $P(3, 2)$ to the circle defined by $4x^2 + 4y^2 = 9$.

(C)

8. Prove that the centre of the circle defined by $x^2 + y^2 = r^2$ lies on the right bisector of the chord determined by $y = mx + k$, given $r^2(m^2 + 1) > k^2$. State why the latter condition is necessary.
9. Determine the length of the tangent from $P_1(x_1, y_1)$ to the circle defined by $x^2 + y^2 = r^2$, if $x_1^2 + y_1^2 > r^2$.

11•22 The equation of the tangent at a point of a circle.

Discovery Exercise 11-16

(B)

Compare your solutions to the following problems with those on page 451.

1. $P(4, 3)$ is a point of the circle whose defining equation is $x^2 + y^2 = 25$.
   (i) Determine the slope OP.
   (ii) Determine the slope of the tangent at P of the circle.
   (iii) Determine the equation of the tangent at P of the circle.
2. For each of the following, determine the equation of the tangent at the given point of the circle defined by the given equation; complete the table.

GIVEN POINT OF THE CIRCLE	EQUATION OF THE CIRCLE	EQUATION OF THE TANGENT AT THE GIVEN POINT
$Q(-4, 3)$	$x^2 + y^2 = 25$	
$R(3, -4)$	$x^2 + y^2 = 25$	
$S(\sqrt{2}, 3)$	$x^2 + y^2 = 11$	
$T(-\sqrt{2}, \sqrt{5})$	$x^2 + y^2 = 7$	
$V(\frac{1}{3}, -\sqrt{5})$	$9x^2 + 9y^2 = 46$	

3. Examine the table completed in 2 and observe how the equation of the tangent is related to the equation of the circle and the coordinates of the point. Using this pattern, write the equation of the tangent at the point $P(x_1,y_1)$ of the circle defined by $x^2 + y^2 = r^2$.

These examples suggest the following conclusion:

The equation of the tangent at the point $P_1(x_1, y_1)$ of the circle defined by $x^2 + y^2 = r^2$ is $x_1 x + y_1 y = r^2$.

Exercise 11-17

(B)

1. Write the equation of the tangent at the given point of the circle defined by each of the following equations:

(i) $P(-2, 7); x^2 + y^2 = 53$ (ii) $A(5, -\frac{1}{2}); x^2 + y^2 = \frac{101}{4}$

(iii) $B(-2, -\frac{1}{3}); 9x^2 + 9y^2 = 37$ (iv) $C(2, -3\sqrt{5}); x^2 + y^2 = 49$

(v) $D(2, -\frac{\sqrt{3}}{2})$; $16x^2 + 16y^2 = 76$

(C)

2. Prove that the equation of the tangent at the point $P_1(x_1, y_1)$ of the circle defined by $x^2 + y^2 = r^2$ is $x_1 x + y_1 y = r^2$.
(Hint: use the fact that $P_1(x_1, y_1)$ is a point of the circle if and only if $x_1^2 + y^2 = r_1^2$.)

Review Exercise 11-18

(B)

For each of the following draw a representative sketch, mark on all given data, and complete the required calculation.

1. A circle, centre O, has radius 5 cm. Calculate the distance from O to a chord 8 cm. long.

2. A circle is drawn on a line 13 in. in length, as diameter. The lengths of two parallel chords are 5 in. and 12 in. Calculate the distance between the chords correct to the nearest tenth foot.

3. ABC is a triangle inscribed in a circle, centre O. BOC is a diameter.
(i) If $\angle OAB = 40°$, calculate the number of degrees in $\angle ACO$.
(ii) If AO is extended to intersect the circle on D, calculate the number of degrees in $\angle ODC$.

4. $ABCD$ is a cyclic quadrilateral; $\angle DAB = 2\angle BCD$ and $\angle ABC = 3\angle CDA$. Calculate the number of degrees in each angle of the quadrilateral.

5. $PQRS$ is a cyclic quadrilateral in which $\angle RSP = 39°$; $\angle QPR = 2\angle PRQ$. Calculate the number of degrees in $\angle PSQ$.

6. $PQRS$ is a cyclic quadrilateral; $\angle QPS = 110°$, $\angle PQS = 30°$, $PS = SR$, and PR intersects QS on T. Calculate the number of degrees in $\angle QTR$ and $\angle PRQ$.

7. TM is a tangent segment to a circle, centre O. The radius of the circle is 5 cm., and OT is 13 cm. Calculate the length of TM to the nearest cm.

8. PQR is a tangent to a circle at Q; PAB is a secant intersecting the circle on A and B; $\angle APQ = 40°$, and $\angle RQB = 80°$. Calculate the number of degrees in $\angle PQA$.
9. AP and AQ are tangents to a circle, centre O, at P and Q respectively. The radius of the circle is 5 cm. CRB is a tangent at R on the minor arc PQ and intersects AP and AQ on C and B respectively. $OA = 13$ cm. Calculate the perimeter of $\triangle ABC$ to the nearest centimetre.
10. PA is a tangent segment of a circle. A is in the circle and $PA = 2$ cm. A secant PBC intersects the circle on B and C. $\angle PAB = 45°$, and $AB \perp PBC$. Calculate (i) $\angle ACB$, (ii) PB correct to the nearest tenth centimetre, (iii) the length of the diameter of the circle correct to the nearest inch.

For each of the following questions use $\pi = 3.14$ and round-off the answer to the nearest tenth unit.

11. The radius of a circle is 30 inches. Calculate the lengths of the arcs with sector angles (i) 60° (ii) 120° (iii) 135° (iv) 150°.
12. The radius of a circle is 2 inches. Calculate the areas of the sectors with sector angles (i) 30° (ii) 90° (iii) 135° (iv) 150°.
13. If the length of an arc with sector angle 30° is 2 cm., calculate the radius of the arc.

Write a proof for each of the following:

14. If two chords of the same circle are equal, they determine equal arcs.
15. $ABCD$ is a parallelogram. A circle with centre B and radius BC intersects CD on E. Prove that A, B, E, D are concyclic points.
16. ABC is an isosceles triangle in which $AB = AC$. A circle is described on AB as diameter intersecting BC on D. Prove that AD bisects $\angle BAC$.
17. DB is a tangent segment and DCA is a secant from a point D to a circle, centre O. Prove that $\triangle BCD$ is equiangular to $\triangle ABD$.
18. $ABCD$ is a quadrilateral in which $AD = AB$ and $CD = BC$. Prove that the quadrilateral is cyclic if $\angle D = 90°$.
19. ABC is an isosceles triangle having $AB = AC$. BD and CE are medians from the ends of the base BC. Prove that the points E, B, C, and D are concyclic.
20. Prove that a rectangle is a cyclic quadrilateral.
21. ABC is a triangle inscribed in a circle, centre P. $AD \perp BC$. Prove that $\angle BAD = \angle PAC$.
22. Prove that the graph of a relation is a circle with centre $O(0, 0)$ and radius $r (r > 0)$ if and only if the defining sentence of the relation may be written in the form $x^2 + y^2 = r^2$, $x, y \in R$.

Write solutions for the following:

23. Write the standard form of the equation of each of the following circles:

(i) centre $O(0, 0)$, radius 7 ;

(ii) centre $O(0, 0)$, and on $A(0, \sqrt{3})$;

(iii) centre $O(0, 0)$, and on $B(5, -4)$;

(iv) centre $O(0, 0)$, tangent to the line defined by $x - 6 = 0, x, y \in R$;

(v) centre $O(0, 0)$, tangent to the line defined by $y + 8 = 0, x, y \in R$.

24. Draw the graph of $A = \{(x, y) \mid x^2 + y^2 \leqq 9, x + y \geqq 1, x, y \in R\}$.

25. Prove that the centre of the circle defined by $x^2 + y^2 = 36, x, y \in R$ lies in the right bisector of the chord determined by $x + y = 2$, $x, y \in R$.

26. Prove that the line on the centre of the circle defined by $x^2 + y^2 = 25$ and on the mid-point of the chord determined by $x - y = 1, x, y \in R$ is perpendicular to the chord.

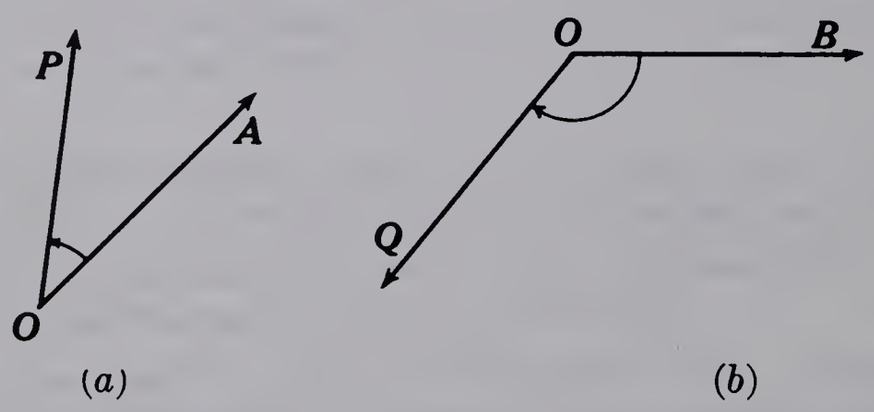
27. Determine the length of the tangent segment from the point $P(5, 2)$ to the circle defined by $x^2 + y^2 = 9, x, y \in R$.

28. Write the equation of the tangent at $P(2, -1)$ of the circle defined by $x^2 + y^2 = 5, x, y \in R$.

Chapter XII

TRIGONOMETRY

12•1 Oriented angles. If a ray OA, *Fig. 12-1*(a), rotates about O to a terminal position OP, then $\angle AOP$ is called an *oriented angle.* To orient an angle, we first order its sides by designating one of them as the *initial*



(a) (b)

Fig. 12-1

side and the other as the *terminal side.* We say that the rotating ray generates the angle, and the direction of rotation is indicated by a curved arrow as shown. In *Fig. 12-1*(a), $\angle AOP$ is generated by a counterclockwise rotation; in *Fig. 12-1* (b), $\angle BOQ$ is generated by a clockwise rotation. The order in which the letters are used in naming oriented $\angle AOP$ indicates that the initial side is OA, the vertex is O, and the terminal side is OP.

DEFINITION. *An oriented angle is an ordered pair of rays having a common vertex.*

A rotation can be zero, part of a revolution, a complete revolution, or more than a revolution. The measurement of the rotation is defined to be the measurement of the oriented angle. We often state the measurement in terms of degrees, minutes, and seconds, where:

$$1 \text{ degree is } \frac{1}{360} \text{ of a complete revolution,}$$

$$60 \text{ minutes } (60') = 1 \text{ degree } (1^\circ),$$

$$60 \text{ seconds } (60'') = 1 \text{ minute } (1').$$

If the rotation is counterclockwise the measure of the angle is positive and the angle is said to be *positively oriented;* if the rotation is clockwise the angle is *negatively oriented* and its measure is a negative real number. Thus, the degree measure of an oriented angle may be any real number including zero.

12•2 Standard position. If coordinate axes are selected, an oriented angle is said to be in standard position when its vertex is at the origin and its initial side is the positive x-axis (*Fig. 12-2*).

The angle is said to be (i) in the quadrant, or (ii) an angle of the quadrant, in which the terminal side lies. Thus $\angle XOA$, *Fig. 12-2*, is said to be in the second quadrant or to be a second quadrant angle.

Unless otherwise specified, an angle in this study is considered to be an oriented angle in standard position.

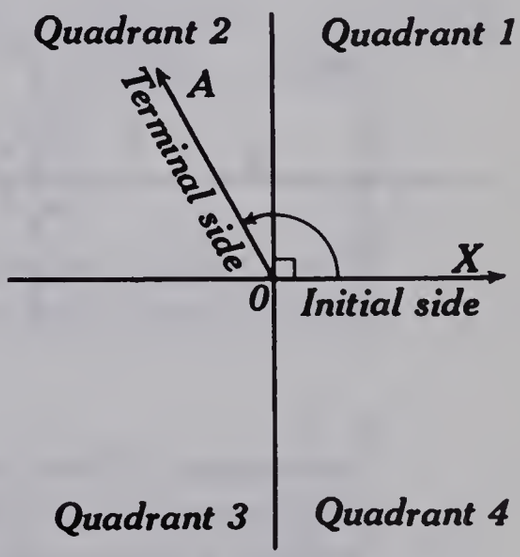


Fig. 12-2

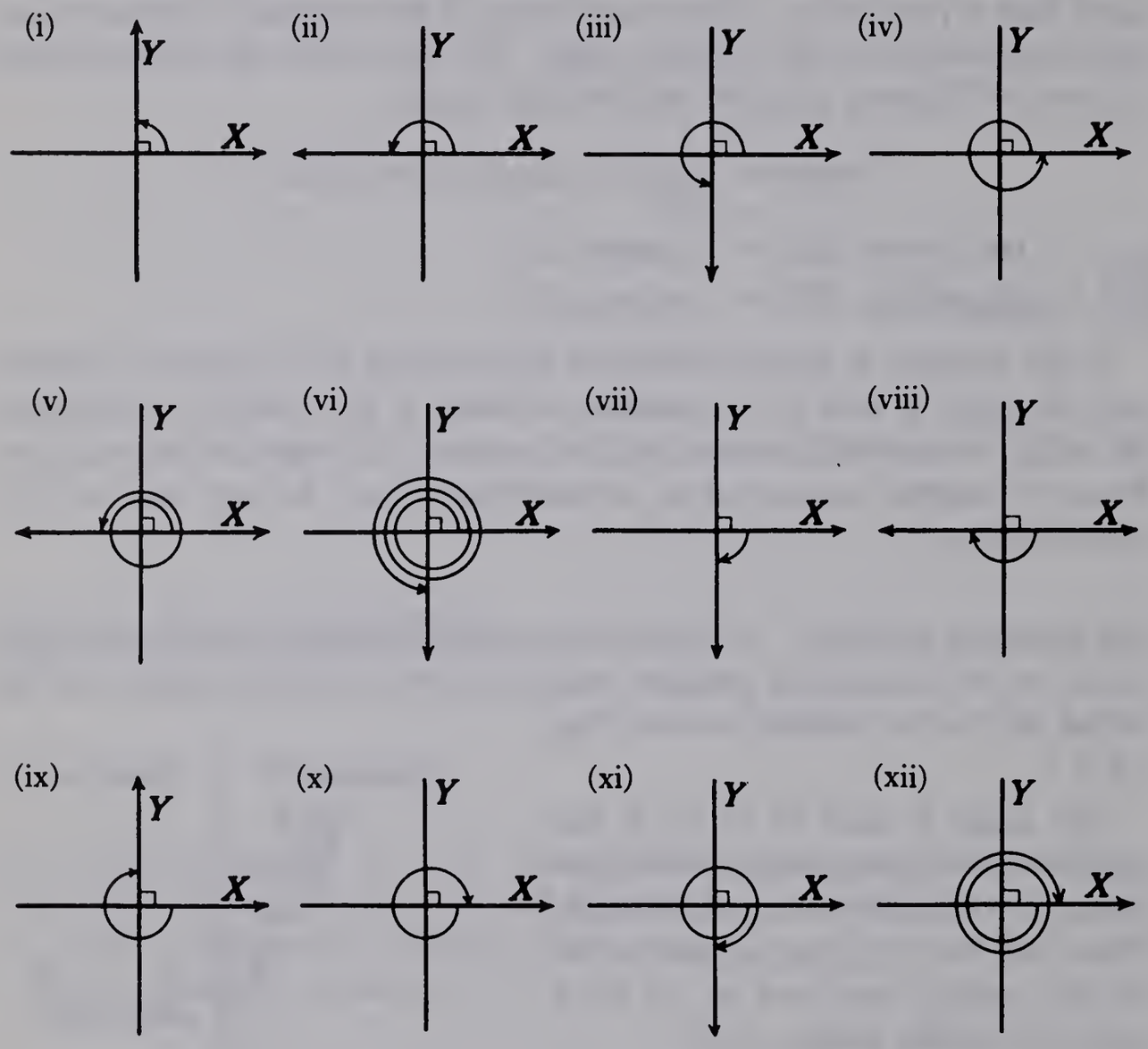
Discovery Exercise 12-1

(*Compare your answers with those on page* 507.)

(B)

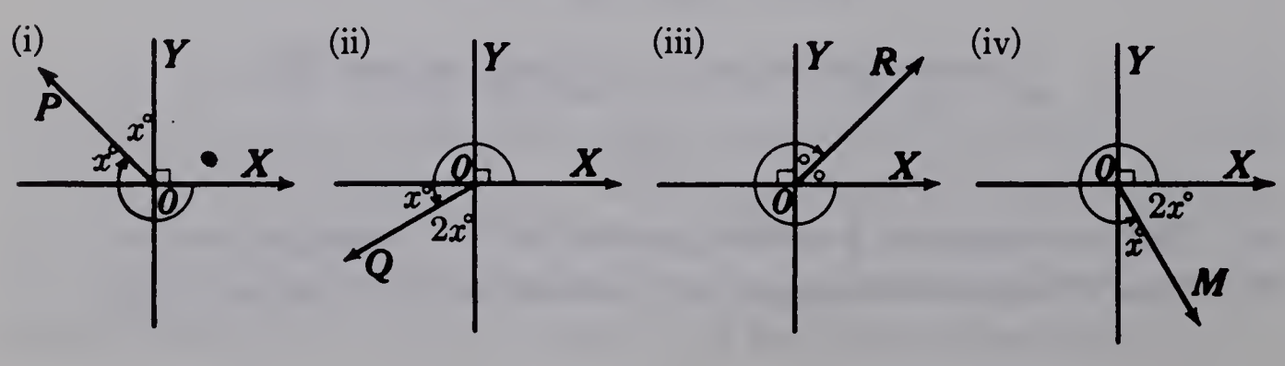
1. Draw a diagram of a positively oriented third quadrant angle.
2. Draw a diagram of a negatively oriented second quadrant angle.

3. In the accompanying diagrams state the measurement in degrees of each of the angles indicated:

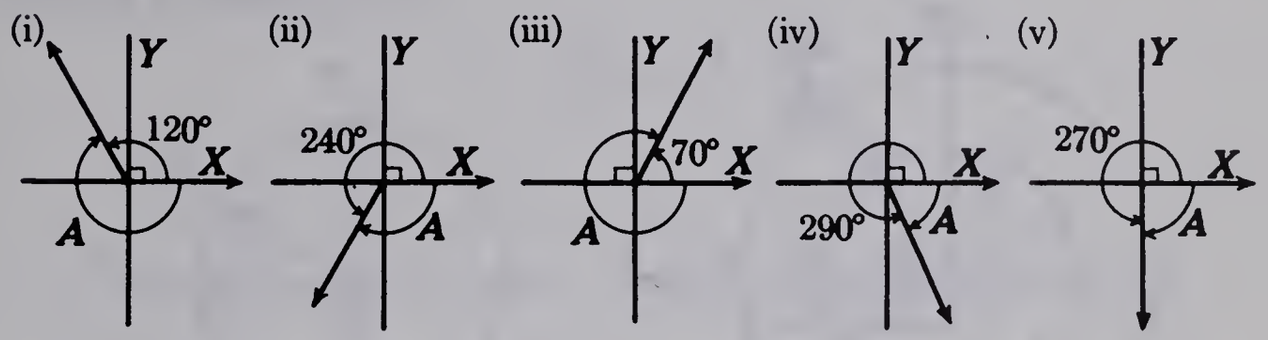


DEFINITION. *If the terminal side of an oriented angle in standard positon is on the x-axis or the y-axis, the angle is called a quadrantal angle.*

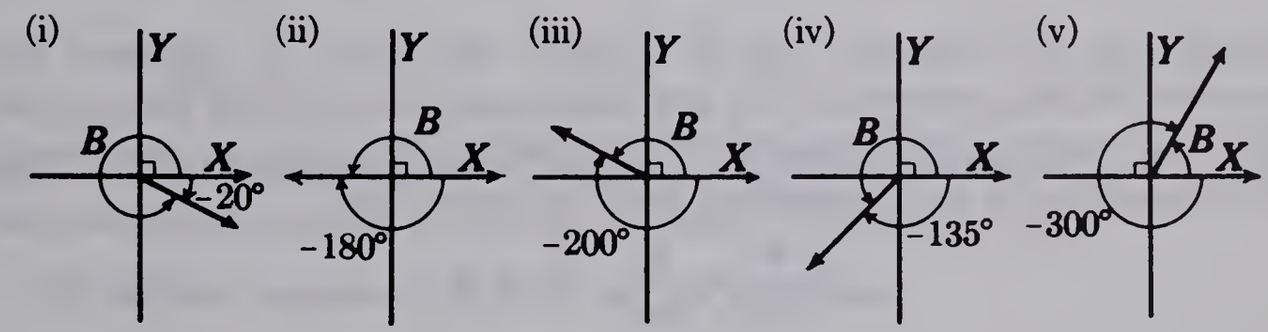
4. In the diagrams below name and state the measurement in degrees of each of the angles indicated:



5. For each of the following positively oriented angles, state the measurement in degrees of the negatively oriented angle A which has the same terminal arm as the given angle:

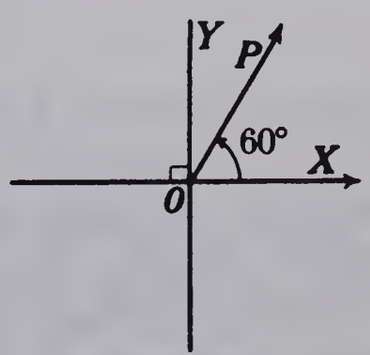


6. For each of the following negatively oriented angles, state the measurement in degrees of the positively oriented angle B which has the same terminal arm as the given angle:



DEFINITION. *Oriented angles in standard position, which have coincident terminal arms, are called coterminal angles.*

7. In the accompanying diagram, $\angle XOP = 60°$. State:



(i) the measurements of three positively oriented angles coterminal with $\angle XOP$;

(ii) the measurements of three negatively oriented angles coterminal with $\angle XOP$;

(iii) an expression for the measurements of all positively oriented angles coterminal with $\angle XOP$;

(iv) an expression for the measurements of all negatively oriented angles coterminal with $\angle XOP$;

(v) an expression for the measurements of all positively or negatively oriented angles coterminal with $\angle XOP$.

12•3 Radian system for measurement of angles. In *Fig 12-3*(*a*), P_1 and Q_1 are points on the initial side of $\angle XOM$ of θ degrees such that $OP_1 = r_1$ units, $OQ_1 = r_2$ units. If P_1 maps onto P_2 and Q_1 onto Q_2 as the initial side

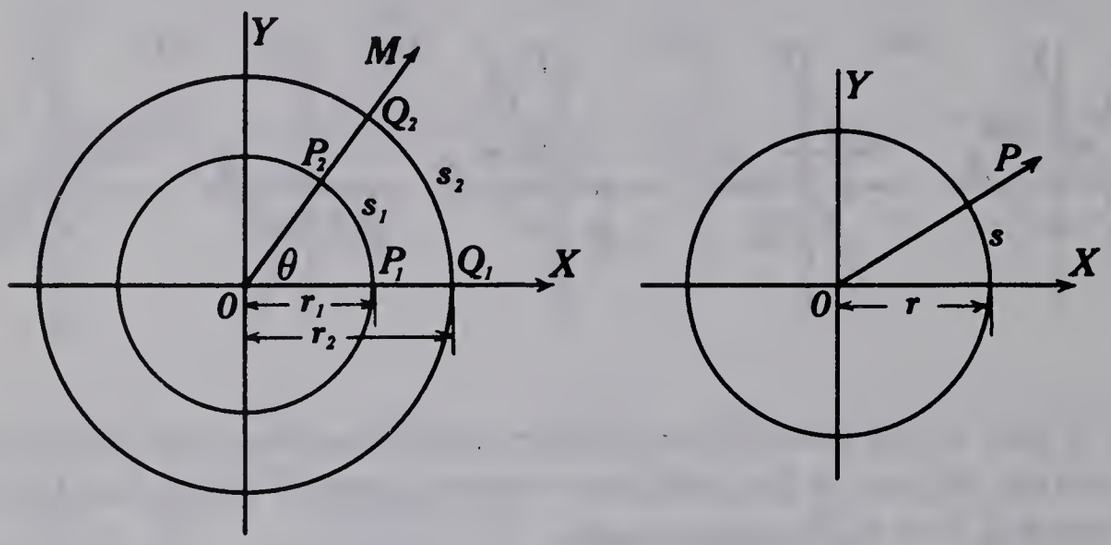


Fig. 12-3

rotates into the terminal side of $\angle XOM$, let s_1 and s_2 represent the measures of the arc lengths P_1P_2 and Q_1Q_2 respectively, in the same units. By a theorem in plane geometry *"Central angles of the same or equal circles are proportional to their subtending arcs"* we know that:

$$\frac{\theta}{360} = \frac{s_1}{2\pi r_1}, \quad r_1 \neq 0 \tag{1}$$

and

$$\frac{\theta}{360} = \frac{s_2}{2\pi r_2}, \quad r_2 \neq 0\,. \tag{2}$$

From (1) and (2);

$$\frac{s_1}{2\pi r_1} = \frac{s_2}{2\pi r_2}$$

$$\therefore \quad \frac{s_1}{r_1} = \frac{s_2}{r_2}.$$

Further, any other two-term ratio obtained in a similar manner for $\angle XOM$ is equal to the ratio $s_1 : r_1$.

In general, for each oriented angle the ratio of arc length to radius is uniquely determined. This ratio is the basis for the radian system of measurement of angles.

DEFINITION. *In Fig 12-3*(*b*) *the ratio* $s : r$ $(r > 0)$ (*arc length to radius*) *is the radian measure of* $\angle XOP$.

If $s = r$ the ratio $s : r$ is 1, and the angle is the unit angle of the system and has a measurement of 1 *radian*. The word radian is derived from the word "radius".

It should be noted that the radian measure of an angle is a two-term ratio and hence a real number. Thus, s and r may be expressed in inches, centimetres, or feet as long as they are both expressed in the same unit.

For negatively oriented angles such as $\angle XOQ$ in *Fig 12-4* the radian measure is defined to be

$$\frac{-s}{r} \text{ or } -\frac{s}{r}.$$

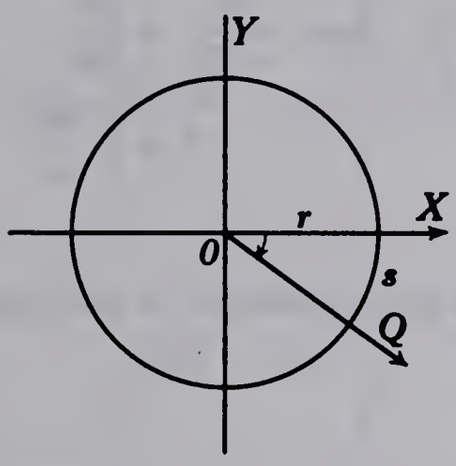


Fig. 12-4

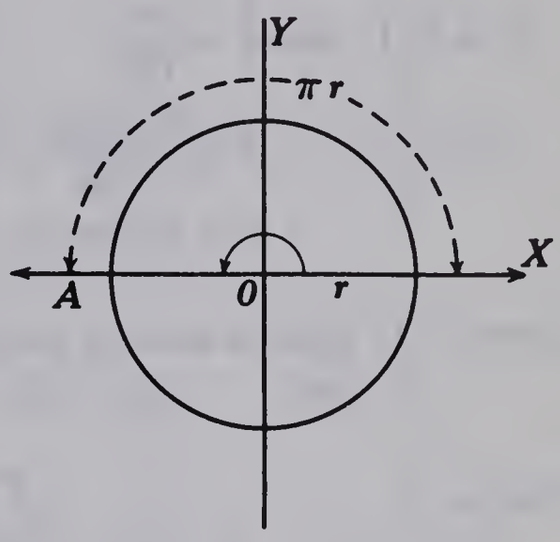


Fig. 12-5

***12·4* Relation between degree measure and radian measure.** In *Fig. 12-5* the semi-circumference of a circle of radius r units is πr units.

The radian measure of $\angle XOA$ is $\frac{\pi r}{r}$ or π radians.

The degree measure of $\angle XOA$ is 180.

Thus the radian measure π corresponds to a degree measure of 180.

The table below lists some radian measures and their corresponding degree measures.

RADIAN MEASURE	DEGREE MEASURE
π	180
2π	360
$\frac{\pi}{2}$	90
$\frac{\pi}{4}$	45
$\frac{\pi}{6}$	30
$\frac{2\pi}{3}$	120

In general, if the radian measure of an angle is represented by θ and the degree measure of the same angle by m, then the relation between the radian and degree measure is given by

$$\frac{\theta}{m} = \frac{\pi}{180} \text{ or } \frac{m}{\theta} = \frac{180}{\pi}.$$

If $\theta = 1$, then $\dfrac{m}{1} = \dfrac{180}{\pi}$.

$$\therefore \quad m \doteq \frac{180}{3.14159}$$

$$m \doteq 57.2958$$

$(57.2958° \doteq 57°\ 17'\ 44.8'')$.

If $\theta = -3$, then $\dfrac{m}{-3} = \dfrac{180}{\pi}$.

$$\therefore \quad m = \frac{-540}{\pi}.$$

If $m = 1$, then $\dfrac{\theta}{1} = \dfrac{\pi}{180}$.

$$\therefore \quad \theta \doteq \frac{3.14159}{180}$$

$$\theta \doteq 0.017453.$$

If $m = -135$, then $\dfrac{\theta}{-135} = \dfrac{\pi}{180}$.

$$\therefore \quad \theta = -\frac{3\pi}{4}.$$

Example 1. Express each of the following angle measurements in radians:

(i) $100°$ (ii) $720°$ (iii) $18°$ (iv) $-60°$.

Solution. $\dfrac{\theta}{m} = \dfrac{\pi}{180}$.

(i) $\dfrac{\theta}{100} = \dfrac{\pi}{180}$, $\therefore \quad \theta = \dfrac{5\pi}{9}$.

(ii) $\dfrac{\theta}{720} = \dfrac{\pi}{180}$, $\therefore \quad \theta = 4\pi$.

(iii) $\dfrac{\theta}{18} = \dfrac{\pi}{180}$, $\therefore \quad \theta = \dfrac{\pi}{10}$.

(iv) $\dfrac{\theta}{-60} = \dfrac{\pi}{180}$, $\therefore \quad \theta = -\dfrac{\pi}{3}$.

Example 2. Express each of the following radian measures as equivalent degree measures:

(i) 2π (ii) $-\dfrac{5\pi}{6}$ (iii) 0.23 (iv) -10.

Solution. $\dfrac{m}{\theta} = \dfrac{180}{\pi}$.

(i) $\dfrac{m}{2\pi} = \dfrac{180}{\pi}$,

$\therefore m = 360$.

(ii) $\dfrac{m}{\frac{-5\pi}{6}} = \dfrac{180}{\pi}$,

$$\therefore \quad m = -150.$$

(iii) $\dfrac{m}{0.23} = \dfrac{180}{\pi}$,

$$\therefore \quad m \doteq \frac{41.4}{3.1416}$$

$$\doteq 13.2.$$

(iv) $\dfrac{m}{-10} = \dfrac{180}{\pi}$,

$$\therefore \quad m = \frac{-1800}{\pi}$$

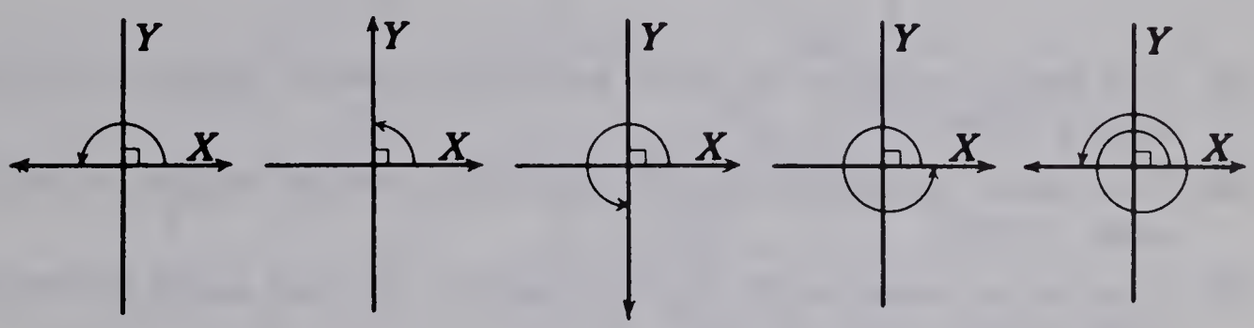
$$\doteq -573.2.$$

Exercise 12-2

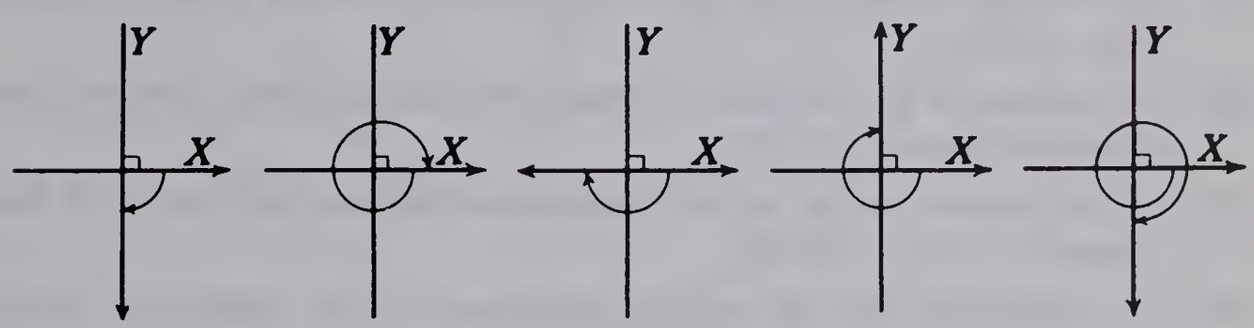
(A)

State the radian measure of the oriented angle indicated by the arrow in each of the following:

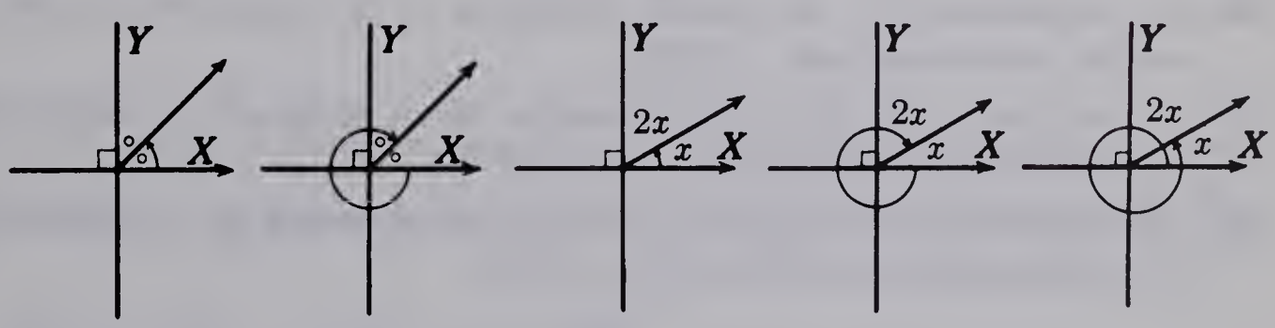
1. **2.** **3.** **4.** **5.**



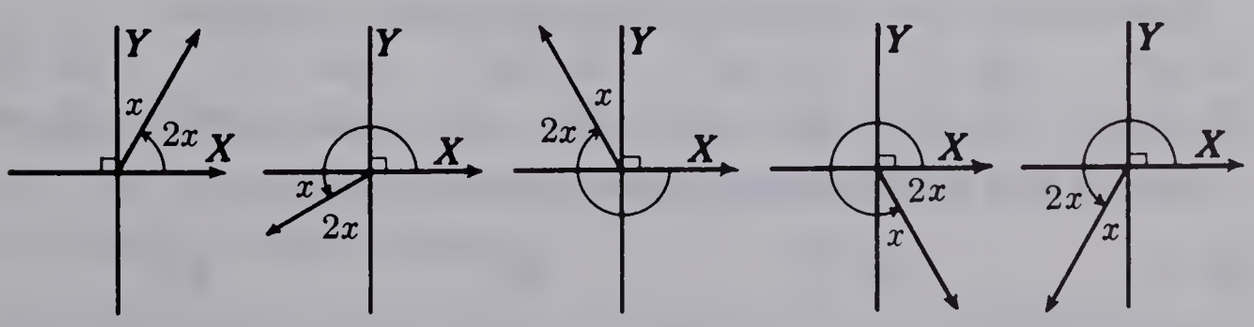
6. **7.** **8.** **9.** **10.**



11. **12.** **13.** **14.** **15.**

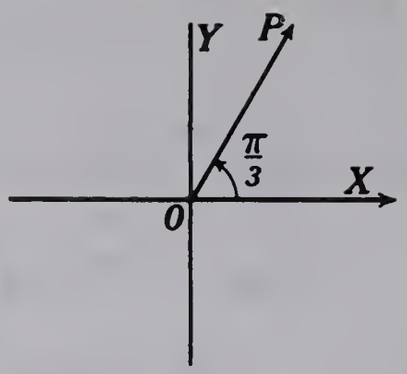


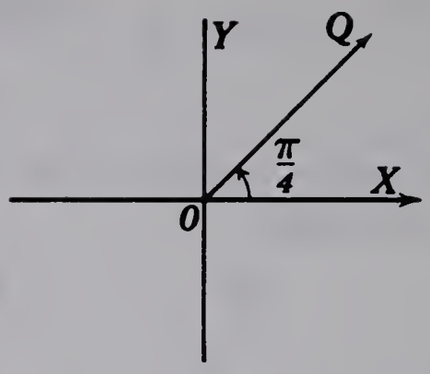
16. **17.** **18.** **19.** **20.**



Using the diagrams at the top of page 354, *in which*

$$\angle XOP = \frac{\pi}{3} \text{ radians}, \quad \angle XOQ = \frac{\pi}{4} \text{ radians, state:}$$





21. The radian measures of three positively oriented angles coterminal with $\angle XOP$.

22. The radian measures of three positively oriented angles coterminal with $\angle XOQ$.

23. The radian measures of three negatively oriented angles coterminal with $\angle XOP$.

24. The radian measures of three negatively oriented angles coterminal with $\angle XOQ$.

25. An expression for the radian measures of all positively oriented angles coterminal with $\angle XOP$.

26. An expression for the radian measures of all positively oriented angles coterminal with $\angle XOQ$.

27. An expression for the radian measures of all negatively oriented angles coterminal with $\angle XOP$.

28. An expression for the radian measures of all negatively oriented angles coterminal with $\angle XOQ$.

29. An expression for the radian measures of all positively or negatively oriented angles coterminal with $\angle XOP$.

30. An expression for the radian measures of all positively or negatively oriented angles coterminal with $\angle XOQ$.

(B)

Express each of the following angle measurements in radians:

31. 90° **32.** 45° **33.** 60° **34.** 360° **35.** 135° **36.** 150°

37. 720° **38.** 10° **39.** −5° **40.** −18° **41.** −180° **42.** −36°

Express each of the following radian measures in degrees:

43. π **44.** $\frac{\pi}{6}$ **45.** $\frac{\pi}{4}$ **46.** $\frac{2\pi}{3}$

47. $\frac{4\pi}{3}$ **48.** -3π **49.** $\frac{2}{5}\pi$ **50.** -10π

51. 10 **52.** −0.5 **53.** 1 **54.** $-\frac{\pi}{90}$

12·5 The sine, cosine, and tangent functions. In *Fig. 12-6* $\angle XOP$ of θ units determines uniquely a point A on the circle with radius r_1 ; a point B on the circle with radius r_2 ; and a point C on the circle with radius r_3.

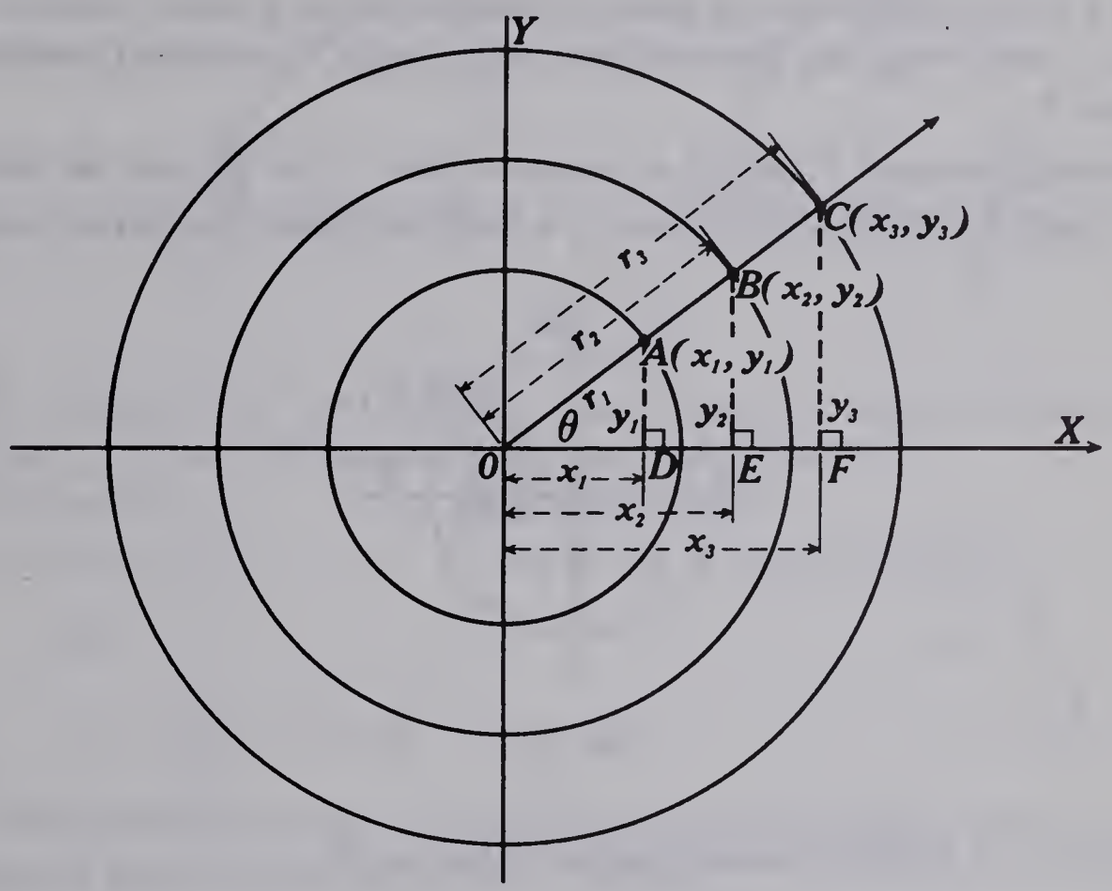


Fig. 12-6

In turn, point A determines uniquely right $\triangle ADO$ and hence the ratio $\frac{y_1}{r_1}$. Similarly, B determines $\triangle BEO$ and ratio $\frac{y_2}{r_2}$; C determines ratio $\frac{y_3}{r_3}$.

Since $\triangle ADO \sim \triangle BEO \sim \triangle CFO$,

$$\therefore \quad \frac{y_1}{r_1} = \frac{y_2}{r_2} = \frac{y_3}{r_3}.$$

Thus, $\angle XOP$ determines the unique ratio $y : r$ $\left(\text{or } \frac{y}{r}\right)$ where y is the ordinate of the point of intersection of the terminal arm of $\angle XOP$ with a circle of radius r units, *Fig. 12-7* .

The association, in this way, of the number of units of measurement of any angle in standard position with the unique ratio $\frac{y}{r}$ for that angle, leads to a function called the *sine function*. The ratio $\frac{y}{r}$ is called *sine* θ (abbreviated *sin* θ).

The sine function is

$$s = \{(\theta, \sin\theta) \mid \sin\theta = \frac{y}{r},\ \theta,\ \sin\theta \in R\},$$

where y is the ordinate of the point of intersection of a circle, centre (0, 0), radius r units, with the terminal arm of an angle in standard position of measure θ.

Similarly, for each θ there is a unique ratio $x : r$ $\left(\text{or } \frac{x}{r}\right)$, and the association of each θ with this ratio leads to a function called the *cosine function.*

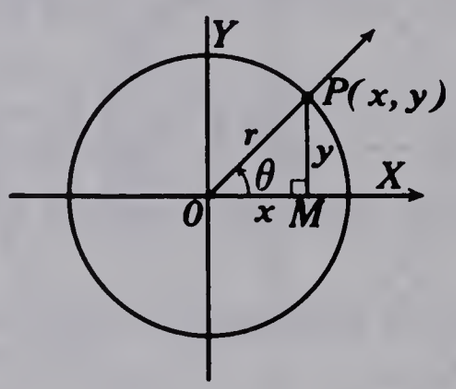


Fig. 12-7

The ratio $\frac{x}{r}$ is called *cosine* θ (abbreviated *cos* θ).

The cosine function is

$$c = \{(\theta, \cos\theta) \mid \cos\theta = \frac{x}{r},\ \theta,\ \cos\theta \in R\},$$

where x is the abscissa of the point of intersection of a circle, centre (0, 0), radius r units, with the terminal arm of an angle in standard position of measure θ.

Also, θ uniquely determines the ratio $y : x$. The association of each θ with this ratio leads to a function called the *tangent function.* The ratio $\frac{y}{x}$ is called *tangent* θ (abbreviated *tan* θ).

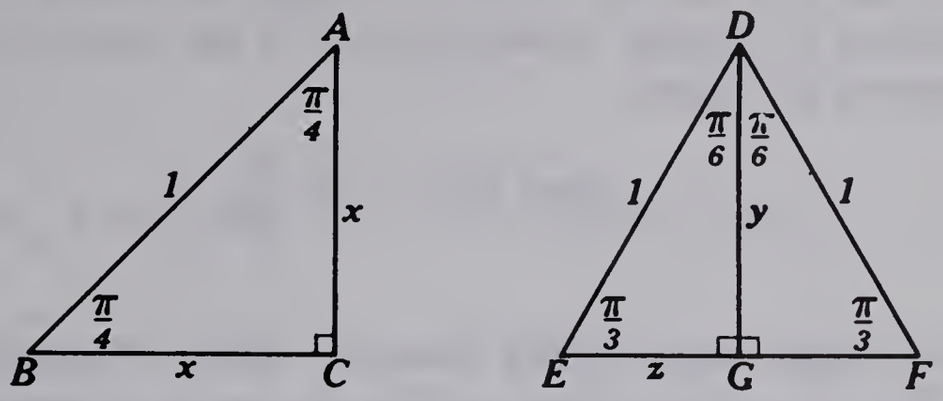
The tangent function is

$$t = \{(\theta, \tan\theta) \mid \tan\theta = \frac{y}{x},\ x \neq 0,\ \theta, \tan\theta \in R\},$$

where (x, y) are the coordinates of the point (if $x \neq 0$) of intersection of a circle, centre (0, 0), radius r units, with the terminal arm of an angle in standard position of measure θ.

The sine, cosine, and tangent functions are often referred to as the *primary trigonometric functions.*

12·6 Sine, cosine, and tangent function values of special acute angles.



(a) (b)

Fig. 12-8

In *Fig 12-8* if x, y, z represent the number of units of length of AC (or BC), DG, and EG respectively, then by geometry:

in *Fig 12-8* (*a*),

$$x^2 + x^2 = 1,$$

$$2x^2 = 1,$$

$$x = \frac{1}{\sqrt{2}} (\because x > 0).$$

in *Fig 12-8* (*b*),

$EG = GF = \frac{1}{2}$ and $y^2 + (\frac{1}{2})^2 = 1$.

$$y^2 = \frac{3}{4},$$

$$y = \frac{\sqrt{3}}{2} (\because y > 0).$$

This information may be applied to determine trigonometric function values of special angles as follows (*Fig.12-9*):

(i) $\frac{\pi}{4}$ (ii) $\frac{\pi}{3}$ (iii) $\frac{\pi}{6}$

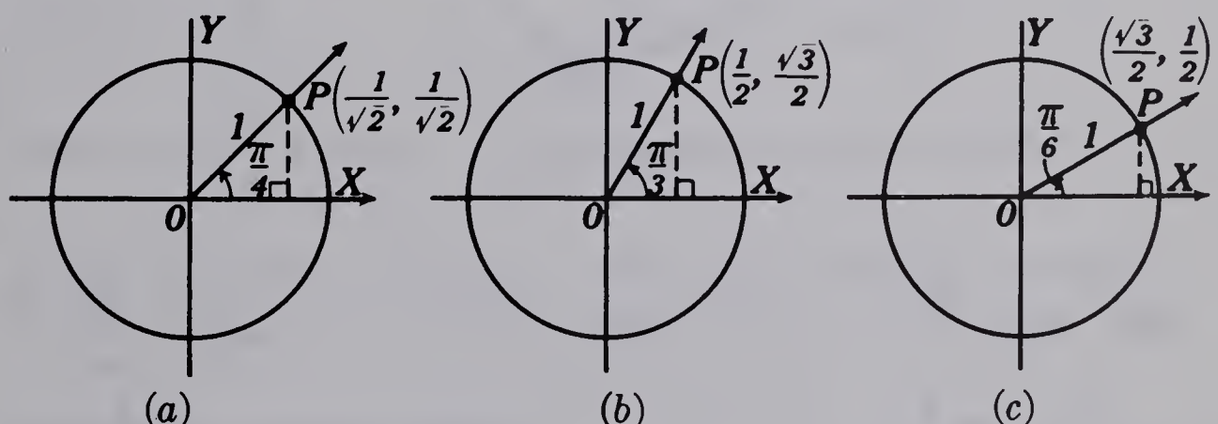


Fig. 12-9

$$\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} \qquad \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2} \qquad \sin\frac{\pi}{6} = \frac{1}{2}$$

$$\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} \qquad \cos\frac{\pi}{3} = \frac{1}{2} \qquad \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2.}$$

$$\tan\frac{\pi}{4} = 1 \qquad \tan\frac{\pi}{3} = \sqrt{3} \qquad \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$$

If the unit of angular measure is not specified, it is understood that the measurement is in radians. Thus, sin 1 means the sine of an angle whose measurement is 1 radian. However, sin 1° is the sine of an angle whose measurement is 1 degree.

$$\text{Thus, } \sin 1^\circ = \sin \frac{\pi}{180}.$$

12·7 **Sine, cosine, and tangent function values of quadrantal angles.** *From Fig.12-10* and the definitions of the sine, cosine, and tangent functions:

(i) $\sin 0 = \frac{0}{1}$, $\cos 0 = \frac{1}{1}$, $\tan 0 = \frac{0}{1}$,

$= 0$. $= 1$. $= 0$.

(ii) $\sin \frac{\pi}{2} = \frac{1}{1}$, $\cos \frac{\pi}{2} = \frac{0}{1}$, $\tan \frac{\pi}{2}$ is undefined.

$= 1$. $= 0$.

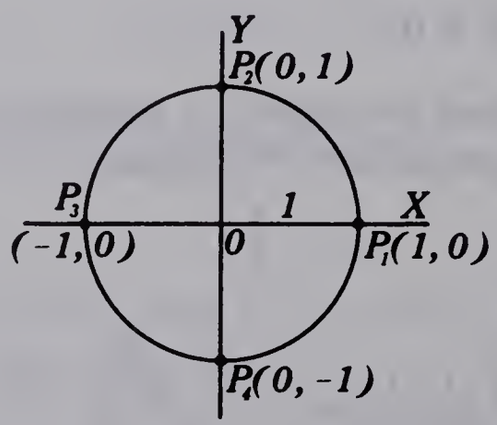


Fig. 12-10

(iii) $\sin \pi = \frac{0}{1}$, $\cos \pi = \frac{-1}{1}$, $\tan \pi = \frac{0}{-1}$,

$= 0$. $= -1$. $= 0$.

(iv) $\sin \frac{3\pi}{2} = \frac{-1}{1}$, $\cos \frac{3\pi}{2} = \frac{0}{1}$, $\tan \frac{3\pi}{2}$ is undefined.

$= -1$. $= 0$.

(v) $\sin 2\pi = \frac{0}{1}$, $\cos 2\pi = \frac{1}{1}$, $\tan 2\pi = \frac{0}{1}$,

$= 0$. $= 1$. $= 0$.

Exercise 12-3

(B)

Evaluate:

1. $\sin\frac{\pi}{3}\cos\frac{\pi}{6}+\cos\frac{\pi}{3}\sin\frac{\pi}{6}$

2. $\tan\frac{\pi}{6}\tan\frac{\pi}{4}\tan\frac{\pi}{3}$

3. $\sin\frac{\pi}{4}\cos\frac{\pi}{6}-\cos\frac{\pi}{4}\sin\frac{\pi}{6}$

4. $\cos\frac{\pi}{4}\cos\frac{\pi}{6}+\sin\frac{\pi}{4}\sin\frac{\pi}{6}$

5. $\left(\sin\frac{\pi}{3}-\sin\frac{\pi}{4}\right)\left(\cos\frac{\pi}{6}+\cos\frac{\pi}{4}\right)$

6. $\cos^2\frac{\pi}{6}-\cos^2\frac{\pi}{4}$

7. $\sin^2\frac{\pi}{6}+\sin^2\frac{\pi}{4}+\sin^2\frac{\pi}{3}$

8. $2\tan^2\frac{\pi}{4}+\tan^2\frac{\pi}{3}$

9. $\tan^2\frac{\pi}{6}+\tan^2\frac{\pi}{4}+\tan^2\frac{\pi}{3}$

10. $\cos\frac{\pi}{3}\cos\frac{\pi}{6}-\sin\frac{\pi}{3}\sin\frac{\pi}{6}$

11. $\sin\frac{\pi}{2}+\sin 0+\sin\frac{\pi}{6}+\sin\frac{\pi}{3}$

12. $3\cos\frac{\pi}{4}-5\tan 0$

13. $\dfrac{\tan\frac{\pi}{3}-\tan\frac{\pi}{6}}{1+\tan\frac{\pi}{3}\tan\frac{\pi}{6}}$

14. $\dfrac{2}{\sin\frac{\pi}{2}}-3\sin 0-1$

15. $\sin^2\frac{\pi}{3}+\cos^2\frac{\pi}{3}$

16. $\sin^2\frac{\pi}{6}+\cos^2\frac{\pi}{6}$

17. $\sin\pi+\cos\frac{3\pi}{2}+\tan 2\pi$

18. $\sin^2\pi+\cos^2\pi+\tan^2\pi$

19. $\frac{1}{2}\tan\pi+\frac{1}{3}\sin 2\pi+\frac{1}{4}\cos 3\pi$

20. $\tan 2\pi+2\sin\frac{\pi}{3}-3\cos\frac{\pi}{2}$

12·8 Signs of the function values.

Quadrant I. Abscissa x and ordinate y of $P\ (x, y)$ are positive.

If $0<\theta<\frac{\pi}{2}$, then $\sin\theta$, $\cos\theta$, and $\tan\theta$ are positive.

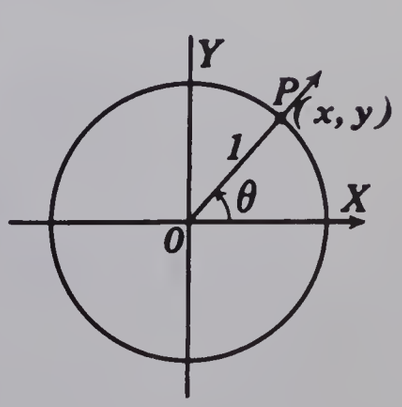
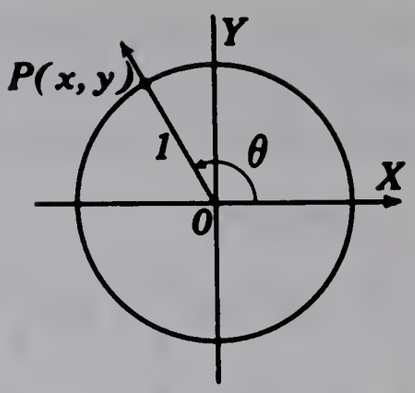


Fig. 12-11

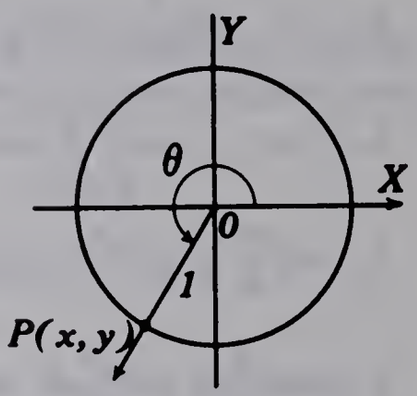
Quadrant II. Abscissa x of $P\ (x, y)$ is negative and ordinate y is positive.

If $\frac{\pi}{2} < \theta < \pi$, then $\sin\theta$ is positive, $\cos\theta$ and $\tan\theta$ are negative.



Quadrant III. Abscissa x and ordinate y of $P\ (x, y)$ are negative.

If $\pi < \theta < \frac{3\pi}{2}$, then $\tan\theta$ is positive, $\sin\theta$ and $\cos\theta$ are negative.



Quadrant IV. Abscissa x of $P\ (x, y)$ is positive and ordinate y is negative.

If $\frac{3\pi}{2} < \theta < 2\pi$, then $\cos\theta$ is positive, $\sin\theta$ and $\tan\theta$ are negative.

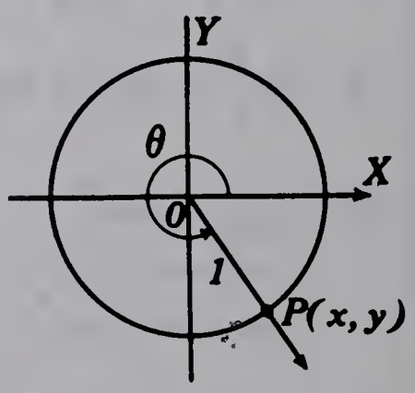


Fig. 12-11

Summary

Quadrant II	*Quadrant* I
$\frac{\pi}{2} < \theta < \pi$	$0 < \theta < \frac{\pi}{2}$
x neg., y pos.	x pos., y pos.
$\sin\theta > 0$	$\sin\theta > 0$
	$\cos\theta > 0$
	$\tan\theta > 0$
Quadrant III	*Quadrant* IV
$\pi < \theta < \frac{3\pi}{2}$	$\frac{3\pi}{2} < \theta < 2\pi$
x neg., y neg.	x pos., y neg.
$\tan\theta > 0$	$\cos\theta > 0$

Exercise 12-4

(A)

State the quadrant in which the terminal arm of each of the following angles lies:

1. $\frac{5\pi}{6}$ **2.** $\left(2\pi - \frac{\pi}{3}\right)$ **3.** $\left(2\pi + \frac{\pi}{10}\right)$ **4.** $\left(\pi + \frac{\pi}{4}\right)$

5. $\frac{3\pi}{4}$ **6.** $\frac{\pi}{3}$ **7.** $-\frac{\pi}{4}$ **8.** $-\left(\pi + \frac{\pi}{4}\right)$

9. $-\left(\frac{3\pi}{2} + \frac{\pi}{3}\right)$ **10.** $\left(-\frac{\pi}{2} - \frac{\pi}{3}\right)$ **11.** $\left(\frac{\pi}{2} + \frac{\pi}{6}\right)$ **12.** $\left(\pi + \frac{\pi}{10}\right)$

13. $-110°$ **14.** $-330°$ **15.** $225°$ **16.** $150°$

State whether each of the following function values is positive or negative:

17. $\sin\frac{\pi}{6}$ **18.** $\sin\left(\frac{\pi}{2} + \frac{\pi}{6}\right)$ **19.** $\cos\left(\pi - \frac{\pi}{6}\right)$

20. $\tan\left(\pi + \frac{\pi}{3}\right)$ **21.** $\sin\left(-\frac{\pi}{6}\right)$ **22.** $\cos\frac{9\pi}{8}$

23. $\tan\left(-\pi - \frac{\pi}{4}\right)$ **24.** $\cos\frac{3\pi}{4}$ **25.** $\sin\left(2\pi - \frac{\pi}{10}\right)$

26. $\sin 200°$ **27.** $\tan 120°$ **28.** $\cos 300°$

29. $\sin 120°$ **30.** $\cos 60°$ **31.** $\tan 225°$

32. $\tan 370°$ **33.** $\sin(-370°)$ **34.** $\cos(-750°)$

12·9 Relating the sine, cosine, or tangent of any oriented angle to those of a positively oriented acute angle.

(i) In *Fig.12-12(a)*, $\angle XOP = (\pi - \theta)$, where $0 < \theta < \frac{\pi}{2}$, that is $\angle XOP$ is a second quadrant angle. $P'(a, b)$ is the image of $P(x, y)$ obtained by employing symmetry of the circle with respect to the y-axis. P' determines the positively oriented $\angle XOP'$.

By geometry, $\angle XOP = \angle OPP' = \angle PP'O = \angle XOP'$.

$$\therefore \angle XOP' = \theta$$

By symmetry, $|y| = |b|$, $|x| = |a|$.

Also, $y > 0$, $b > 0$, $x < 0$, $a > 0$.

$\therefore y = b$. $\therefore x = -a$.

$$\therefore \sin(\pi - \theta) = \frac{y}{1} \qquad \cos(\pi - \theta) = \frac{x}{1} \qquad \tan(\pi - \theta) = \frac{y}{x}$$

$$= \frac{b}{1} \qquad = \frac{-a}{1} \qquad = \frac{b}{-a}$$

$$= \sin\theta. \qquad = -\cos\theta. \qquad = -\tan\theta.$$

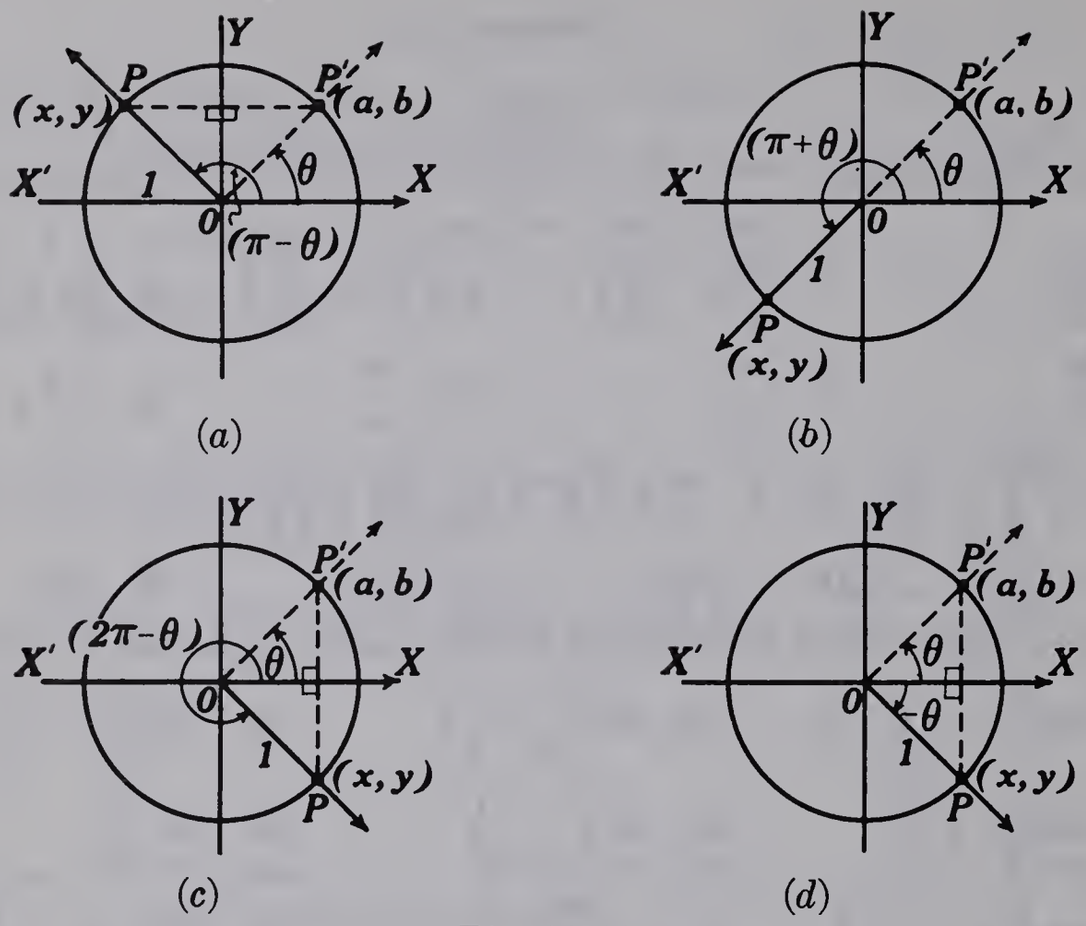


Fig. 12-12

Thus the sine, cosine, and tangent of $(\pi - \theta)$ are related to the sine, cosine, and tangent respectively, of the positively oriented acute angle θ. The absolute value of a function of $(\pi - \theta)$ is equal to the same function of the positively oriented acute angle θ.

(ii) *Fig. 12-12* (*b*) $P'(a, b)$ is the image of $P(x, y)$ obtained by employing symmetry of the circle with respect to the origin. P' determines the positively oriented acute angle XOP'.

By geometry, $\angle XOP' = \theta$.

Employing a method similar to that of (i) we may show that:

$$\sin(\pi + \theta) = -\sin\theta,$$
$$\cos(\pi + \theta) = -\cos\theta,$$
$$\tan(\pi + \theta) = \tan\theta.$$

(iii) In *Fig. 12-12* (*c*) $P'(a, b)$ is the image of $P(x, y)$ obtained by employing symmentry of the circle with respect to the x-axis. P' determines the positively oriented acute angle XOP'.

By geometry, $\angle XOP' = \theta$.

As in (i) and (ii) we may show that:

$$\sin(2\pi - \theta) = -\sin\theta,$$
$$\cos(2\pi - \theta) = \cos\theta,$$
$$\tan(2\pi - \theta) = -\tan\theta.$$

(iv) In *Fig. 12-12 (d)* $\angle XOP$ is a negatively oriented angle, $-\theta$.
P' is the image of P by symmetry with respect to the x-axis and $\angle XOP' = \theta$.

As in (i) and (ii) we may show that

$$\sin(-\theta) = -\sin\theta,$$
$$\cos(-\theta) = \cos\theta,$$
$$\tan(-\theta) = -\tan\theta.$$

In general, the absolute value of a trigonometric function of $(\pi - \theta)$, $(\pi + \theta)$, $(2\pi - \theta)$ or $(-\theta)$ is equal to the same function of the positively oriented acute angle θ. The angle θ is sometimes referred to as the positively oriented acute angle related to $(\pi - \theta)$, $(\pi + \theta)$, $(2\pi - \theta)$, or $(-\theta)$.

Example 1. Find the function value $\cos\frac{4\pi}{3}$.

Solution. or

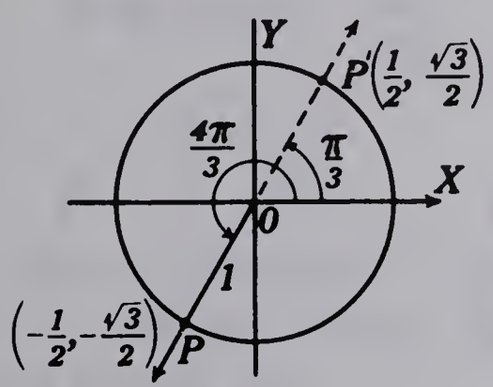


Fig. 12-13

$$\cos\frac{4\pi}{3} = -\frac{1}{2}.$$

$$\cos\left(\frac{4\pi}{3}\right) = \cos\left(\pi + \frac{\pi}{3}\right)$$
$$= -\cos\frac{\pi}{3}$$
$$= -\frac{1}{2}.$$

Example 2. Find the function value $\sin\frac{3\pi}{4}$.

Solution. or

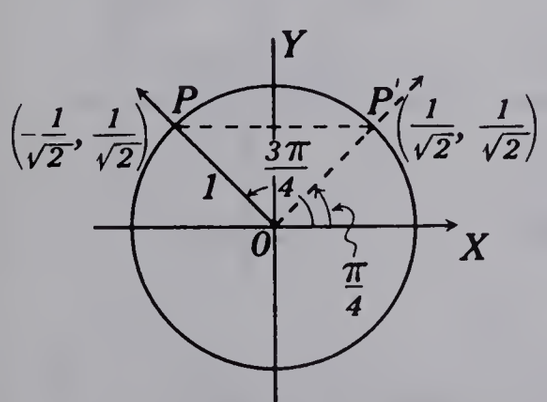


Fig. 12-14

$$\sin\frac{3\pi}{4} = \frac{1}{\sqrt{2}}.$$

$$\sin\frac{3\pi}{4} = \sin\left(\pi - \frac{\pi}{4}\right)$$
$$= \sin\frac{\pi}{4}$$
$$= \frac{1}{\sqrt{2}}.$$

Example 3. Find the function value cos 870°.

Solution.

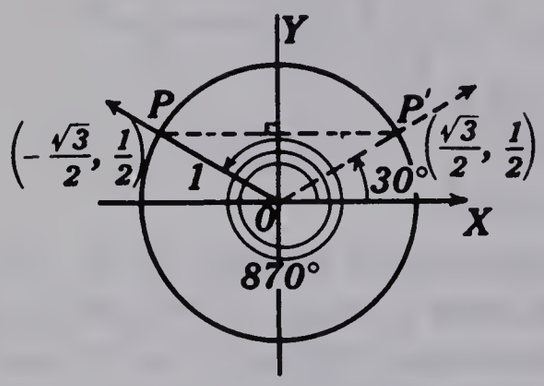


Fig. 12-15

$$\cos 870° = -\frac{\sqrt{3}}{2}.$$

or

$$\begin{aligned}\cos 870° &= \cos(720 + 150)° \\ &= \cos 150° \\ &= \cos(180 - 30)° \\ &= -\cos 30° \\ &= -\frac{\sqrt{3}}{2}.\end{aligned}$$

Example 4. Find the function value tan (− 240°).

Solution.

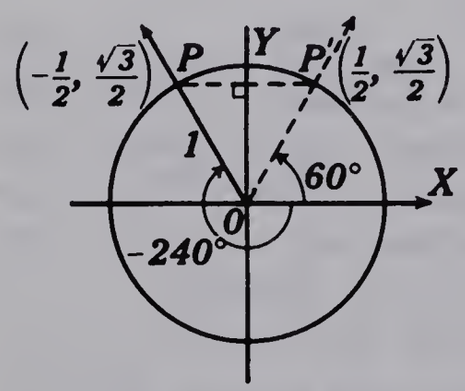


Fig. 12-16

$$\tan(-240°) = \frac{\frac{\sqrt{3}}{2}}{-\frac{1}{2}} = -\sqrt{3}.$$

or

$$\begin{aligned}\tan(-240°) &= \tan 120° \\ &= \tan(180 - 60)° \\ &= -\tan 60° \\ &= -\frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} \\ &= -\sqrt{3}.\end{aligned}$$

Exercise 12-5

(B)

Find the following function values:

1. $\sin\frac{3\pi}{4}$ **2.** $\tan\frac{4\pi}{3}$ **3.** $\tan(-120°)$

4. $\cos(-60°)$ 5. $\sin\frac{2\pi}{3}$ 6. $\tan 300°$

7. $\sin\frac{4\pi}{3}$ 8. $\tan 150°$ 9. $\cos(-240°)$

10. $\sin\frac{50\pi}{3}$ 11. $\cos(-1500°)$ 12. $\cos\left(-\frac{11\pi}{6}\right)$

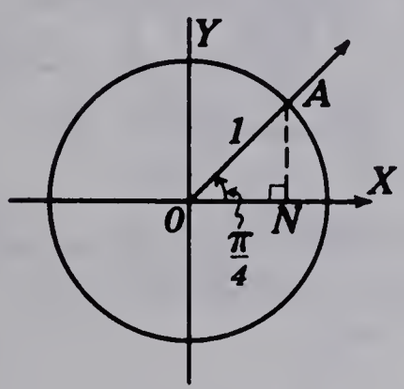
13. $\tan 930°$ 14. $\sin 600°$ 15. $\tan\left(-\pi-\frac{\pi}{3}\right)$

12·10 Ordered pairs determined by the trigonometric functions.

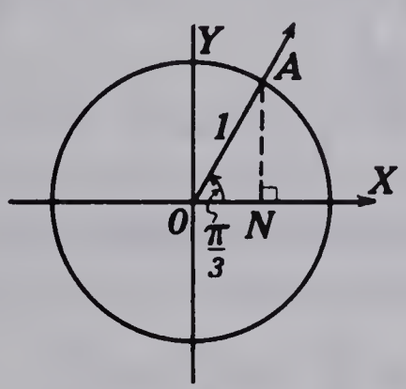
Discovery Exercise 12-6

(B)

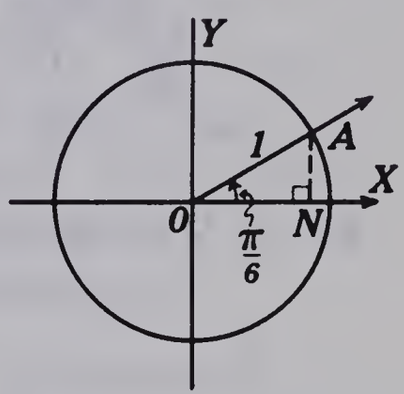
1. In each of the following, state the coordinates of point A if the radius of the circle is 1 unit.



(i)

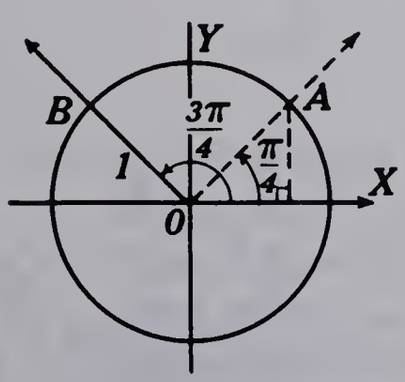


(ii)

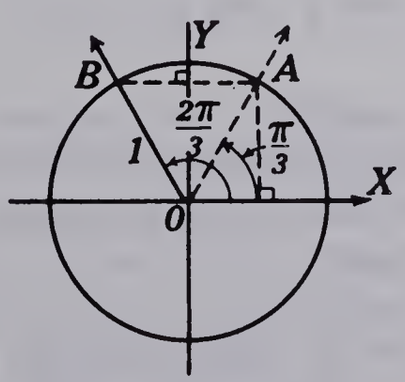


(iii)

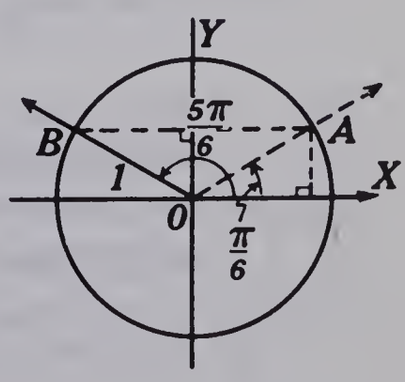
2. In each of the following, state the coordinates of point B if the radius of the circle is 1 unit:



(i)



(ii)



(iii)

3. (i) Copy and complete the diagram on the left following 3(vii) on page 366 by including the coordinates of the points A, B, C, D; the radius of the circle is 1 unit.

(ii) Find $\sin\frac{\pi}{4}$, $\sin\frac{3\pi}{4}$, $\sin\frac{5\pi}{4}$, $\sin\frac{7\pi}{4}$.

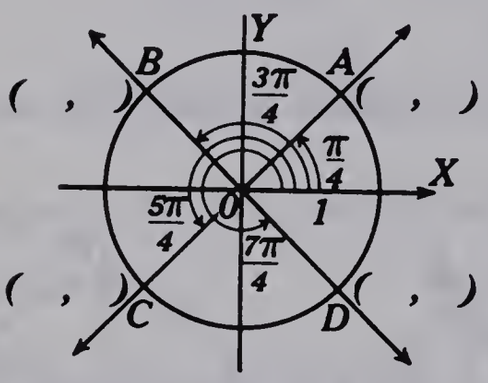
(iii) List four ordered pairs of the sine function.

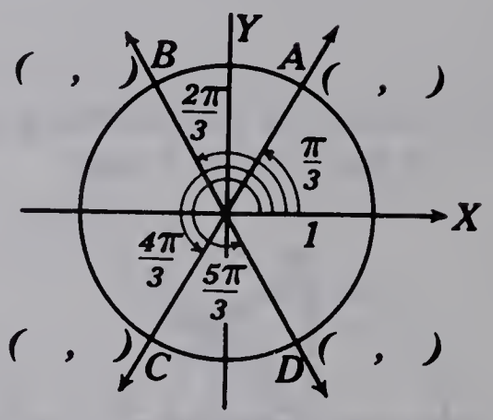
(iv) Find $\cos\frac{\pi}{4}$, $\cos\frac{3\pi}{4}$, $\cos\frac{5\pi}{4}$, $\cos\frac{7\pi}{4}$.

(v) List four ordered pairs of the cosine function.

(vi) Find $\tan\frac{\pi}{4}$, $\tan\frac{3\pi}{4}$, $\tan\frac{5\pi}{4}$, $\tan\frac{7\pi}{4}$.

(vii) List four ordered pairs of the tangent function.





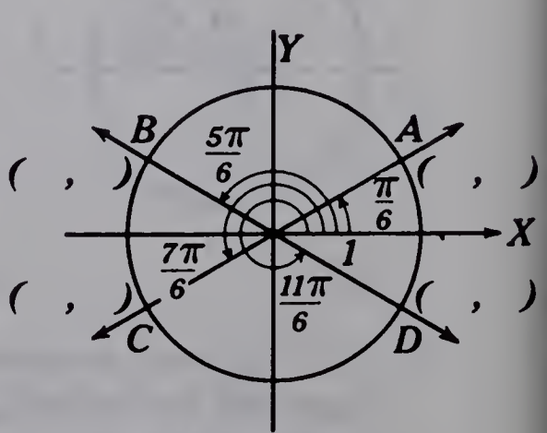
4. (i) Copy and complete the diagram at the right above by including the coordinates of points A to D; radius of circle is 1 unit.

(ii) Find $\sin\frac{\pi}{3}$, $\sin\frac{2\pi}{3}$, $\sin\frac{4\pi}{3}$, $\sin\frac{5\pi}{3}$.

(iii) Find $\cos\frac{\pi}{3}$, $\cos\frac{2\pi}{3}$, $\cos\frac{4\pi}{3}$, $\cos\frac{5\pi}{3}$.

(iv) Find $\tan\frac{\pi}{3}$, $\tan\frac{2\pi}{3}$, $\tan\frac{4\pi}{3}$, $\tan\frac{5\pi}{3}$.

5. (i) Copy and complete the diagram at the right by including the coordinates of points A to D; radius of circle is 1 unit.

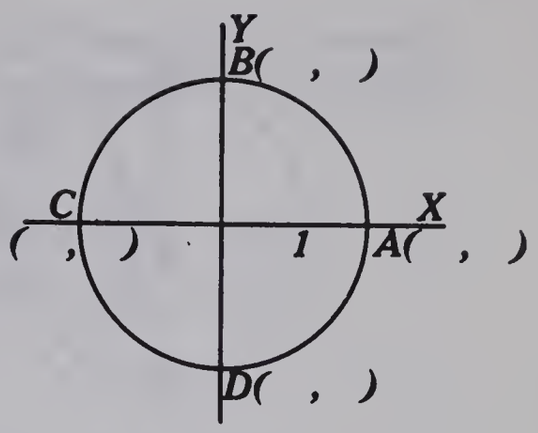


(ii) Find $\sin\frac{\pi}{6}$, $\sin\frac{5\pi}{6}$, $\sin\frac{7\pi}{6}$, $\sin\frac{11\pi}{6}$.

(iii) Find $\cos\frac{\pi}{6}$, $\cos\frac{5\pi}{6}$, $\cos\frac{7\pi}{6}$, $\cos\frac{11\pi}{6}$.

(iv) Find $\tan\frac{\pi}{6}$, $\tan\frac{5\pi}{6}$, $\tan\frac{7\pi}{6}$, $\tan\frac{11\pi}{6}$.

6. (i) Copy and complete the diagram at the right by including the co-ordinates of points A to D; radius of the circle is 1 unit.



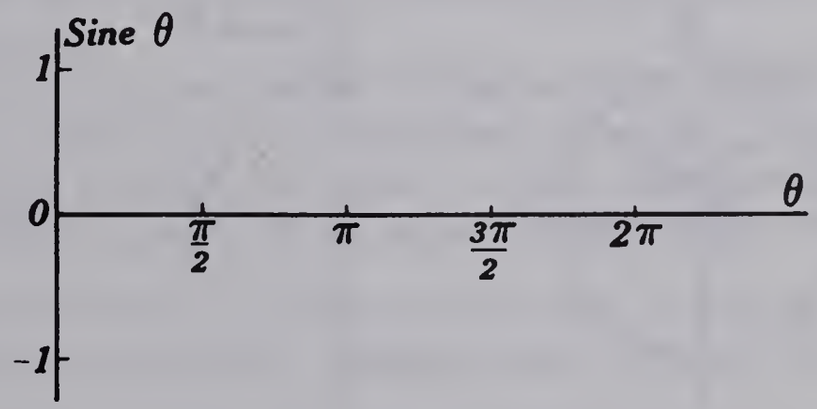
(ii) Find $\sin 0$, $\sin\frac{\pi}{2}$, $\sin\pi$, $\sin\frac{3\pi}{2}$ $\sin 2\pi$.

(iii) Find $\cos 0$, $\cos\frac{\pi}{2}$, $\cos\pi$, $\cos\frac{3\pi}{2}$, $\cos 2\pi$.

7. (i) Copy and complete the following table using the information obtained in questions 3 to 6; approximate $\sin\theta$ to one decimal place, where necessary.

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
$\sin\theta$																	

(ii) Plot the points which correspond to the ordered pairs in the table using coordinate axes indicated below. Draw a smooth curve through these points to obtain the graph of the sine function.



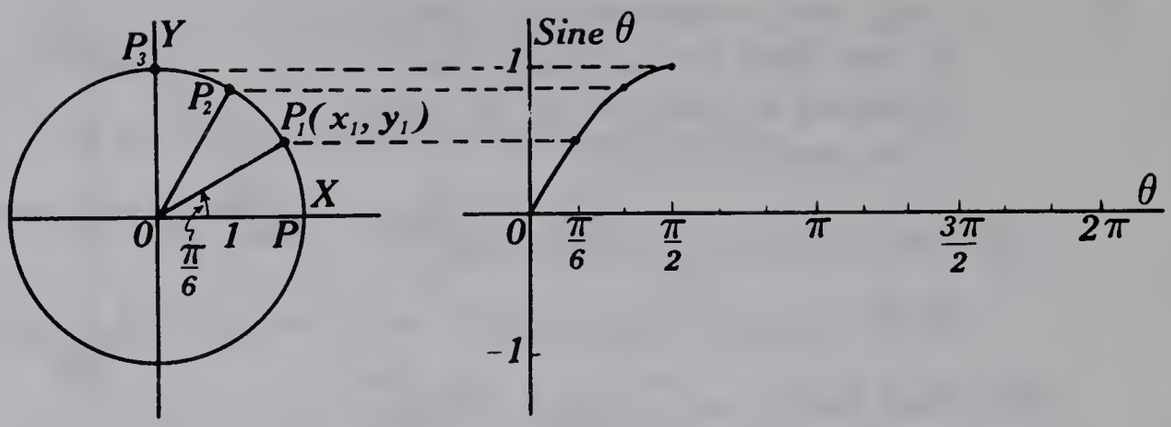
(iii) If the terminal ray is permitted to rotate, generating angles greater than 2π, predict the nature of the extension of the graph.

8. (i) Construct a table of ordered pairs of the cosine function in the same manner as for 7 (i). Approximate values of $\cos\theta$ to one decimal place where necessary.

(ii) Draw the graph of the cosine function.

(iii) What relation appears to exist between the graphs of the sine and cosine functions?

9. In the accompanying diagram, at the top of page 368, OP of unit length rotates counter-clockwise about O.



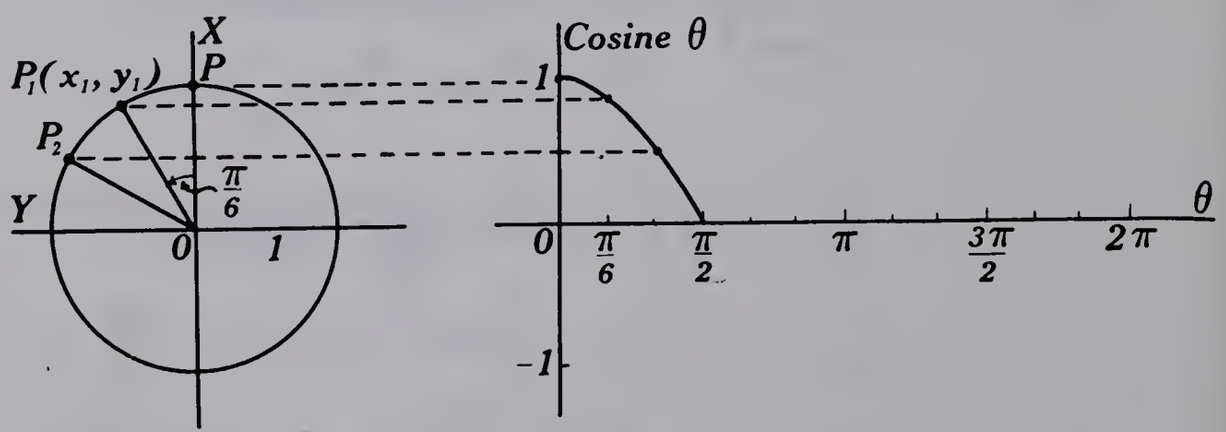
Since $\sin \angle POP_1 = \frac{y_1}{1}$ = ordinate of P_1,

a graph of the sine function may be obtained by plotting the vertical displacement of the point P (ordinate of P) against the angle through which OP has rotated as illustrated. Copy and complete the graph by plotting the ordinates of P which correspond to the following values of θ:

$$0, \ \frac{\pi}{6}, \ \frac{\pi}{3}, \ \frac{\pi}{2}, \ \frac{2\pi}{3}, \ \frac{5\pi}{6}, \ \pi, \ \frac{7\pi}{6}, \ \frac{4\pi}{3}, \ \frac{3\pi}{2}, \ \frac{5\pi}{3}, \ \frac{11\pi}{6} \ 2\pi.$$

10. In the accompanying diagram, OP of unit length rotates counterclockwise about O.

If the coordinate axes are arranged as indicated in the above diagram, then



$$\cos \angle POP_1 = \frac{x_1}{1}$$

$$= \frac{\text{vertical displacement of } P_1}{1}.$$

Thus a graph of the cosine function may be obtained by plotting the vertical displacement of the point P against the angle through which OP has rotated as illustrated.

Copy and complete the graph by plotting the vertical displacement of P which corresponds to the following values of θ:

$$0, \ \frac{\pi}{6}, \ \frac{\pi}{3}, \ \frac{\pi}{2}, \ \frac{2\pi}{3}, \ \frac{5\pi}{6}, \ \pi, \ \frac{7\pi}{6}, \ \frac{4\pi}{3}, \ \frac{3\pi}{2}, \ \frac{5\pi}{3}, \ \frac{11\pi}{6}, \ 2\pi.$$

12·11 **The cosecant, secant, and cotangent functions (supplementary).** The *cosecant* (csc), *secant* (sec) and *cotangent* (cot) functions are defined as follows:

(i) *cosecant function*

$$\text{csc} = \left\{(\theta, y) \mid y = \csc\theta = \frac{1}{\sin\theta},\ \sin\theta \neq 0,\ \theta, y \in R\right\};$$

(ii) *secant function*

$$\text{sec} = \left\{(\theta, y) \mid y = \sec\theta = \frac{1}{\cos\theta},\ \cos\theta \neq 0,\ \theta, y \in R\right\};$$

(iii) *cotangent function*

$$\text{cot} = \left\{(\theta, y) \mid y = \cot\theta = \frac{1}{\tan\theta},\ \tan\theta \neq 0,\ \theta, y \in R\right\}.$$

The cosecant, secant, and cotangent functions are sometimes referred to as the *reciprocal functions*, or as the *secondary trigonometric functions*.

12·12 **Using the tables of function values to determine a function value for a particular angle.**

It is not convenient to determine the trigonometric function values for any particular angle by geometric methods. Instead, we may refer to the Tables of Values of the Trigonometric Functions (pages 394-95) which are prepared from formulas.

The trigonometric function values of angles whose degree measures are integers from 0 to 90 are given in the table on page 394. These function values are expressed as decimals, correct to four decimal places. The corresponding radian measures of the angles are also shown.

By answering the following questions discover how to use the Table of Trigonometric Ratios on page 394. *Compare your solutions with those on page* 451.

1. Write each of the following trigonometric ratios from the table on pages 394-95:

(i) sin 37° (ii) sec 80° (iii) tan 2° (iv) csc 29°

2. For each of the following, determine the measurement of the angle to the nearest degree:

(i) $\tan\theta° = 2.0000$ (ii) $\cos\alpha° = 0.8018$

3. As θ increases from 0 to $\frac{\pi}{2}$:

(i) which trigonometric ratios of θ increase?

(ii) which trigonometric ratios of θ decrease?

The method of using the tables to find function values of angles greater than 90° is illustrated in the following examples.

Example 1. Using the table of trigonometric function values on pages 394-395 determine:

(i) $\sin 140°$ (ii) $\cos 215°$ (iii) $\tan(-50°)$.

Solution.

(i) (ii) (iii)

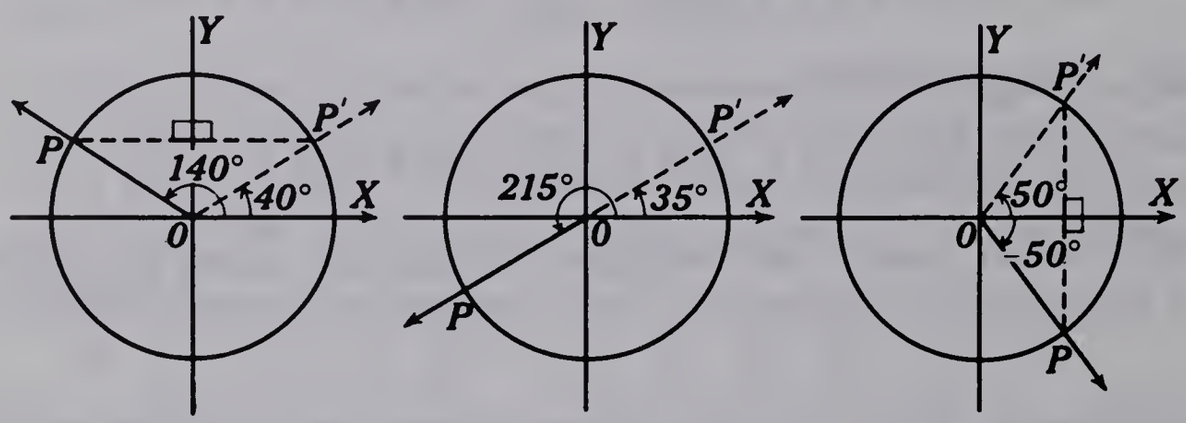


Fig. 12-17

From *Fig 12-17* (i), $\sin 140° = \sin 40°$.
From table page 444 $\sin 40° \doteq 0.6428$.
$\therefore \sin 140° \doteq 0.6428$.

From *Fig 12-17* (ii), $\cos 215° = -\cos 35°$.
From tables, $\cos 35° \doteq 0.8192$.
$\therefore \cos 215° \doteq -0.8192$.

From *Fig 12-17* (iii), $\tan(-50°) = -\tan 50°$.
From tables: $\tan 50° \doteq 1.192$.
$\therefore \tan(-50°) \doteq -1.192$.

Example 2. Using the table of trigonometric function values, determine:
(i) $\cos 820°$ (ii) $\sin(-230°)$.

Solution.

(i) (ii)

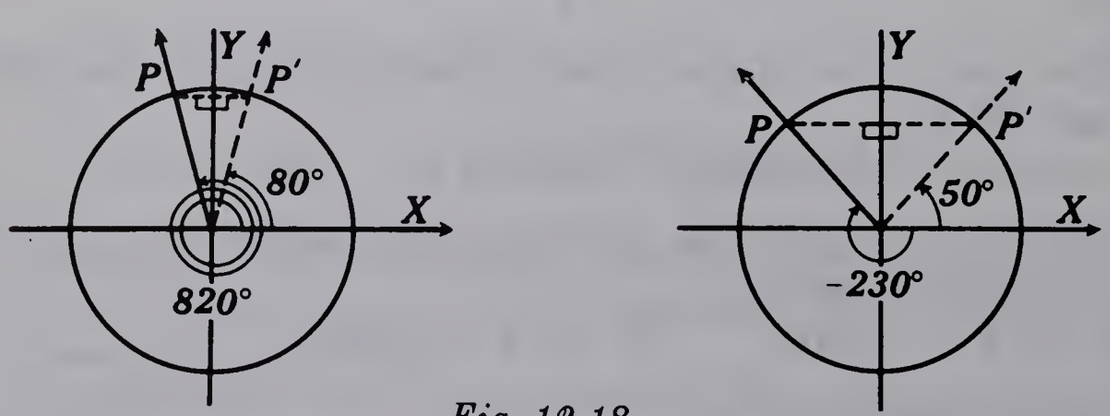


Fig. 12-18

From *Fig 12-18* (i), $\cos 820° = -\cos 80°$.

From *Fig 12-18* (ii), $\sin(-230°) = \sin 50°$.

From the table,
$\cos 80° \doteq 0.1736$.
$\therefore \cos 820° \doteq -0.1736$.

From the table,
$\sin 50° \doteq 0.7660$.
$\therefore \sin(-230°) \doteq 0.7660$.

Exercise 12-7

(A)

Find from the tables:

1. $\sin 0°$ **2.** $\sin 30°$ **3.** $\sin 50°$

4. $\sin 65°$ **5.** $\sin 75°$ **6.** $\sin 90°$

7. As an angle increases in measurement from 0° to 90°, how does the sine of the angle change?

Find from the tables:

8. $\cos 0°$ **9.** $\cos 30°$ **10.** $\cos 50°$

11. $\cos 65°$ **12.** $\cos 75°$ **13.** $\cos 90°$

14. As an angle increases in measurement from 0° to 90°, how does the cosine of the angle change?

15. By an examination of the tangent table state how the tangent of an angle changes as the angle increases in measurement from 0° to 90°.

(B)

From the tables on page 394, *find:*

16. $\cos 29°$ **17.** $\tan 75°$ **18.** $\sin 58°$

19. $\csc 21°$ **20.** $\cot 63°$ **21.** $\sin \frac{\pi}{36}$

22. $\sec 39°$ **23.** $\tan 42°$ **24.** $\cos 80°$

25. $\csc 52°$ **26.** $\sec 1.5359$ **27.** $\cot \frac{\pi}{4}$

From the tables on page 394 *find the measurement of the indicated angles to the nearest degree and state the corresponding radian measurement:*

28. $\tan \alpha° = 1.150$ **29.** $\sin \beta° = .9075$

30. $\cos \theta° = .8935$ **31.** $\cot \gamma° = .40$

32. $\sin \theta° = .5770$ **33.** $\sec \beta° = 9.20$

34. $\cot \gamma° = .750$ **35.** $\csc \alpha° = 2.816$

36. $\tan \theta° = .091$ **37.** $\cos \beta° = .202$

Using the table of trigonometric function values on pages 394-95 find:

38. $\sin\frac{3\pi}{4}$ **39.** $\cos\frac{2\pi}{3}$ **40.** $\tan\frac{5\pi}{4}$

41. $\cos 225°$ **42.** $\tan(-130°)$ **43.** $\tan(-120°)$

44. $\sin\frac{4\pi}{3}$ **45.** $\tan 150°$ **46.** $\cos(-240°)$

(C)

47. $\sin\frac{50\pi}{3}$ **48.** $\cos(-1500°)$ **49.** $\tan 3720°$

50. $\cos\left(-\frac{11\pi}{6}\right)$ **51.** $\sin 600°$ **52.** $\cos\left(-\frac{\pi}{7}\right)$

53. $\cos\left(-\frac{3\pi}{8}\right)$ **54.** $\sin(-100°)$ **55.** $\sin(180+20)°$

12·13 **Trigonometric function values of the acute angles of a right triangle; complementary angles.** In *Fig. 12-19(a)*, $\triangle ABC$ is right-angled at B. There is a ray in the first quadrant, *Fig. 12-19(b)*, with its end point on O, such that $OM = CB$, $MP = BA$, and $MP \perp OX$.

$\because\ OM = CB,\ MP = BA,\ \angle PMO = \angle ABC = 90°,$

$\therefore\ \triangle OMP \cong \triangle CBA.$

$\therefore\ OP = CA,\ \angle MOP = \angle BCA.$

If the coordinates of P are (x, y), and the length of OP is r units, then:

$OM = |x| = x,$ $\qquad MP = |y| = y.$

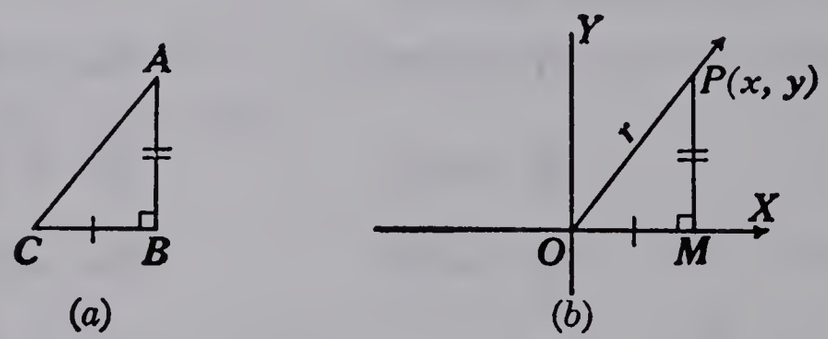


Fig. 12-19

$$\therefore \sin\angle C = \sin\angle MOP = \frac{y}{r} = \frac{MP}{OP} = \frac{BA}{CA},$$

$$\cos\angle C = \cos\angle MOP = \frac{x}{r} = \frac{OM}{OP} = \frac{CB}{CA},$$

$$\tan\angle C = \tan\angle MOP = \frac{y}{x} = \frac{MP}{OM} = \frac{BA}{CB}.$$

We commonly refer to the interior angles of a triangle by naming only the vertex. It is common practice to use the letter naming the vertex to represent the measure of the angle. Thus, *sin A* means the sine of the interior angle of the triangle whose vertex is A; its measure is represented by A.

The sides of $\triangle ABC$ are often described with reference to $\angle ACB$ as follows:

CB is the side adjacent to $\angle ACB$
or *the adjacent side*;

BA is the side opposite $\angle ACB$
or *the opposite side*;

CA is *the hypotenuse*.

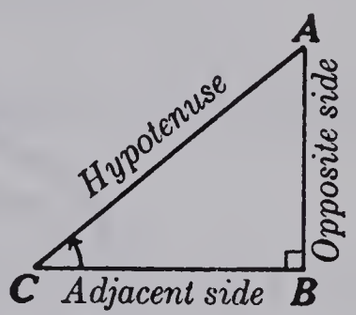


Fig. 12-20

Thus, for a right triangle ABC:

$$\sin C = \frac{\text{opposite side}}{\text{hypotenuse}} \qquad \csc C = \frac{\text{hypotenuse}}{\text{opposite side}}$$

$$\cos C = \frac{\text{adjacent side}}{\text{hypotenuse}} \qquad \sec C = \frac{\text{hypotenuse}}{\text{adjacent side}}$$

$$\tan C = \frac{\text{opposite side}}{\text{adjacent side}} \qquad \cot C = \frac{\text{adjacent side}}{\text{opposite side}}$$

The terms *opposite side, adjacent side, hypotenuse* mean *the measures of these sides in the same unit.*

Write solutions for the following problems and compare then with those on page 452.

1. (i) Calculate the six trigonometric function values of each acute angle in $\triangle ABC$ (*Fig. 12-21*).

(ii) Make a list of the pairs of equal trigonometric ratios.

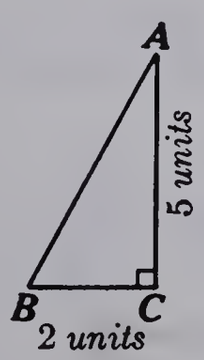


Fig. 12-21

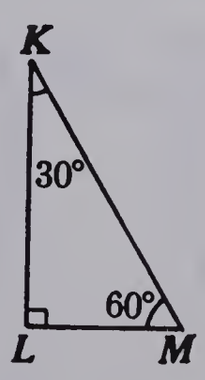


Fig. 12-22

2. In *Fig. 12-22*, $\angle K = 30°$, $\angle L = 90°$, $\angle M = 60°$.

(i) Determine the ratio $LM : KM : KL$.

(ii) Determine the six trigonometric function values of $\angle K$ and $\angle M$.

(iii) Make a list of the pairs of equal trigonometric functions.

3. From these results, make a conjecture about the relation between the trigonometric function of complementary angles.

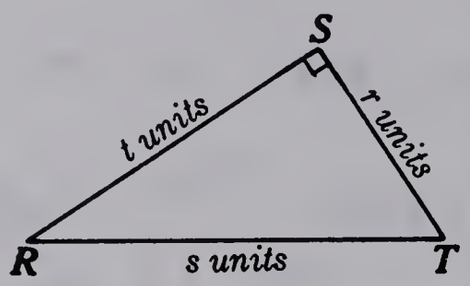


Fig. 12-23

4. Test this conjecture (question 3) for $\triangle RST$ (*Fig. 12-23*) in which $\angle S = 90°$, $RS = t$ units, $ST = r$ units, and $RT = s$ units.

From question 4, the following results are obtained:

If R and T are complementary angles,

sine R = *co*sine T	*co*sine R = sine T
tangent R = *co*tangent T	*co*tangent R = tangent T
secant R = *co*secant T	*co*secant R = secant T

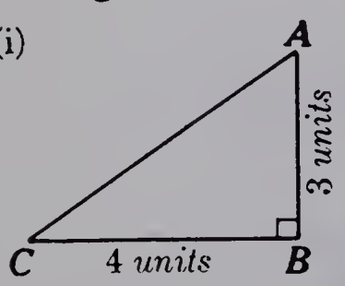
In general, a trigonometric ratio of an acute angle is equal to the co-ratio of its complement.

Exercise 12-8

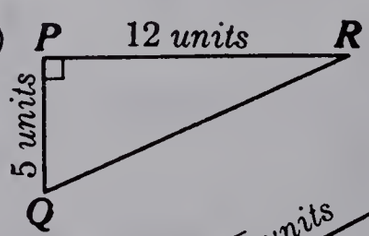
(A)

1. State the trigonometric function values of each acute angle in the given triangles:

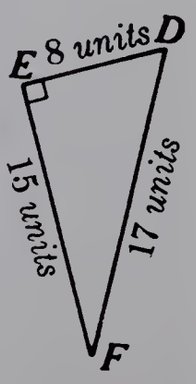
(i)



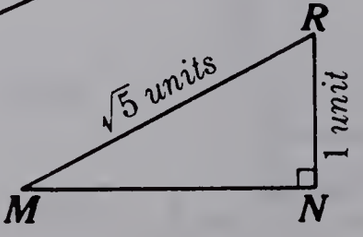
(ii)



(iii)



(iv)



(B)

2. Given the following information about $\triangle ABC$, find the trigonometric function values of each acute angle of $\triangle ABC$:

(i) $\angle B = 90°$, $BC = 2$ units, $AB = 3$ units

(ii) $\angle A = 90°$, $AB = 5$ units, $BC = \sqrt{29}$ units

(iii) $\angle C = 90°$, $AC = 3$ units, $\tan \angle A = 1\frac{1}{3}$

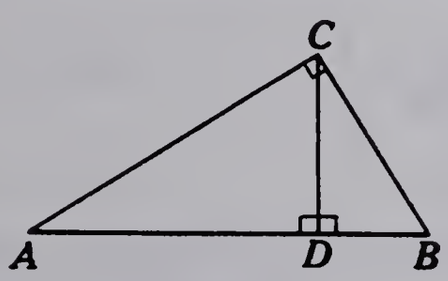
(iv) $\angle C = 90°$, $BC = 2$ units, $\sin \angle B = \dfrac{1}{\sqrt{2}}$

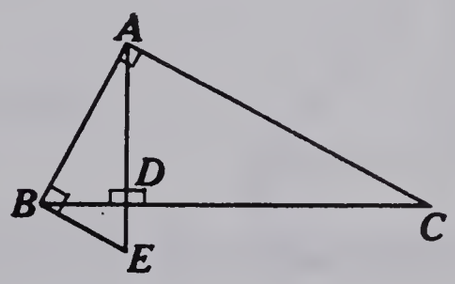
3. $\triangle PQR$ is equilateral. $PT \perp QR$. Determine the six trigonometric function values of:

(i) $\angle PQT$ (ii) $\angle QPT$

4. In the diagram at the left below, $\angle ACB = 90°$, $CD \perp AB$. State the six trigonometric function values of $\angle A$ using:

(i) the sides of $\triangle ABC$ (ii) the sides of $\triangle ADC$





5. By referring to the diagram at the right above, determine the trigonometric function values of:

(i) $\angle ACB$ (ii) $\angle AEB$ (iii) $\angle BAE$

Give two answers for each.

6. Express each of the following as a function of the complementary angle:

(i) sec 50° (ii) cos 75° (iii) tan 48°

(iv) sin 57° (v) csc 60° (vi) cot 66°

7. $ABCD$ is a square in which the sides are each 1 unit in length. Determine the six trigonometric function values of $\angle ABD$.

8. In $\triangle ABC$, $\angle B = 90°$. From the additional information in each part below, calculate the six trigonometric function values of $\angle A$ and of $\angle C$:

(i) $AB = 20$ units, $\sin C = \dfrac{4}{5}$

(ii) $AB = 1$ unit, $\tan C = .5$

(iii) $\tan \angle A = \dfrac{8}{15}$, $\sin C = \dfrac{15}{17}$ (iv) $AC : AB = 29 : 20$

***12·14* Solution of right triangles.** The Pythagorean Theorem relates the measurements of the sides of a right triangle.

In *Fig. 12-24*, the length of BC, the side opposite vertex A, is a units; the length of AC, the side opposite vertex B, is b units; the length of AB, the side opposite vertex C, is c units.

$$\because \ \angle C = 90°, \ \therefore \ c^2 = a^2 + b^2$$
$$\text{or } a^2 = c^2 - b^2$$
$$\text{or } b^2 = c^2 - a^2.$$

In this Chapter, a, b, and c represent the number of units in the measurements of the sides opposite $\angle A$, $\angle B$, and $\angle C$, respectively. It is understood that each is expressed in the same unit.

The trigonometric functions may be used to relate the sides of a right triangle to the angles of the triangle:

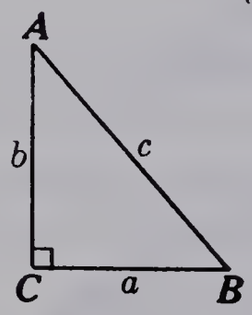


Fig. 12-24

$$\sin B = \frac{b}{c} \qquad \sin A = \frac{a}{c}$$
$$\cos B = \frac{a}{c} \qquad \cos A = \frac{b}{c}$$
$$\tan B = \frac{b}{a} \qquad \tan A = \frac{a}{b}$$

These relationships can be used to *solve a triangle:* that is, to determine the measurements of certain parts of a triangle when the measurements of the other parts are given. In order to solve a *right triangle,* we must be given the measurements of

(i) two sides of the triangle or (ii) one side and one acute angle.

Write solutions to the following problems to discover how to solve right triangles for each of these cases; compare them with those given on page 452.

1. In $\triangle ABC$, $\angle B = 90°$, $a = 2.6$, $b = 4.2$.

(i) Draw a diagram to represent $\triangle ABC$.

(ii) Using the Pythagorean Theorem, calculate c to one decimal place.

(iii) Calculate $\sin A$ to four decimal places.

(iv) Using the table on page 394, find $\angle A$ to the nearest degree.

(v) Calculate the measurement of $\angle C$ to the nearest degree.

As an alternative method of determining $\angle C$*:*

(vi) Find $\tan C$ to four decimal places (from the table on page 394).

(vii) Using the relation $\frac{c}{a} = \tan C$ or $c = a \tan C$, find c to one decimal place.

(viii) How can $\tan A$ be used to calculate c?

2. In $\triangle PQR$, $\angle Q = 90°$, $\angle P = 32°$, $r = 7.4$.

(i) Draw a diagram to represent $\triangle PQR$.

(ii) From the table on page 433, find $\tan P$ to four decimal places.

(iii) From the relation $\tan P = \frac{p}{r}$ or $p = r \tan P$, calculate p to one decimal place.

(iv) Use the relation $\sec P = \frac{q}{r}$ or $q = r \sec P$ to calculate q to one decimal place.

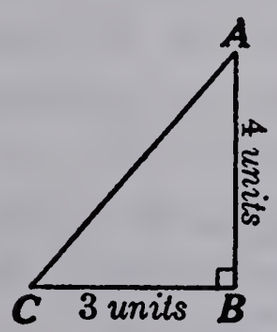
(v) State the measurement of $\angle R$.

Exercise 12-9

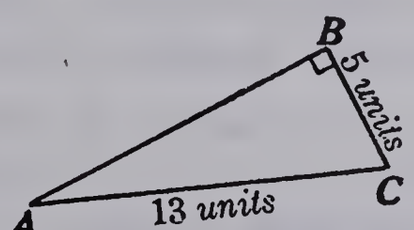
(A)

1. Give a relationship from which $\angle A$ can be determined for each of the following triangles:

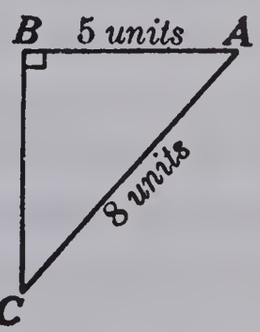
(i)



(ii)

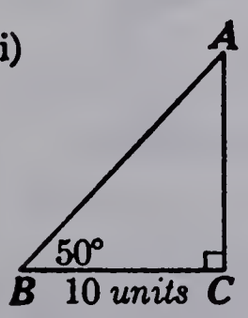


(iii)

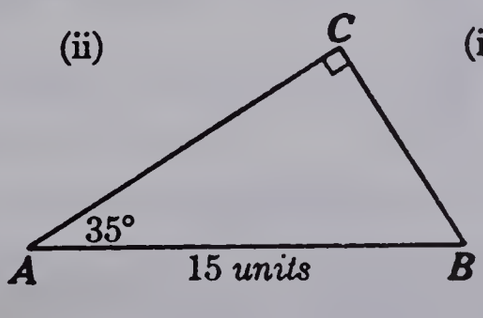


2. For each of the following triangles, state a trigonometric ratio which could be used to determine b:

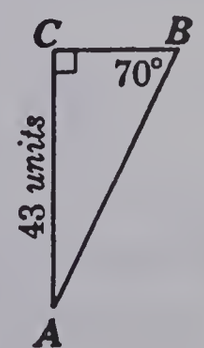
(i)



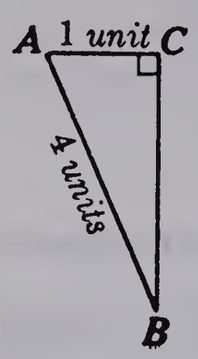
(ii)

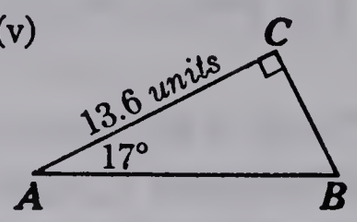


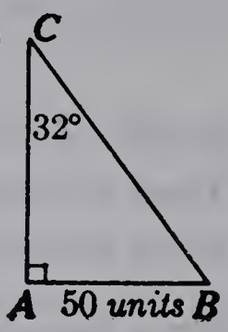
(iii)



2. For each of the following triangles, state a trigonometric ratio which could be used to determine b:

(iv) 

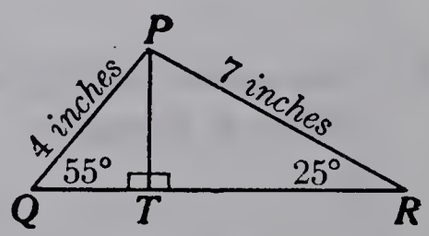
(v) 

(vi) 

(B)

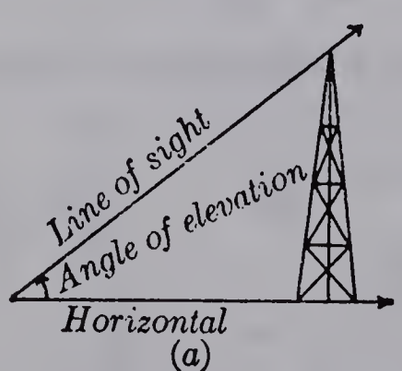
3. From the given information, solve each of the following triangles. State all angle measurements to the nearest degree and all lengths to one decimal place:

(i) $\angle A = 90°$, $b = 12$, $a = 13$
(ii) $\angle B = 90°$, $a = 2.5$, $c = 1.5$
(iii) $\angle B = 90°$, $\angle A = 58°$, $c = 10$
(iv) $\angle C = 90°$, $\angle B = 29°$, $b = 5.6$
(v) $\angle B = 90°$, $c = 31$, $b = 76$
(vi) $\angle C = 90°$, $\angle A = 23°$, $a = 5$
(vii) $\angle A = 90°$, $\angle C = 40°$, $a = 8$
(viii) $\angle B = 90°$, $a = 23$, $c = 24$
(ix) $\angle C = 90°$, $\angle B = 72°$, $c = 8.4$
(x) $\angle A = 90°$, $\angle C = 15°$, $b = 20$

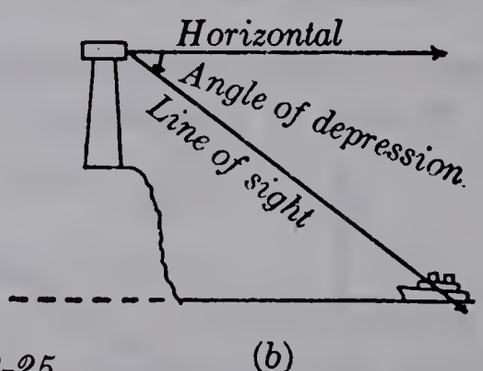


4. From the information in the diagram at the right above, calculate the length of QR.

12·15 **Problems involving the solution of right triangles.** In referring to angles in a vertical plane, the two terms following are frequently used.



(a)



(b)

Fig. 12-25

(i) If the object observed is above the horizontal, *Fig12-25* (*a*), the angle determined by the horizontal and the line of sight is called the *angle of elevation.*

(ii) If the object observed is below the horizontal, *Fig 12-25(b)*, the angle determined by the horizontal and the line of sight is called the *angle of depression.*

In referring to angles in a horizontal plane, the direction of an object from an observer is obtained by measuring the angle determined by the line of sight and one of the compass directions: north, south, east, or west. This angle is called the *bearing* of the object with respect to the *compass direction.* In *Fig. 12-26 (a)*, the object has a bearing 15° *east of north*, which is usually expressed N 15° E.

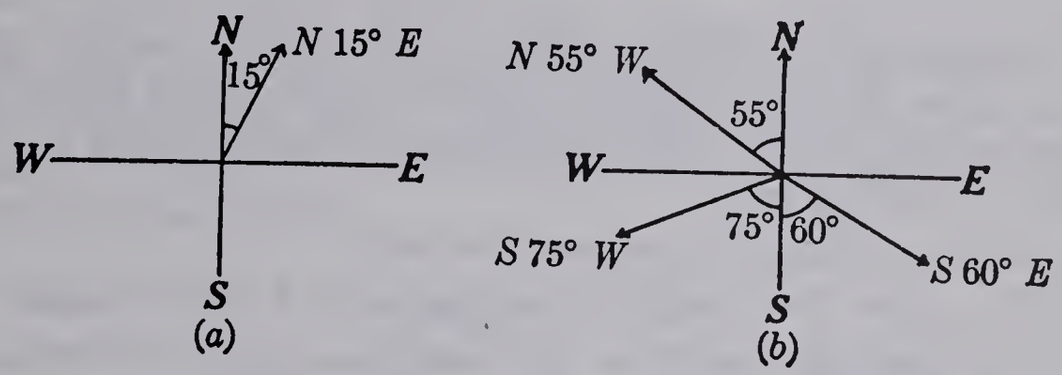


Fig. 12-26

Other compass bearings are shown in *Fig 12-26(b)*.

Many problems in surveying, navigation, and engineering involve the calculation of heights, distances, and angles.

Discover how to solve problems involving heights, distances, and angles by answering the following questions. Compare your solutions with those on page 453.

1. The angle of elevation of the top of a flagpole from a point 108 feet from its base is 48°. Find the height of the flagpole.

(i) Make a diagram to represent the data.

(ii) What measurement are we asked to find? Represent this measurement by h feet.

(iii) State a value for tan 48° in terms of h.

(iv) Using the Trigonometric Tables on page 394, solve this equation for h.

(v) Write a conclusion to the problem, giving your answer to the nearest foot.

2. From the top of a lighthouse which is 250 feet above sea level, the angle of depression of a small ship is observed to be 32°. Calculate the distance of the ship from the base of the lighthouse.

(i) Make a diagram to represent the data.

(ii) Make a statement in which the required measurement in the problem is represented by x feet.

(iii) Name the right triangle in the diagram and name an angle of this triangle which has a measurement of 32°.

(iv) State a value for cot 32° in terms of x.

(v) Using the Trigonometric Tables on page 394, solve this equation for x.

(vi) Write a conclusion to the problem, giving your answer to one decimal place.

3. A ship travels N 32° E for 2 hours at 14 knots (1 nautical mile per hour). The ship then changes course to S 58° E and travels in this direction for 3 hours. Find the distance in nautical miles of the ship from the starting point.

(i) Make a diagram to represent the data.

(ii) Show that the triangle in the diagram is a right triangle.

(iii) Calculate the length of the third side using the Pythagorean Theorem.

(iv) Write a conclusion to the problem giving your answer to the nearest nautical mile.

Exercise 12-10

(B)

In each of the following, state angle measurements to the nearest degree and lengths to the nearest whole number of units:

1. The angle of elevation of the top of a building from a point 1,000 ft. from its foot is 35°. Find the height of the building.

2. The string of a kite is 500 ft. long. Find the height of the kite when its angle of elevation is 40°.

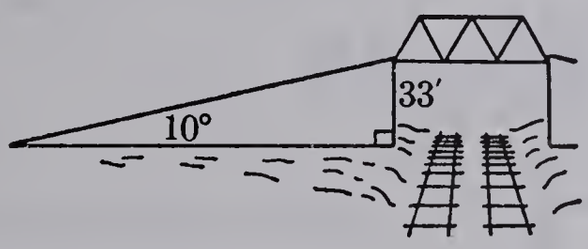
3. A tree casts a shadow 30 ft. long when the angle of elevation of the sun is 62°. Find the height of the tree.

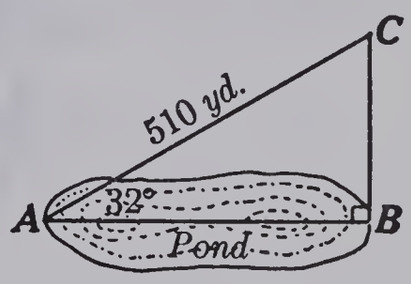
4. From the top of a 300-ft. tower, the angle of depression of an object is 32°. How far is the object from the tower?

5. The combined height of a flagpole and the building on which it stands is 160 ft. At a point 115 ft. from the foot of the building, the angle of elevation of the top of the building is 42°. Find the length of the flagpole.

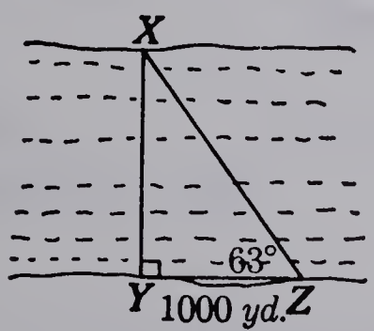
6. Three towns A, B, and C are located so that B is due north of C, A is 31 miles due west of C, and the bearing of B from A is N 56° E. Find the distance from town A to town B.

7. Two ships A and B start from the same harbour jetty. A sails S 45° E at 16 knots, and B sails N 45° E at 17 knots. Find their distance apart after 7 hours.
8. The angle of inclination of a road is 8°. Find the rise for 500 feet measured along the road.
9. At a distance of 14,000 ft. from the foot of the Empire State Building, the angle of elevation of its top is 5°. Find the height of the building.
10. From the top of a perpendicular cliff 325 ft. above water level, the angle of depression of a ship at sea is 32°. Find the distance of the ship from the foot of the cliff.
11. The bridge over the railroad track, shown in the diagram at the left below, is 33 ft. high. If the approach to the bridge must have an angular elevation of 10°, find the length of the approach measured along the road surface.





12. To find the distance from A to B across a pond, a surveyor makes the measurements shown in the diagram at the right above. Find AB.
13. To find the distance from X to Y across a river, a surveyor makes the measurements shown in the diagram at the left below. Find XY.



14. The angles of elevation of the top of a mountain from points on opposite sides of it, on the same level, are 35° and 47°. The top of the mountain is 1,510 ft. above the level of the two points. Find the length of a tunnel through the mountain between the two points.

12·16 Coordinates of a point in terms of r and θ. In *Fig. 12-27* the rectangular coordinates of P are (x, y), and $\angle XOP = \theta$, $OP = r$ units.

$$\because \frac{x}{r} = \cos\theta, \quad \therefore x = r\cos\theta.$$

$$\because \frac{y}{r} = \sin\theta, \quad \therefore y = r\sin\theta.$$

Thus, the coordinates of P in terms of r and θ are $(r\cos\theta, r\sin\theta)$.

This leads to a fundamental trigonometric relation:

$$\begin{aligned} \because r^2 &= x^2 + y^2 \\ &= r^2\cos^2\theta + r^2\sin^2\theta, \text{ it follows that} \\ 1 &= \cos^2\theta + \sin^2\theta \text{ is true for all } \theta \in R. \end{aligned}$$

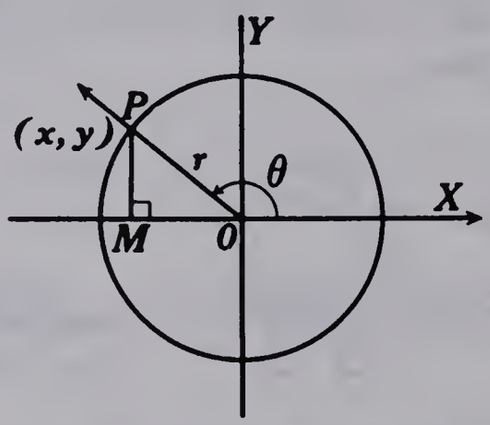


Fig. 12-27

12·17 Law of Sines for a triangle. In order to use our knowledge of trigonometry to calculate lengths of sides and measurements of angles of a triangle, we must first orient an angle of the triangle.

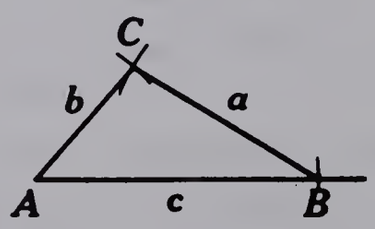


Fig. 12-28

Thus, for $\triangle ABC$, *Fig. 12-28* $\angle BAC$ may be positively oriented by locating the origin of the coordinate axes at A and the positive x-axis along AB, as shown in *Fig 12-29*(*a*).

In *Fig 12-29* (*b*), $\angle CBA$ is positively oriented; in *Fig 12-29* (*c*), $\angle BCA$ is positively oriented. Using the result of Section 12·16, in each case, the coordinates of C, A, B, respectively, are as indicated. We commonly

refer to the interior angles of a triangle by naming only the vertex. It is also common practice to use the letter naming the vertex to represent the measure of the angle. Thus, sin A means the sine of the interior angle of the triangle whose vertex is A; its measure is represented by A.

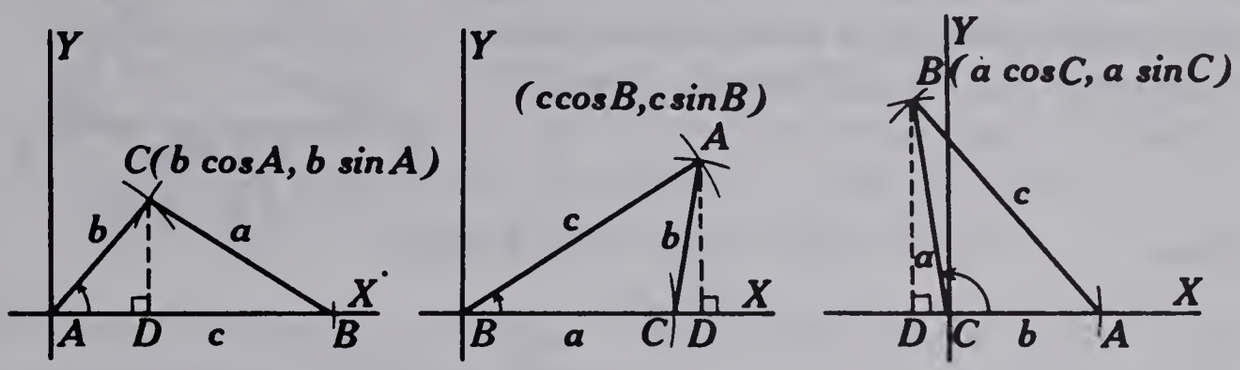


Fig. 12-29

In *Fig. 12-29* (*a*) the number of square units, $\triangle$, in the area of $\triangle ABC$ is given by

$$\begin{aligned}\triangle &= \tfrac{1}{2}c(CD)\\ &= \tfrac{1}{2}c(b\sin A)\\ &= \tfrac{1}{2}cb\sin A\,.\end{aligned}$$

Similarly, in *Fig. 12-29* (*b*),

$$\begin{aligned}\triangle &= \tfrac{1}{2}a(AD)\\ &= \tfrac{1}{2}a(c\sin B)\\ &= \tfrac{1}{2}ac\sin B\,.\end{aligned}$$

Also, in *Fig. 12-29* (*c*),

$$\begin{aligned}\triangle &= \tfrac{1}{2}b(BD)\\ &= \tfrac{1}{2}b(a\sin C)\\ &= \tfrac{1}{2}ba\sin C\,.\end{aligned}$$

Thus, $\tfrac{1}{2}cb\sin A = \tfrac{1}{2}ac\sin B = \tfrac{1}{2}ba\sin C\,.$

Dividing by $\tfrac{1}{2}abc$,

$$\therefore\quad \frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}\,.$$

Since none of these numerators is zero:

$$\therefore\quad \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}\,.$$

This relation is called the *Law of Sines for triangles.*

12•18 **Solving oblique triangles using the Law of Sines.** To solve a triangle means to find the measurements of the remaining parts of a triangle, given the measurements of certain parts.

Example. A ship is sailing a course from A to B (*Fig. 12-30*) past a lighthouse, C. If $\angle A = 55°$, $\angle B = 45°$, $AB = 6.5$ miles, find, to the nearest tenth of a mile, AC and BC, the distances from the lighthouse to A and B.

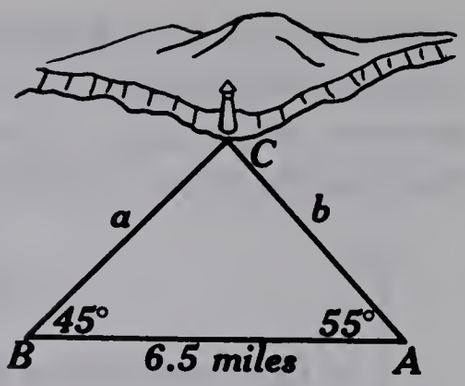


Fig. 12-30

Solution. $\angle C = [180 - (45 + 55)]°$
$= 80°.$

$$\frac{b}{\sin B} = \frac{c}{\sin C}.$$

$$\therefore \quad b = \frac{c \sin B}{\sin C}$$

$$= \frac{6.5 \sin 45°}{\sin 80°}$$

$$\doteq \frac{6.5 \times 0.7071}{0.9848}$$

$$\doteq 4.66.$$

$$\frac{a}{\sin A} = \frac{c}{\sin C}.$$

$$a = \frac{c \sin A}{\sin C}$$

$$= \frac{6.5 \sin 55°}{\sin 80°}$$

$$\doteq \frac{6.5 \times 0.8192}{0.9848}$$

$$\doteq 5.40.$$

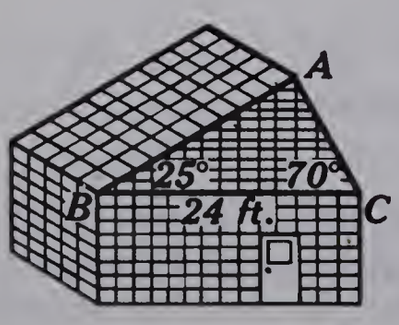
$\therefore$ $AC = 4.7$ miles (to the nearest tenth of a mile).
$BC = 5.4$ miles (to the nearest tenth of a mile).

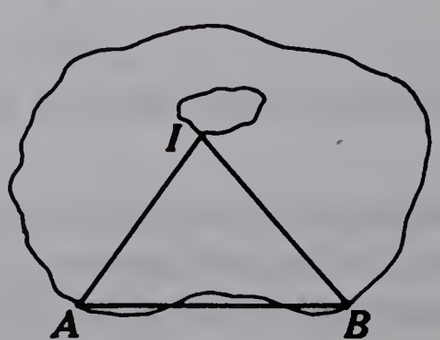
Exercise 12-11

(B)

Solve each of the following, expressing lengths correct to two digits:

1. $\angle A = 66°$, $\angle B = 36°$, $AB = 8.4$ in.
2. $\angle A = 127°$, $\angle B = 43°$, $BC = 66$ ft.
3. $\angle A = 32°$, $\angle C = 74°$, $AC = 44$ ft.
4. A greenhouse is 24 ft. wide. One side of the roof makes an angle of 25° with the horizontal, while the other makes an angle of 70°. Find the rafter lengths, AB and AC, correct to the nearest tenth of a foot.





5. Two docks, A and B, are 1400 yards apart. How far is an island dock, I, from A and B, if $\angle IBA = 40°$ and $\angle IAB = 60°$? (Give your answer correct to the nearest yard.)

6. Two boys, A and B, flying a kite wish to find its distance from them and its vertical height. B holds the string, and A steps off a path beneath the kite a distance of 200 yards. Facing B, A finds the angle of elevation of the kite to be 40°. If the kite string makes an angle of 35° with the ground, find the distance of the kite from B and its vertical height. (Express your answers to two digits.)

12·19 **Law of Cosines for a triangle.** In *Fig. 12-31* a $\triangle ABC$ is oriented as in the determination of the Law of Sines.

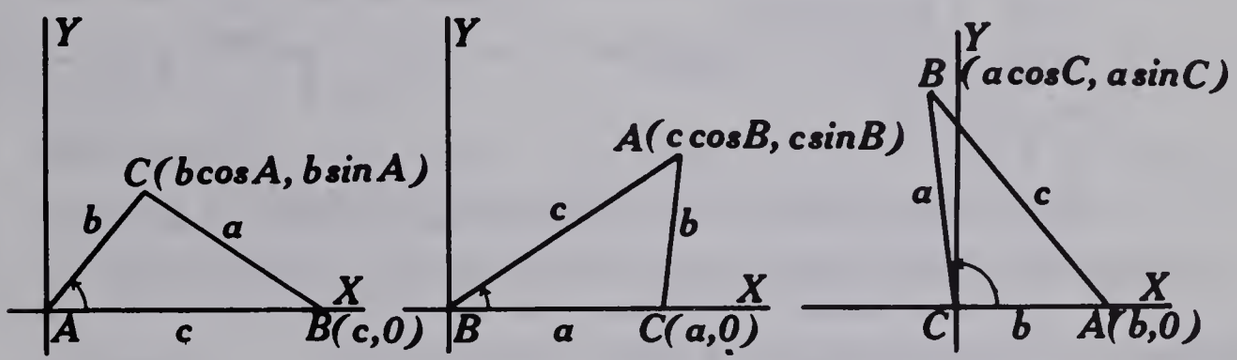


Fig. 12-31

In *Fig. 12-31* (*a*) the coordinates of C and B are indicated. By the formula for the distance between two points, the length of BC or a is given by:

$$a = \sqrt{(b\cos A - c)^2 + (b\sin A - 0)^2}.$$

$$\therefore\ a^2 = b^2\cos^2 A - 2bc\cos A + c^2 + b^2\sin^2 A$$

$$= b^2(\cos^2 A + \sin^2 A) + c^2 - 2bc\cos A\ ,$$

$$\text{or}\quad a^2 = b^2 + c^2 - 2bc\cos A\ .$$

Similarly,

$$b^2 = c^2 + a^2 - 2ca\cos B\ ;$$

$$c^2 = a^2 + b^2 - 2ab\cos C\ .$$

This relation is called the *Law of Cosines.*

The following are equivalent forms of the Law of Cosines:

$$\text{(i)}\quad a^2 = b^2 + c^2 - 2bc\cos A \quad\leftrightarrow\quad \cos A = \frac{b^2 + c^2 - a^2}{2bc}\ ;$$

$$\text{(ii)}\quad b^2 = c^2 + a^2 - 2ca\cos B \quad\leftrightarrow\quad \cos B = \frac{c^2 + a^2 - b^2}{2ca}\ ;$$

$$\text{(iii)}\quad c^2 = a^2 + b^2 - 2ab\cos C \quad\leftrightarrow\quad \cos C = \frac{a^2 + b^2 - c^2}{2ab}\ .$$

***12·20* Solving oblique triangles using the Law of Cosines.** The Law of Cosines enables us to find the length of a side of a triangle if we are given the lengths of the other two sides and the measurement of the angle contained by them. It also enables us to find the measurements of the angles of a triangle if we are given the lengths of the three sides of the triangle.

Example 1. Solve $\triangle ABC$ (*Fig. 12-32*), given $\angle A = 34°$, $AB = 5$ in., $AC = 6$ in.

Solution. $b = 6$, $c = 5$, $\angle A = 34°$.

(i) $a^2 = b^2 + c^2 - 2bc\cos A$

$= 36 + 25 - 2(6)(5)\cos 34°$

$\doteq 61 - 60(0.8290)$

$\doteq 61 - 49.74 \doteq 11.26$.

$a \doteq \sqrt{11.26}$

$\doteq 3.36$.

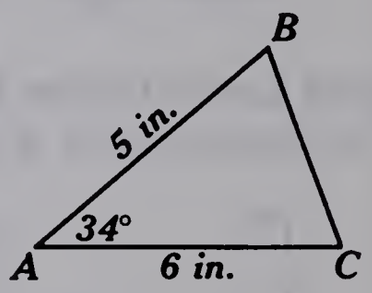


Fig. 12-32

$\therefore$ the length of BC is 3.4 in. (to the nearest tenth of an inch).

(ii) Use the Law of Sines to compute the measurements of $\angle B$ and $\angle C$.

Example 2. Find the length AB of the pond in *Fig. 12-33* given the measurements indicated; give your answer to the nearest rod.

Solution. $a = 40$, $b = 30$, $\angle C = 88°$.

$c^2 = a^2 + b^2 - 2ab\cos C$

$= 1600 + 900 - 2400\cos 88°$

$\doteq 2500 - 2400(0.0349) \doteq 2416$.

$c \doteq 49.2$.

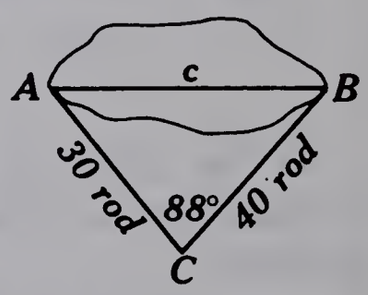


Fig. 12-33

$\therefore$ the length of the pond is 49 rods (to the nearest rod).

Example 3. The distance between the posts at A and B (*Fig. 12-34*) of a hockey goal is 6 feet. A boy scores a goal by shooting the puck along the ice from a point 20 feet from B and 25 feet from A. Within what angle, to the nearest degree, must he make his shot?

Solution. From the Law of Cosines,

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}.$$

$$\therefore \cos C = \frac{400 + 625 - 36}{1000}$$

$$\doteq 0.989.$$

$\therefore$ $\angle C \doteq 8\frac{1}{2}$ degrees.

The boy must make his shot within an angle of 8°.

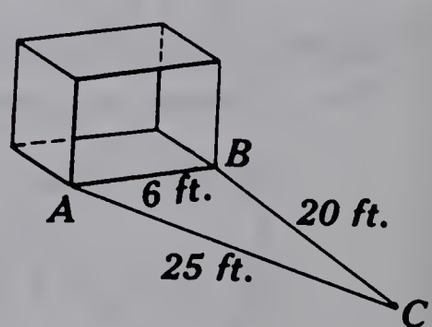


Fig. 12-34

Exercise 12-12

(B)

1. In $\triangle ABC$, find a, to the nearest tenth of an inch, if $\angle A = 15°$, $AC = 2$ in., $AB = 3$ in.
2. In $\triangle ABC$, find c, to the nearest tenth of a foot, if $\angle C = 85°$, $BC = 5$ ft., $AC = 8$ ft.
3. In $\triangle ABC$, $a = 70$, $b = 90$, $c = 100$; find the smallest angle to the nearest degree.
4. The lengths of the sides of a triangle are 2 units, $\sqrt{2}$ units, and $(1 + \sqrt{3})$ units; find, to the nearest degree, the two angles which are less than the greatest angle.
5. If the sides of a triangle have lengths 3 in., 5 in., 7 in., show that the greatest angle is 120°.
6. Solve $\triangle ABC$, if $B = 70°$, $AB = 5.0$ yd., $BC = 2.\dot{5}$ yd. Express angles to the nearest degree and lengths to the nearest tenth of a yard.
7. Two roads diverge from a village A at an angle whose measurement is 15°. Two boys leave A at the same time, one at 3 miles per hour and the other at 5 miles per hour. Find, to the nearest tenth of a mile, their distance apart after two hours.
8. The ship Queen Mary is 1018 feet long. What angle would she subtend when viewed from a point at sea which is $\frac{1}{3}$ mile from her bow and $\frac{1}{4}$ mile from her stern? (Give your answer to the nearest degree.)

Practice Exercise 12-13

(B)

Find the positively oriented acute angle related to:

1. $150°$
2. $750°$
3. $-\frac{5\pi}{6}$
4. $1050°$
5. $\frac{6\pi}{5}$
6. $-6\pi + \frac{\pi}{3}$
7. $\text{₱}200°$
8. $-30°$

Find the following function values:

9. $\sin 240°$
10. $\cos(-225°)$
11. $\tan\frac{2\pi}{3}$
12. $\sin\frac{3\pi}{2}$
13. $\tan(-120°)$
14. $\sin\left(2\pi + \frac{5\pi}{6}\right)$
15. $\cos 960°$
16. $\tan 12\pi$
17. $\sin 480°$

Practice Exercise 12-14

(B)

Evaluate:

1. $\sin\frac{3\pi}{2} + 2\cos\frac{\pi}{2} + \frac{3}{\cos 2\pi}$

2. $\dfrac{\sin 180° + \cos 270° - \csc 90°}{\sin 360° + \cos 0°}$

3. $10\cot^2\frac{\pi}{4}\sec^3\frac{\pi}{6}\sin 0$

4. $\dfrac{1}{\tan^2\frac{\pi}{6}} + \sin^2 0 - \dfrac{3}{4\sin^2\frac{\pi}{4}} + \tan 0$

5. $\sin^2\frac{\pi}{6} + \sin^2 0 + \sin^2\frac{\pi}{2} + \sin^2\frac{\pi}{4} + \sin^2\frac{\pi}{3}$

6. $\tan^2 0 + \tan^2\frac{\pi}{6} + \tan^2\frac{\pi}{4} + \tan^2\frac{\pi}{3}$

7. If $A = \frac{\pi}{2}$ and $B = \frac{\pi}{6}$, show that
$\sin(A - B) = \sin A \cos B - \cos A \sin B$.

Review Exercise 12-15

(B)

Express the following degree measurements in radians:

1. 30° **2.** 135° **3.** 270° **4.** 40° **5.** − 10°
6. 140° **7.** 1° **8.** − 170° **9.** 450° **10.** 720°

Express each of the following radian measurements in degrees:

11. $\frac{\pi}{6}$ **12.** $\frac{5}{4}\pi$ **13.** 1 **14.** 10 **15.** $\frac{3\pi}{5}$

16. $-\frac{7\pi}{6}$ **17.** $-\frac{3\pi}{4}$ **18.** $\frac{7\pi}{3}$ **19.** − 2 **20.** 8π

Determine each of the following:

21. $3\sec 0° + 2\tan 45° + 3\cot 60°$

22. $\cot^2 30°\tan^2 45° + \sqrt{3}\csc 90°\cot 30°$

23. $\cos 0°\sin^2 45° - \cot 90°\sin 45°$

24. $2\csc 90° - 3\sin 0° - 1$

25. $\dfrac{\sin^2 60°\tan^2 0° + \cot^2 60°}{\frac{1}{2} + \sin 30°\sin 90°}$

26. Given that $\cos\alpha = -\frac{2}{3}$, find (i) $\tan\alpha$ (ii) $\csc\alpha$.

Prove the following identities:

27. $(\csc^2\theta - 1)(1 - \cos^2\theta) = \cos^2\theta$

28. $\dfrac{\sin^2\alpha}{\cot^2\alpha} = \tan^2\alpha - \sin^2\alpha$

In each of the following, state angle measurements to the nearest degree and lengths to the nearest whole number of units:

29. From the top of a tower 215 ft. high, a forest ranger sighted a fire. If the angle of depression of the fire was 17°, how far was the fire from the tower?

30. The navigator on a ship finds that the angle of elevation of the top of a lighthouse is 35°. The ship navigation chart shows that the lighthouse is 157 ft. above water level. If the measurement of the angle of elevation is made at a point 20 ft. above water level, how far is the ship from the lighthouse?

31. An overpass must clear a through highway by 40 feet. If the approaches to the overpass must have an elevation of 15°, find the length of the approach.

32. A pendulum 4.2 feet long swings through an angle of 18° on each side of the vertical. How high does the end of the pendulum rise above its lowest point? (Express your answer to one decimal place.)

33. Evaluate
$(\tan 120° - \tan 135°)(\tan 120° + \tan 135°)$.

Determine the amplitude, period, and phase shift of the graph of the function defined by each of the following:

34. $y = \cos\frac{x}{2}$ **35.** $y = \frac{1}{2}\cos x$ **36.** $y = -3\cos 2x$

TABLES

Mantissas of Common Logarithms

Values of the Exponential Function
$t = \{(x, y) \mid y = 10^x, x \in R\}$

Values of the Trigonometric Functions

Amount of 1

Present value of 1

Powers, roots, reciprocals

Mantissas of Common Logarithms

	0	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396

Mantissas of Common Logarithms

	0	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996

Values of the Exponential Function

$$t = \{(x, y) \mid y = 10^x, x \in R\}$$

x	0	1	2	3	4	5	6	7	8	9
.00	1000	1002	1005	1007	1009	1012	1014	1016	1019	1021
.01	1023	1026	1028	1030	1033	1035	1038	1040	1042	1045
.02	1047	1050	1052	1054	1057	1059	1062	1064	1067	1069
.03	1072	1074	1076	1079	1081	1084	1086	1089	1091	1094
.04	1096	1099	1102	1104	1107	1109	1112	1114	1117	1119
.05	1122	1125	1127	1130	1132	1135	1138	1140	1143	1146
.06	1148	1151	1153	1156	1159	1161	1164	1167	1169	1172
.07	1175	1178	1180	1183	1186	1189	1191	1194	1197	1199
.08	1202	1205	1208	1211	1213	1216	1219	1222	1225	1227
.09	1230	1233	1236	1239	1242	1245	1247	1250	1253	1256
.10	1259	1262	1265	1268	1271	1274	1276	1279	1282	1285
.11	1288	1291	1294	1297	1300	1303	1306	1309	1312	1315
.12	1318	1321	1324	1327	1330	1334	1337	1340	1343	1346
.13	1349	1352	1355	1358	1361	1365	1368	1371	1374	1377
.14	1380	1384	1387	1390	1393	1396	1400	1403	1406	1409
.15	1413	1416	1419	1422	1426	1429	1432	1435	1439	1442
.16	1445	1449	1452	1455	1459	1462	1466	1469	1472	1476
.17	1479	1483	1486	1489	1493	1496	1500	1503	1507	1510
.18	1514	1517	1521	1524	1528	1531	1535	1538	1542	1545
.19	1549	1552	1556	1560	1563	1567	1570	1574	1578	1581
.20	1585	1589	1592	1596	1600	1603	1607	1611	1614	1618
.21	1622	1626	1629	1633	1637	1641	1644	1648	1652	1656
.22	1660	1663	1667	1671	1675	1679	1683	1687	1690	1694
.23	1698	1702	1706	1710	1714	1718	1722	1726	1730	1734
.24	1738	1742	1746	1750	1754	1758	1762	1766	1770	1774
.25	1778	1782	1786	1791	1795	1799	1803	1807	1811	1816
.26	1820	1824	1828	1832	1837	1841	1845	1849	1854	1858
.27	1862	1866	1871	1875	1879	1884	1888	1892	1897	1901
.28	1905	1910	1914	1919	1923	1928	1932	1936	1941	1945
.29	1950	1954	1959	1963	1968	1972	1977	1982	1986	1991
.30	1995	2000	2004	2009	2014	2018	2023	2028	2032	2037
.31	2042	2046	2051	2056	2061	2065	2070	2075	2080	2084
.32	2089	2094	2099	2104	2109	2113	2118	2123	2128	2133
.33	2138	2143	2148	2153	2158	2163	2168	2173	2178	2183
.34	2188	2193	2198	2203	2208	2213	2218	2223	2228	2234
.35	2239	2244	2249	2254	2259	2265	2270	2275	2280	2286
.36	2291	2296	2301	2307	2312	2317	2323	2328	2333	2339
.37	2344	2350	2355	2360	2366	2371	2377	2382	2388	2393
.38	2399	2404	2410	2415	2421	2427	2432	2438	2443	2449
.39	2455	2460	2466	2472	2477	2483	2489	2495	2500	2506
.40	2512	2518	2523	2529	2535	2541	2547	2553	2559	2564
.41	2570	2576	2582	2588	2594	2600	2606	2612	2618	2624
.42	2630	2636	2642	2649	2655	2661	2667	2673	2679	2685
.43	2692	2698	2704	2710	2716	2723	2729	2735	2742	2748
.44	2754	2761	2767	2773	2780	2786	2793	2799	2805	2812
.45	2818	2825	2831	2838	2844	2851	2858	2864	2871	2877
.46	2884	2891	2897	2904	2911	2917	2924	2931	2938	2944
.47	2951	2958	2965	2972	2979	2985	2992	2999	3006	3013
.48	3020	3027	3034	3041	3048	3055	3062	3069	3076	3083
.49	3090	3097	3105	3112	3119	3126	3133	3141	3148	3155

Values of the Exponential Function

$$t = \{(x, y) \mid y = 10^x, x \in R\}$$

x	0	1	2	3	4	5	6	7	8	9
.50	3162	3170	3177	3184	3192	3199	3206	3214	3221	3228
.51	3236	3243	3251	3258	3266	3273	3281	3289	3296	3304
.52	3311	3319	3327	3334	3342	3350	3357	3365	3373	3381
.53	3388	3396	3404	3412	3420	3428	3436	3443	3451	3459
.54	3467	3475	3483	3491	3499	3508	3516	3524	3532	3540
.55	3548	3556	3565	3573	3581	3589	3597	3606	3614	3622
.56	3631	3639	3648	3656	3664	3673	3681	3690	3698	3707
.57	3715	3724	3733	3741	3750	3758	3767	3776	3784	3793
.58	3802	3811	3819	3828	3837	3846	3855	3864	3873	3882
.59	3890	3899	3908	3917	3926	3936	3945	3954	3963	3972
.60	3981	3990	3999	4009	4018	4027	4036	4046	4055	4064
.61	4074	4083	4093	4102	4111	4121	4130	4140	4150	4159
.62	4169	4178	4188	4198	4207	4217	4227	4236	4246	4256
.63	4266	4276	4285	4295	4305	4315	4325	4335	4345	4355
.64	4365	4375	4385	4395	4406	4416	4426	4436	4446	4457
.65	4467	4477	4487	4498	4508	4519	4529	4539	4550	4560
.66	4571	4581	4592	4603	4613	4624	4634	4645	4656	4667
.67	4677	4688	4699	4710	4721	4732	4742	4753	4764	4775
.68	4786	4797	4808	4819	4831	4842	4853	4864	4875	4887
.69	4898	4909	4920	4932	4943	4955	4966	4977	4989	5000
.70	5012	5023	5035	5047	5058	5070	5082	5093	5105	5117
.71	5129	5140	5152	5164	5176	5188	5200	5212	5224	5236
.72	5248	5260	5272	5284	5297	5309	5321	5333	5346	5358
.73	5370	5383	5395	5408	5420	5433	5445	5458	5470	5483
.74	5495	5508	5521	5534	5546	5559	5572	5585	5598	5610
.75	5623	5636	5649	5662	5675	5689	5702	5715	5728	5741
.76	5754	5768	5781	5794	5808	5821	5834	5848	5861	5875
.77	5888	5902	5916	5929	5943	5957	5970	5984	5998	6012
.78	6026	6039	6053	6067	6081	6095	6109	6124	6138	6152
.79	6166	6180	6194	6209	6223	6237	6252	6266	6281	6295
.80	6310	6324	6339	6353	6368	6383	6397	6412	6427	6442
.81	6457	6471	6486	6501	6516	6531	6546	6561	6577	6592
.82	6607	6622	6637	6653	6668	6683	6699	6714	6730	6745
.83	6761	6776	6792	6808	6823	6839	6855	6871	6887	6902
.84	6918	6934	6950	6966	6982	6998	7015	7031	7047	7063
.85	7079	7096	7112	7129	7145	7161	7178	7194	7211	7228
.86	7244	7261	7278	7295	7311	7328	7345	7362	7379	7396
.87	7413	7430	7447	7464	7482	7499	7516	7534	7551	7568
.88	7586	7603	7621	7638	7656	7674	7691	7709	7727	7745
.89	7762	7780	7798	7816	7834	7852	7870	7889	7907	7925
.90	7943	7962	7980	7998	8017	8035	8054	8072	8091	8110
.91	8128	8147	8166	8185	8204	8222	8241	8260	8279	8299
.92	8318	8337	8356	8375	8395	8414	8433	8453	8472	8492
.93	8511	8531	8551	8570	8590	8610	8630	8650	8670	8690
.94	8710	8730	8750	8770	8790	8810	8831	8851	8872	8892
.95	8913	8933	8954	8974	8995	9016	9036	9057	9078	9099
.96	9120	9141	9162	9183	9204	9226	9247	9268	9290	9311
.97	9333	9354	9376	9397	9419	9441	9462	9484	9506	9528
.98	9550	9572	9594	9616	9638	9661	9683	9705	9727	9750
.99	9772	9795	9817	9840	9863	9886	9908	9931	9954	9977

Values of the Trigonometric Functions

DEGREES	RADIANS	SIN	COS	TAN	CSC	SEC	COT
0	.0000	.0000	1.0000	.0000		1.0000	
1	.0175	.0175	.9998	.0175	57.30	1.0002	57.29
2	.0349	.0349	.9994	.0349	28.65	1.0006	28.64
3	.0524	.0523	.9986	.0524	19.11	1.0014	19.08
4	.0698	.0698	.9976	.0699	14.34	1.0024	14.30
5	.0873	.0872	.9962	.0875	11.474	1.0038	11.430
6	.1047	.1045	.9945	.1051	9.5668	1.0055	9.5144
7	.1222	.1219	.9925	.1228	8.2055	1.0075	8.1443
8	.1396	.1392	.9903	.1405	7.1853	1.0098	7.1154
9	.1571	.1564	.9877	.1584	6.3925	1.0125	6.3138
10	.1745	.1736	.9848	.1763	5.7588	1.0154	5.6713
11	.1920	.1908	.9816	.1944	5.2408	1.0187	5.1446
12	.2094	.2079	.9781	.2126	4.8097	1.0223	4.7046
13	.2269	.2250	.9744	.2309	4.4454	1.0263	4.3315
14	.2443	.2419	.9703	.2493	4.1336	1.0306	4.0108
15	.2618	.2588	.9659	.2679	3.8637	1.0353	3.7321
16	.2793	.2756	.9613	.2867	3.6280	1.0403	3.4874
17	.2967	.2924	.9563	.3057	3.4203	1.0457	3.2709
18	.3142	.3090	.9511	.3249	3.2361	1.0515	3.0777
19	.3316	.3256	.9455	.3443	3.0716	1.0576	2.9042
20	.3491	.3420	.9397	.3640	2.9238	1.0642	2.7475
21	.3665	.3584	.9336	.3839	2.7904	1.0711	2.6051
22	.3840	.3746	.9272	.4040	2.6695	1.0785	2.4751
23	.4014	.3907	.9205	.4245	2.5593	1.0864	2.3559
24	.4189	.4067	.9135	.4452	2.4586	1.0946	2.2460
25	.4363	.4226	.9063	.4663	2.3662	1.1034	2.1445
26	.4538	.4384	.8988	.4877	2.2812	1.1126	2.0503
27	.4712	.4540	.8910	.5095	2.2027	1.1223	1.9626
28	.4887	.4695	.8829	.5317	2.1301	1.1326	1.8807
29	.5061	.4848	.8746	.5543	2.0627	1.1434	1.8040
30	.5236	.5000	.8660	.5774	2.0000	1.1547	1.7321
31	.5411	.5150	.8572	.6009	1.9416	1.1666	1.6643
32	.5585	.5299	.8480	.6249	1.8871	1.1792	1.6003
33	.5760	.5446	.8387	.6494	1.8361	1.1924	1.5399
34	.5934	.5592	.8290	.6745	1.7883	1.2062	1.4826
35	.6109	.5736	.8192	.7002	1.7435	1.2208	1.4281
36	.6283	.5878	.8090	.7265	1.7013	1.2361	1.3764
37	.6458	.6018	.7986	.7536	1.6616	1.2521	1.3270
38	.6632	.6157	.7880	.7813	1.6243	1.2690	1.2799
39	.6807	.6293	.7771	.8098	1.5890	1.2868	1.2349
40	.6981	.6428	.7660	.8391	1.5557	1.3054	1.1918
41	.7156	.6561	.7547	.8693	1.5243	1.3250	1.1504
42	.7330	.6691	.7431	.9004	1.4945	1.3456	1.1106
43	.7505	.6820	.7314	.9325	1.4663	1.3673	1.0724
44	.7679	.6947	.7193	.9657	1.4396	1.3902	1.0355

Values of the Trigonometric Functions

DEGREES	RADIANS	SIN	COS	TAN	CSC	SEC	COT
45	.7854	.7071	.7071	1.0000	1.4142	1.4142	1.0000
46	.8029	.7193	.6947	1.0355	1.3902	1.4396	.9657
47	.8203	.7314	.6820	1.0724	1.3673	1.4663	.9325
48	.8378	.7431	.6691	1.1106	1.3456	1.4945	.9004
49	.8552	.7547	.6561	1.1504	1.3250	1.5243	.8693
50	.8727	.7660	.6428	1.1918	1.3054	1.5557	.8391
51	.8901	.7771	.6293	1.2349	1.2868	1.5890	.8098
52	.9076	.7880	.6157	1.2799	1.2690	1.6243	.7813
53	.9250	.7986	.6018	1.3270	1.2521	1.6616	.7536
54	.9425	.8090	.5878	1.3764	1.2361	1.7013	.7265
55	.9599	.8192	.5736	1.4281	1.2208	1.7435	.7002
56	.9774	.8290	.5592	1.4826	1.2062	1.7883	.6745
57	.9948	.8387	.5446	1.5399	1.1924	1.8361	.6494
58	1.0123	.8480	.5299	1.6003	1.1792	1.8871	.6249
59	1.0297	.8572	.5150	1.6643	1.1666	1.9416	.6009
60	1.0472	.8660	.5000	1.7321	1.1547	2.0000	.5774
61	1.0647	.8746	.4848	1.8040	1.1434	2.0627	.5543
62	1.0821	.8829	.4695	1.8807	1.1326	2.1301	.5317
63	1.0996	.8910	.4540	1.9626	1.1223	2.2027	.5095
64	1.1170	.8988	.4384	2.0503	1.1126	2.2812	.4877
65	1.1345	.9063	.4226	2.1445	1.1034	2.3662	.4663
66	1.1519	.9135	.4067	2.2460	1.0946	2.4586	.4452
67	1.1694	.9205	.3907	2.3559	1.0864	2.5593	.4245
68	1.1868	.9272	.3746	2.4751	1.0785	2.6695	.4040
69	1.2043	.9336	.3584	2.6051	1.0712	2.7904	.3839
70	1.2217	.9397	.3420	2.7475	1.0642	2.9238	.3640
71	1.2392	.9455	.3256	2.9042	1.0576	3.0716	.3443
72	1.2566	.9511	.3090	3.0777	1.0515	3.2361	.3249
73	1.2741	.9563	.2924	3.2709	1.0457	3.4203	.3057
74	1.2915	.9613	.2756	3.4874	1.0403	3.6280	.2867
75	1.3090	.9659	.2588	3.7321	1.0353	3.8637	.2679
76	1.3265	.9703	.2419	4.0108	1.0306	4.1336	.2493
77	1.3439	.9744	.2250	4.3315	1.0263	4.4454	.2309
78	1.3614	.9781	.2079	4.7046	1.0223	4.8097	.2126
79	1.3788	.9816	.1908	5.1446	1.0187	5.2408	.1944
80	1.3963	.9848	.1736	5.6713	1.0154	5.7588	.1763
81	1.4137	.9877	.1564	6.3138	1.0125	6.3925	.1584
82	1.4312	.9903	.1392	7.1154	1.0098	7.1853	.1405
83	1.4486	.9925	.1219	8.1443	1.0075	8.2055	.1228
84	1.4661	.9945	.1045	9.5144	1.0055	9.5668	.1051
85	1.4835	.9962	.0872	11.43	1.0038	11.474	.0875
86	1.5010	.9976	.0698	14.30	1.0024	14.34	.0699
87	1.5184	.9986	.0523	19.08	1.0014	19.11	.0524
88	1.5359	.9994	.0349	28.64	1.0006	28.65	.0349
89	1.5533	.9998	.0175	57.29	1.0002	57.30	.0175
90	1.5708	1.0000	.0000		1.0000		.0000

Amount of 1

Yrs.	½%	1%	1½%	2%	2½%	3%	3½%	Yrs.
1	1.00500	1.01000	1.01500	1.02000	1.02500	1.03000	1.03500	1
2	1.01003	1.02010	1.03023	1.04040	1.05063	1.06090	1.07123	2
3	1.01508	1.03030	1.04568	1.06121	1.07689	1.09273	1.10872	3
4	1.02015	1.04060	1.06136	1.08243	1.10381	1.12551	1.14752	4
5	1.02525	1.05101	1.07728	1.10408	1.13141	1.15927	1.18769	5
6	1.03038	1.06152	1.09344	1.12616	1.15969	1.19405	1.22926	6
7	1.03553	1.07214	1.10984	1.14869	1.18869	1.22987	1.27228	7
8	1.04071	1.08286	1.12649	1.17166	1.21840	1.26677	1.31681	8
9	1.04591	1.09369	1.14339	1.19509	1.24886	1.30477	1.36290	9
10	1.05114	1.10462	1.16054	1.21899	1.28008	1.34392	1.41060	10
11	1.05640	1.11567	1.17995	1.24337	1.31209	1.38423	1.45997	11
12	1.06168	1.12683	1.19562	1.26824	1.34489	1.42576	1.51107	12
13	1.06699	1.13809	1.21355	1.29361	1.37851	1.46853	1.56396	13
14	1.07232	1.14947	1.23176	1.31948	1.41297	1.51259	1.61869	14
15	1.07768	1.16097	1.25023	1.34587	1.44830	1.55797	1.67535	15
16	1.08307	1.17258	1.26899	1.37279	1.48451	1.60471	1.73399	16
17	1.08849	1.18430	1.28802	1.40024	1.52162	1.65285	1.79468	17
18	1.09393	1.19615	1.30734	1.42825	1.55966	1.70243	1.85749	18
19	1.09940	1.20811	1.32695	1.45681	1.59865	1.75351	1.92250	19
20	1.10490	1.22019	1.34686	1.48595	1.63862	1.80611	1.98979	20
21	1.11042	1.23239	1.36706	1.51567	1.67958	1.86029	2.05943	21
22	1.11597	1.24472	1.38756	1.54598	1.72157	1.91610	2.13151	22
23	1.12155	1.25716	1.40838	1.57690	1.76461	1.97359	2.20611	23
24	1.12716	1.26973	1.42950	1.60844	1.80873	2.03279	2.28333	24
25	1.13280	1.28243	1.45095	1.64061	1.85394	2.09378	2.36324	25
26	1.13846	1.29526	1.47271	1.67342	1.90029	2.15659	2.44596	26
27	1.14415	1.30821	1.49480	1.70689	1.94780	2.22129	2.53157	27
28	1.14987	1.32129	1.51722	1.74102	1.99750	2.28793	2.62017	28
29	1.15562	1.33450	1.53998	1.77584	2.04641	2.35657	2.71188	29
30	1.16140	1.34785	1.56308	1.81136	2.09757	2.42726	2.80679	30
31	1.16721	1.36133	1.58653	1.84759	2.15001	2.50008	2.90503	31
32	1.17304	1.37494	1.61032	1.88454	2.20376	2.57508	3.00671	32
33	1.17891	1.38869	1.63448	1.92223	2.25885	2.65234	3.11194	33
34	1.18480	1.40258	1.65900	1.90668	2.31532	2.73191	3.22086	34
35	1.19073	1.41660	1.68388	1.99989	2.37321	2.81386	3.33359	35
36	1.19668	1.43077	1.70914	2.03989	2.43254	2.89828	3.45027	36
37	1.20266	1.44508	1.73478	2.08069	2.49335	2.98523	3.57103	37
38	1.20868	1.45953	1.76080	2.12230	2.55568	3.07478	3.69601	38
39	1.21472	1.47412	1.78721	2.16474	2.61957	3.16703	3.82537	39
40	1.22079	1.48886	1.81402	2.20804	2.68506	3.26404	3.95926	40

Amount of 1

Yrs.	4%	4½%	5%	5½%	6%	7%	8%	Yrs.
1	1.04000	1.04500	1.05000	1.05500	1.06000	1.07000	1.08000	1
2	1.08160	1.09203	1.10250	1.11303	1.12360	1.14490	1.16640	2
3	1.12486	1.14117	1.15763	1.17424	1.19102	1.22504	1.25971	3
4	1.16986	1.19252	1.21551	1.23882	1.26248	1.31080	1.36049	4
5	1.21665	1.24618	1.27628	1.30696	1.33823	1.40255	1.46933	5
6	1.26532	1.30226	1.34010	1.37884	1.41852	1.50073	1.58687	6
7	1.31593	1.36086	1.40710	1.45468	1.50363	1.60578	1.71382	7
8	1.36857	1.42210	1.47746	1.53469	1.59385	1.71819	1.85093	8
9	1.42331	1.48610	1.55133	1.61909	1.68948	1.83846	1.99900	9
10	1.48024	1.55297	1.62889	1.70814	1.79085	1.96715	2.15893	10
11	1.53945	1.62285	1.71034	1.80209	1.89830	2.10485	2.33164	11
12	1.60103	1.69588	1.79586	1.90121	2.01220	2.25219	2.51817	12
13	1.66507	1.77220	1.88565	2.00577	2.13293	2.40985	2.71962	13
14	1.73168	1.85194	1.97993	2.11609	2.26090	2.57853	2.93719	14
15	1.80094	1.93528	2.07893	2.23248	2.39656	2.75903	3.17217	15
16	1.87298	2.02237	2.18287	2.35526	2.54035	2.95216	3.42594	16
17	1.94790	2.11338	2.29202	2.48480	2.69277	3.15881	3.70002	17
18	2.02582	2.20848	2.40662	2.62147	2.85434	3.37993	3.99602	18
19	2.10685	2.30786	2.52695	2.76565	3.02560	3.61653	4.31570	19
20	2.19112	2.41171	2.65330	2.91776	3.20714	3.86968	4.66096	20
21	2.27877	2.52024	2.78596	3.07823	3.39956	4.14056	5.03383	21
22	2.36992	2.63365	2.92526	3.24754	3.60354	4.43040	5.43654	22
23	2.46472	2.75217	3.07152	3.42615	3.81975	4.74053	5.87146	23
24	2.56330	2.87601	3.22510	3.61459	4.04893	5.07237	6.34118	24
25	2.66584	3.00543	3.38635	3.81339	4.29187	5.42743	6.84848	25
26	2.77247	3.14068	3.55567	4.02313	4.54938	5.80735	7.39635	26
27	2.88337	3.28201	3.73346	4.24440	4.82235	6.21387	7.98806	27
28	2.99870	3.42970	3.92013	4.47784	5.11169	6.64884	8.62711	28
29	3.11865	3.58404	4.11614	4.72412	5.41839	7.11426	9.31727	29
30	3.24340	3.74532	4.32194	4.98395	5.74349	7.61226	10.06266	30
31	3.37313	3.91386	4.53804	5.25807	6.08810	8.14511	10.86767	31
32	3.50806	4.08998	4.76494	5.54726	6.45339	8.71527	11.73708	32
33	3.64838	4.27403	5.00319	5.85236	6.84059	9.32534	12.67605	33
34	3.79432	4.44636	5.25335	6.17424	7.25103	9.97811	13.69013	34
35	3.94609	4.66735	5.51602	6.51383	7.68609	10.67658	14.78534	35
36	4.13093	4.87738	5.79182	6.87209	8.14725	11.42394	15.96817	36
37	4.26809	5.09686	6.08141	7.25005	8.63609	12.22362	17.24563	37
38	4.43881	5.32622	6.38548	7.64880	9.15425	13.07927	18.62528	38
39	4.61637	5.56590	6.70475	8.06949	9.70351	13.99482	20.11530	39
40	4.80102	5.81636	7.03999	8.51331	10.28572	14.97446	21.72452	40

Present value of 1

Yrs.	½%	1%	1½%	2%	2½%	3%	3½%	Yrs.
1	.99502	.99010	.98522	.98039	.97561	.97087	.96618	1
2	.99007	.98030	.97066	.96117	.95181	.94260	.93351	2
3	.98515	.97059	.95632	.94232	.92860	.91514	.90194	3
4	.98025	.96098	.94218	.92385	.90595	.88849	.87144	4
5	.97537	.95147	.92826	.90573	.88385	.86261	.84197	5
6	.97052	.94205	.91454	.88797	.86230	.83748	.81350	6
7	.96569	.93272	.90103	.87056	.84127	.81309	.78599	7
8	.96089	.92348	.88771	.85349	.82075	.78941	.75941	8
9	.95610	.91434	.87459	.83676	.80073	.76642	.73373	9
10	.95135	.90529	.86167	.82035	.78120	.74409	.70892	10
11	.94661	.89632	.84893	.80426	.76214	.72242	.68495	11
12	.94191	.88745	.83639	.78849	.74356	.70138	.66178	12
13	.93722	.87866	.82403	.77303	.72542	.68095	.63940	13
14	.93256	.86996	.81185	.75788	.70773	.66112	.61778	14
15	.92792	.86135	.79985	.74301	.69047	.64186	.59689	15
16	.92330	.85282	.78803	.72845	.67362	.62317	.57671	16
17	.91871	.84438	.77639	.71416	.65720	.60502	.55720	17
18	.91414	.83602	.76491	.70016	.64117	.58739	.53836	18
19	.90959	.82774	.75361	.68643	.62553	.57029	.52016	19
20	.90506	.81954	.74247	.67297	.61027	.55368	.50257	20
21	.90056	.81143	.73150	.65978	.59539	.52755	.48557	21
22	.89608	.80340	.72069	.64684	.58086	.52189	.46915	22
23	.89162	.79544	.71004	.63416	.56670	.50669	.45329	23
24	.88719	.78757	.69954	.62172	.55288	.49193	.43796	24
25	.88277	.77977	.68921	.60953	.53939	.47761	.42315	25
26	.87838	.77205	.67902	.59758	.52623	.46369	.40884	26
27	.87401	.76440	.66899	.58586	.41340	.45019	.39501	27
28	.86966	.75684	.65910	.57437	.50088	.43708	.38165	28
29	.86533	.74934	.64936	.56311	.48866	.42435	.36875	29
30	.86103	.74192	.63976	.55207	.47674	.41199	.35628	30
31	.85675	.73458	.63031	.54125	.46511	.39999	.34423	31
32	.85248	.72730	.62099	.53063	.45377	.38834	.33259	32
33	.84824	.72010	.61182	.52023	.44270	.37703	.32134	33
34	.84402	.71297	.60277	.51003	.43191	.36604	.31048	34
35	.83982	.70591	.59387	.50003	.42137	.35538	.29998	35
36	.83564	.69892	.58509	.49022	.41109	.34503	.28983	36
37	.83149	.69200	.57644	.48061	.40107	.33498	.28003	37
38	.82735	.68515	.56792	.47119	.39128	.32523	.27056	38
39	.82323	.67837	.55953	.46195	.38174	.31575	.26141	39
40	.81914	.67165	.55126	.45289	.37243	.30656	.25257	40

Present value of 1

Yrs.	4%	4½%	5%	5½%	6%	7%	8%	Yrs.
1	.96154	.95694	.95238	.94787	.94340	.93458	.92593	1
2	.92456	.91573	.90703	.89845	.89000	.87344	.85734	2
3	.88900	.87630	.86384	.85161	.83962	.81630	.79383	3
4	.85480	.83856	.82270	.80722	.79209	.76290	.73503	4
5	.82193	.80245	.78353	.76513	.74726	.71299	.68058	5
6	.79031	.76790	.74622	.72525	.70496	.66634	.63017	6
7	.75992	.73483	.71068	.68744	.66506	.62275	.58349	7
8	.73069	.70319	.67684	.65160	.62741	.58201	.54027	8
9	.70259	.67290	.64461	.61763	.59190	.54393	.50025	9
10	.67556	.64393	.61391	.58543	.55839	.50835	.46319	10
11	.64958	.61620	.58468	.55491	.52679	.47509	.42888	11
12	.62460	.58966	.55684	.52598	.49697	.44401	.39711	12
13	.60057	.56427	.53032	.49856	.46884	.41496	.36770	13
14	.57748	.53997	.50507	.47257	.44230	.38782	.34046	14
15	.55526	.51672	.48102	.44793	.41727	.36245	.31524	15
16	.53391	.49447	.45811	.42458	.39365	.33873	.29189	16
17	.51337	.47318	.43630	.40245	.37136	.31657	.27027	17
18	.49363	.45280	.41552	.38147	.35034	.29586	.25025	18
19	.47464	.43330	.39573	.36158	.33051	.27651	.23171	19
20	.45639	.41464	.37689	.34273	.31180	.25842	.21455	20
21	.43883	.39679	.35894	.32486	.29416	.24151	.19866	21
22	.42196	.37970	.34185	.30793	.27751	.22571	.18394	22
23	.40573	.36335	.32557	.29187	.26180	.21095	.17032	23
24	.39012	.34770	.31007	.27666	.24698	.19715	.15770	24
25	.37512	.33273	.29530	.26223	.23300	.18425	.14602	25
26	.36069	.31840	.28124	.24856	.21981	.17220	.13520	26
27	.34682	.30469	.26785	.23560	.20737	.16093	.12519	27
28	.33348	.29157	.25509	.22332	.19563	.15040	.11591	28
29	.32065	.27902	.24295	.21168	.18456	.14056	.10733	29
30	.30832	.26700	.23138	.20064	.17411	.13137	.09938	30
31	.29646	.25550	.22036	.19018	.16425	.12277	.09202	31
32	.28506	.24450	.20987	.18027	.15496	.11474	.08420	32
33	.27409	.23397	.19987	.17087	.14619	.10723	.07889	33
34	.26355	.22390	.19035	.16196	.13791	.10022	.07305	34
35	.25342	.21425	.18129	.15352	.13011	.09366	.06763	35
36	.24367	.20503	.17266	.14552	.12274	.08754	.06262	36
37	.23430	.19620	.16444	.13793	.11579	.08181	.05799	37
38	.22529	.18775	.15661	.13074	.10924	.07646	.05369	38
39	.21662	.17967	.14915	.12392	.10306	.07146	.04971	39
40	.20829	.17193	.14205	.11746	.09722	.06678	.04603	40

Powers, Roots, Reciprocals

n	n^2	$\sqrt{n}$	n^3	$\sqrt[3]{n}$	$\frac{1}{n}$	n	n^2	$\sqrt{n}$	n^3	$\sqrt[3]{n}$	$\frac{1}{n}$
1	1	1.000	1	1.000	1.0000	51	2,601	7.141	132,651	3.708	.0196
2	4	1.414	8	1.260	.5000	52	2,704	7.211	140,608	3.733	.0192
3	9	1.732	27	1.442	.3333	53	2,809	7.280	148,877	3.756	.0189
4	16	2.000	64	1.587	.2500	54	2,916	7.348	157,464	3.780	.0185
5	25	2.236	125	1.710	.2000	55	3,025	7.416	166,375	3.803	.0182
6	36	2.449	216	1.817	.1667	56	3,136	7.483	175,616	3.826	.0179
7	49	2.646	343	1.913	.1429	57	3,249	7.550	185,193	3.849	.0175
8	64	2.828	512	2.000	.1250	58	3,364	7.616	195,112	3.871	.0172
9	81	3.000	729	2.080	.1111	59	3,481	7.681	205,379	3.893	.0169
10	100	3.162	1,000	2.154	.1000	60	3,600	7.746	216,000	3.915	.0167
11	121	3.317	1,331	2.224	.0909	61	3,721	7.810	226,981	3.936	.0164
12	144	3.464	1,728	2.289	.0833	62	3,844	7.874	238,328	3.958	.0161
13	169	3.606	2,197	2.351	.0769	63	3,969	7.937	250,047	3.979	.0159
14	196	3.742	2,744	2.410	.0714	64	4,096	8.000	262,144	4.000	.0156
15	225	3.873	3,375	2.466	.0667	65	4,225	8.062	274,625	4.021	.0154
16	256	4.000	4,096	2.520	.0625	66	4,356	8.124	287,496	4.041	.0152
17	289	4.123	4,913	2.571	.0588	67	4,489	8.185	300,763	4.062	.0149
18	324	4.243	5,832	2.621	.0556	68	4,624	8.246	314,432	4.082	.0147
19	361	4.359	6,859	2.668	.0526	69	4,761	8.307	328,509	4.102	.0145
20	400	4.472	8,000	2.714	.0500	70	4,900	8.367	343,000	4.121	.0143
21	441	4.583	9,261	2.759	.0476	71	5,041	8.426	357,911	4.141	.0141
22	484	4.690	10,648	2.802	.0455	72	5,184	8.485	373,248	4.160	.0139
23	529	4.796	12,167	2.844	.0435	73	5,329	8.544	389,017	4.179	.0137
24	576	4.899	13,824	2.884	.0417	74	5,476	8.602	405,224	4.198	.0135
25	625	5.000	15,625	2.924	.0400	75	5,625	8.660	421,875	4.217	.0133
26	676	5.099	17,576	2.962	.0385	76	5,776	8.718	438,976	4.236	.0132
27	729	5.196	19,683	3.000	.0370	77	5,929	8.775	456,533	4.254	.0130
28	784	5.292	21,952	3.037	.0357	78	6,084	8.832	474,552	4.273	.0128
29	841	5.385	24,389	3.072	.0345	79	6,241	8.888	493,039	4.291	.0127
30	900	5.477	27,000	3.107	.0333	80	6,400	8.944	512,000	4.309	.0125
31	961	5.568	29,791	3.141	.0323	81	6,561	9.000	531,441	4.327	.0123
32	1,024	5.657	32,768	3.175	.0312	82	6,724	9.055	551,368	4.344	.0122
33	1,089	5.745	35,937	3.208	.0303	83	6,889	9.110	571,787	4.362	.0120
34	1,156	5.831	39,304	3.240	.0294	84	7,056	9.165	592,704	4.380	.0119
35	1,225	5.916	42,875	3.271	.0286	85	7,225	9.220	614,125	4.397	.0118
36	1,296	6.000	46,656	3.302	.0278	86	7,396	9.274	636,056	4.414	.0116
37	1,369	6.083	50,653	3.332	.0270	87	7,569	9.327	658,503	4.431	.0115
38	1,444	6.164	54,872	3.362	.0263	88	7,744	9.381	681,472	4.448	.0114
39	1,521	6.245	59,319	3.391	.0256	89	7,921	9.434	704,969	4.465	.0112
40	1,600	6.325	64,000	3.420	.0250	90	8,100	9.487	729,000	4.481	.0111
41	1,681	6.403	68,921	3.448	.0244	91	8,281	9.539	753,571	4.498	.0110
42	1,764	6.481	74,088	3.476	.0238	92	8,464	9.592	778,688	4.514	.0109
43	1,849	6.557	79,507	3.503	.0233	93	8,649	9.644	804,357	4.531	.0108
44	1,936	6.633	85,184	3.530	.0227	94	8,836	9.695	830,584	4.547	.0106
45	2,025	6.708	91,125	3.557	.0222	95	9,025	9.747	857,375	4.563	.0105
46	2,116	6.782	97,336	3.583	.0217	96	9,216	9.798	884,736	4.579	.0104
47	2,209	6.856	103,823	3.609	.0213	97	9,409	9.849	912,673	4.595	.0103
48	2,304	6.928	110,592	3.634	.0208	98	9,604	9.899	941,192	4.610	.0102
49	2,401	7.000	117,649	3.659	.0204	99	9,801	9.950	970,299	4.626	.0101
50	2,500	7.071	125,000	3.684	.0200	100	10,000	10.000	1,000,000	4.642	.0100

SOLUTIONS

Chapter 1

Section 1.4 (page 6).

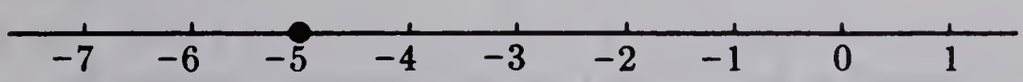
1. $\{x \mid \frac{1}{3}x - \frac{4}{5}x = \frac{7}{3}, x \in R\}$

The defining equation

$$\begin{aligned} & \tfrac{1}{3}x - \tfrac{4}{5}x = \tfrac{7}{3} \\ \leftrightarrow\ & 5x - 12x = 35 \\ \leftrightarrow\ & -7x = 35 \\ \leftrightarrow\ & x = -5 . \end{aligned}$$

$\therefore$ the solution set is $\{-5\}$.

The graph of the solution set is:



2. $3ax + 4bx = c,\ x \in R,\ a, b, c \in R$

$\leftrightarrow (3a + 4b)x = c$

$\leftrightarrow x = \dfrac{c}{3a + 4b},\ 3a + 4b \neq 0 .$

If $c \neq 0$ and $3a + 4b = 0$, the equation has no solution.

If $c = 0$ and $3a + 4b = 0$, the solution set is R.

Thus the only case in which a unique solution occurs is when $3a + 4b \neq 0$, in which case the solution set is $\left\{\dfrac{c}{3a + 4b}\right\}$.

Verification.

$$\text{L.S.} = 3a\left(\frac{c}{3a + 4b}\right) + 4b\left(\frac{c}{3a + 4b}\right) \qquad \text{R.S.} = c .$$

$$= \frac{3ac + 4bc}{3a + 4b}$$

$$= \frac{(3a + 4b)c}{3a + 4b}$$

$$= c .$$

Section 1.6 (page 10).

1. $2x^3 - 11x^2 + 5x + 4$

The integral factors of 4 are $\pm 1, \pm 2, \pm 4$.

If $x = 1$, the expression $= 2 - 11 + 5 + 4 = 0$.

$\therefore x - 1$ is a factor of the expression.

$$\begin{array}{r} 2x^2 - 9x - 4 \\ x - 1 \overline{)\ 2x^3 - 11x^2 + 5x + 4} \\ \underline{2x^3 - 2x^2} \\ -9x^2 + 5x \\ \underline{-9x^2 + 9x} \\ -4x + 4 \\ \underline{-4x + 4} \\ 0 \end{array}$$

$\therefore 2x^3 - 11x^2 + 5x + 4 = (x - 1)(2x^2 - 9x - 4).$

2. (i) $x^3 - y^3$

The factors of y^3 are $\pm y$, $\pm y^2$, $\pm y^3$

If $x = y$, the expression $= y^3 - y^3$
$= 0$.

$\therefore x - y$ is a factor of the expression.

$$
\begin{array}{r|l}
 & x^2 + xy + y^2 \\
\hline
x - y & x^3 \qquad\qquad - y^3 \\
 & \underline{x^3 - x^2y} \\
 & \quad x^2y \\
 & \quad \underline{x^2y - xy^2} \\
 & \qquad\quad xy^2 - y^3 \\
 & \qquad\quad \underline{xy^2 - y^3} \\
 & \qquad\qquad 0
\end{array}
$$

$\therefore x^3 - y^3 = (x - y)(x^2 + xy + y^2)$.

(ii) $x^3 + y^3$

If $x = -y$, the expression $= (-y)^3 + y^3$
$= 0$.

$\therefore x + y$ is a factor of the expression.

$$
\begin{array}{r|l}
 & x^2 - xy + y^2 \\
\hline
x + y & x^3 \qquad\qquad + y^3 \\
 & \underline{x^3 + x^2y} \\
 & \quad -x^2y \\
 & \quad \underline{-x^2y - xy^2} \\
 & \qquad\quad xy^2 + y^3 \\
 & \qquad\quad \underline{xy^2 + y^3} \\
 & \qquad\qquad 0
\end{array}
$$

$\therefore x^3 + y^3 = (x + y)(x^2 - xy + y^2)$.

3. If $y = -3$, then $y^3 + 2y^2 - 7y - 12 = -27 + 18 + 21 - 12 = 0$.
$\therefore y + 3$ is a factor of the expression.

Chapter 2

Section 2.2 (page 19).

1. $2^2 \cdot 2^4 = 2^{2+4}$
$= 2^6$
$= 64$

2. $(-3)^4(-3)^0 = (-3)^{4+0}$
$= (-3)^4$
$= 81$

3. $5^3 \times 5^{-2} = 5^{3+(-2)}$
$= 5^1$
$= 5$

4. $(-16)^3(-16)^{-3} = (-16)^{3+(-3)}$
$= (-16)^0$
$= 1$

5. $x^3 \cdot x^{-4} = x^{3+(-4)}$
$= x^{-1}$ or $\frac{1}{x}$

6. $x^{-2} \cdot x^{-3} = x^{-2+(-3)}$
$= x^{-5}$ or $\frac{1}{x^5}$

Section 2.3 (page 20).

1. $3^5 \div 3^2 = 3^{5-2}$
$= 3^3$
$= 27$

2. $(-2)^3 \div (-2)^0 = (-2)^{3-0}$
$= (-2)^3$
$= -8$

3. $(-5)^2 \div (-5)^{-3} = (-5)^{2-(-3)}$
$= (-5)^5$
$= -3125$

4. $x^0 \div x^{-2} = x^{0-(-2)}$
$= x^2$

5. $y^{-2} \div y^{-4} = y^{-2-(-4)}$
$= y^2$

6. $z^0 \div z^0 = z^{0-0}$
$= z^0$
$= 1$

Section 2.4 (page 21).

The index laws apply only to factors involving the *same* base.

1. $(a^2b^3c^4)(a^0b^{-2}c^2)$
$= a^{2+0}b^{3-2}c^{4+2}$
$= a^2bc^6$

2. $3m^{-2}p^0q^2 \times 4m^4p^{-3}q^{-4}$
$= 12m^{-2+4}p^{0+(-3)}q^{2+(-4)}$
$= 12m^2p^{-3}q^{-2}$ or $\frac{12m^2}{p^3q^2}$

3. $(x^4y^2z^0) \div (x^2y^4z^3)$
$= x^{4-2}y^{2-4}z^{0-3}$
$= x^2y^{-2}z^{-3}$ or $\frac{x^2}{y^2z^3}$

4. $(12r^4s^{-5}) \div (3r^{-2}s^5)$
$= 4r^{4-(-2)}s^{-5-5}$
$= 4r^6s^{-10}$ or $\frac{4r^6}{s^{10}}$

Section 2.8 (page 23).

1. $3(a^2)^{10}$
$= 3a^{2\times10}$
$= 3a^{20}$

2. $(3m^2n^3p^4)^3$
$= 3^3m^{2\times3}n^{3\times3}p^{4\times3}$
$= 27m^6n^9p^{12}$

3. $(4x^{-2}z^0y^2)^{-3}$
$= 4^{-3}x^6z^0y^{-6}$
$= \frac{1}{64}x^6y^{-6}$ or $\frac{x^6}{64y^6}$

4. $\left(\frac{r^2}{t^3}\right)$
$= \frac{r^{2\times3}}{t^{3\times3}}$
$= \frac{r^6}{t^9}$

5. $\left(\frac{6s^3t^{-4}}{3mn^2}\right)^{-1}$
$= \frac{3mn^2}{6s^3t^{-4}}$
(the reciprocal of the given expression)

6. $\left(\frac{a^0}{b^0}\right)^3$
$= \frac{a^0}{b^0}$
1

Section 2.11 (page 28).

1. $3\sqrt{2} + 4\sqrt{8} - 5\sqrt{18}$
$= 3\sqrt{2} + 4\sqrt{4\times2} - 5\sqrt{9\times2}$
$= 3\sqrt{2} + 8\sqrt{2} - 15\sqrt{2}$
$= -4\sqrt{2}$ (*D*)

2. $3\sqrt{8} + 5\sqrt{32} - 2\sqrt{72}$
$= 3\sqrt{4\times2} + 5\sqrt{16\times2} - 2\sqrt{36\times2}$
$= 6\sqrt{2} + 20\sqrt{2} - 12\sqrt{2}$
$= 14\sqrt{2}$ (*D*)

3. $\sqrt{75} - \sqrt{27} + \sqrt{200} = \sqrt{25\times3} - \sqrt{9\times3} + \sqrt{100\times2}$
$= 5\sqrt{3} - 3\sqrt{3} + 10\sqrt{2}$
$= 2\sqrt{3} + 10\sqrt{2}$ (*D*)

Section 2.12 (page 30).

1. (i) $\frac{\sqrt{3}}{\sqrt{7}} = \sqrt{\frac{3}{7}}$

(ii) $\frac{\sqrt{5}}{\sqrt{10}} = \sqrt{\frac{5}{10}}$
$= \sqrt{\frac{1}{2}}$

(iii) $\frac{2\sqrt{7}}{\sqrt{3}} = \frac{\sqrt{28}}{\sqrt{3}}$
$= \sqrt{\frac{28}{3}}$

(iv) $\frac{3\sqrt{5}}{2\sqrt{11}} = \frac{\sqrt{45}}{\sqrt{44}}$
$= \sqrt{\frac{45}{44}}$

2. (i) $\frac{\sqrt{12}}{\sqrt{3}} = \sqrt{\frac{12}{3}}$
$= \sqrt{4} = 2$

or $\because \sqrt{3}\times\sqrt{4} = \sqrt{12}$
$\therefore \sqrt{12} \div \sqrt{3} = \sqrt{4} = 2$

(ii) $4\sqrt{15} \div 2\sqrt{5} = \frac{4\sqrt{15}}{2\sqrt{5}}$
$= 2\sqrt{\frac{15}{5}} = 2\sqrt{3}$

or $\because 2\sqrt{5}\times2\sqrt{3} = 4\sqrt{15}$
$\therefore 4\sqrt{15} \div 2\sqrt{5} = 2\sqrt{3}$

Section 2.14 (page 32).

1. (i) $\sqrt{2}(\sqrt{3} + \sqrt{5})$
$= \sqrt{6} + \sqrt{10}$ (*D*)

(ii) $3\sqrt{2}(\sqrt{7} + 5\sqrt{11})$
$= 3\sqrt{14} + 15\sqrt{22}$ (*D*)

2. $(3\sqrt{2} + 5\sqrt{3})^2 = (3\sqrt{2} + 5\sqrt{3})(3\sqrt{2} + 5\sqrt{3})$
$= 3\sqrt{2}(3\sqrt{2} + 5\sqrt{3}) + 5\sqrt{3}(3\sqrt{2} + 5\sqrt{3})$ (*D*)
$= 18 + 15\sqrt{6} + 15\sqrt{6} + 75$ (*D*)
$= 93 + 30\sqrt{6}$

3. (i) $(3\sqrt{2} + 5\sqrt{3})^2 = (3\sqrt{2})^2 + 2(3\sqrt{2})(5\sqrt{3}) + (5\sqrt{3})^2$
$= 18 + 30\sqrt{6} + 95$
$= 93 + 30\sqrt{6}$

(ii) $(\sqrt{5} - 2\sqrt{7})^2 = (\sqrt{5})^2 - 2(\sqrt{5})(2\sqrt{7}) + (2\sqrt{7})^2$
$= 5 - 4\sqrt{35} + 28$
$= 33 - 4\sqrt{35}$

4.
$$\begin{array}{r} \sqrt{2} + \sqrt{3} \\ 2\sqrt{2} - \sqrt{3} \\ \hline 4 + 2\sqrt{6} \\ -\sqrt{6} - 3 \\ \hline 4 + \sqrt{6} - 3 \end{array} \Big\} D$$
$$= 1 + \sqrt{6}$$

5. $(\sqrt{2} - \sqrt{3})(\sqrt{2} + \sqrt{3}) = (\sqrt{2} - \sqrt{3})\sqrt{2} + (\sqrt{2} - \sqrt{3})\sqrt{3}$ (D)
$= 2 - \sqrt{6} + \sqrt{6} - 3$ (D)
$= -1$

6. (i) $(\sqrt{2} - \sqrt{3})(\sqrt{2} + \sqrt{3})$
$= (\sqrt{2})^2 - (\sqrt{3})^2$
$= 2 - 3 = -1$

(ii) $(3\sqrt{2} + 4\sqrt{5})(3\sqrt{2} - 4\sqrt{5})$
$= (3\sqrt{2})^2 - (4\sqrt{5})^2$
$= 18 - 80 = -62$

(iii) $(\sqrt{x} + \sqrt{y})(\sqrt{x} - \sqrt{y})$
$= x - y$

(iv) $(a\sqrt{x} + b\sqrt{y})(a\sqrt{x} - b\sqrt{y})$
$= a^2x - b^2y$

Section 2.15 (page 33).

1. $\dfrac{\sqrt{2}}{3\sqrt{7} - \sqrt{5}}$
$= \dfrac{\sqrt{2}(3\sqrt{7} + \sqrt{5})}{(3\sqrt{7} - \sqrt{5})(3\sqrt{7} + \sqrt{5})}$
$= \dfrac{3\sqrt{14} + \sqrt{10}}{9 \cdot 7 - 5}$
$= \dfrac{3\sqrt{14} + \sqrt{10}}{63 - 5}$
$= \dfrac{3\sqrt{14} + \sqrt{10}}{58}$

2. $\dfrac{3\sqrt{3}}{3\sqrt{5} - 4\sqrt{2}}$
$= \dfrac{3\sqrt{3}(3\sqrt{5} + 4\sqrt{2})}{(3\sqrt{5} - 4\sqrt{2})(3\sqrt{5} + 4\sqrt{2})}$
$= \dfrac{9\sqrt{15} + 12\sqrt{6}}{9 \cdot 5 - 16 \cdot 2}$
$= \dfrac{9\sqrt{15} + 12\sqrt{6}}{45 - 32}$
$= \dfrac{9\sqrt{15} + 12\sqrt{6}}{13}$

3. $\dfrac{3 + \sqrt{11}}{2\sqrt{7} + 3\sqrt{6}}$
$= \dfrac{(3 + \sqrt{11})(2\sqrt{7} - 3\sqrt{6})}{(2\sqrt{7} + 3\sqrt{6})(2\sqrt{7} - 3\sqrt{6})}$
$= \dfrac{6\sqrt{7} - 9\sqrt{6} + 2\sqrt{77} - 3\sqrt{66}}{4 \cdot 7 - 9 \cdot 6}$
$= \dfrac{6\sqrt{7} - 9\sqrt{6} + 2\sqrt{77} - 3\sqrt{66}}{-26}$

4. $\dfrac{5\sqrt{2} - 3\sqrt{3}}{2\sqrt{5} + 3\sqrt{2}}$
$= \dfrac{(5\sqrt{2} - 3\sqrt{3})(2\sqrt{5} - 3\sqrt{2})}{(2\sqrt{5} + 3\sqrt{2})(2\sqrt{5} - 3\sqrt{2})}$
$= \dfrac{10\sqrt{10} - 15 \cdot 2 - 6\sqrt{15} + 9\sqrt{6}}{4 \cdot 5 - 9 \cdot 2}$
$= \dfrac{10\sqrt{10} - 6\sqrt{15} + 9\sqrt{6} - 30}{2}$

Section 2.17 (page 37).

1. $\sqrt{b+5} + \sqrt{b} - \sqrt{4b+9} = 0$

$$\therefore \sqrt{b+5} + \sqrt{b} = \sqrt{4b+9}$$
$$\therefore b + 5 + 2\sqrt{b^2+5b} + b = 4b + 9$$
$$\therefore 2\sqrt{b^2+5b} = 2b + 4$$
$$\therefore \sqrt{b^2+5b} = b + 2$$
$$\therefore b^2 + 5b = b^2 + 4b + 4$$
$$\therefore b = 4$$

Verification. L.S. $= \sqrt{4+5} + \sqrt{4} - \sqrt{16+9}$ R.S. $= 0$
$= 3 + 2 - 5 = 0$

$\therefore$ the root is 4.

2. $\sqrt{x^2+7x+17} - \sqrt{x^2+3x+5} = 2$

$$\therefore \sqrt{x^2+7x+17} = 2 + \sqrt{x^2+3x+5}$$
$$\therefore x^2 + 7x + 17 = 4 + 4\sqrt{x^2+3x+5} + x^2 + 3x + 5$$
$$\therefore 4x + 8 = 4\sqrt{x^2+3x+5}$$
$$\therefore x + 2 = \sqrt{x^2+3x+5}$$
$$\therefore x^2 + 4x + 4 = x^2 + 3x + 5$$
$$\therefore x = 1$$

Verification. L.S. $= \sqrt{1+7+17} - \sqrt{1+3+5}$ R.S. $= 2$
$= 5 - 3$
$= 2$

$\therefore$ the root is 1.

Chapter 3

Section 3.2 (page 41).

1. $A \times A = \{(1, 1), (3, 1), (5, 1), (1, 3), (3, 3), (5, 3), (1, 5), (3, 5), (5, 5)\}$

$\therefore B = \{(3, 1), (5, 1), (5, 3)\}$

2. $N = \{1, 2, 3, 4, \ldots\}$
$N \times N$ cannot be listed, nor can the required subset. The required subset can be indicated as follows:
$\{(1, 3), (2, 4), (3, 5), (4, 6), (5, 7), \ldots\}$

3. $A \times A = \{(8, 8), (27, 8), (64, 8), (8, 27), (27, 27), (64, 27), (8, 64), (27, 64), (64, 64)\}$

$C = \phi$

Section 3.4 (page 49).

1. $A = \{0, 1, 2, 3, 4, 5\}$

if $x = 0$, $y = 0^2 + 1 = 1$
if $x = 1$, $y = 1^2 + 1 = 2$
if $x = 2$, $y = 2^2 + 1 = 5$
if $x = 3$, $y = 3^2 + 1 = 10$
if $x = 4$, $y = 4^2 + 1 = 17$
if $x = 5$, $y = 5^2 + 1 = 26$

Since both x, y must be elements of A,
$\therefore B = \{(0, 1), (1, 2), (2, 5)\}$.
Domain of B is $\{0, 1, 2\}$.
Range of B is $\{1, 2, 5\}$.

2. x, y are elements of $\{0, 1, 2, 3, \ldots\}$.
if $x = 0$, $y = 2 \times 0 - 1 = -1$
if $x = 1$, $y = 2 \times 1 - 1 = 1$
if $x = 2$, $y = 2 \times 2 - 1 = 3$
if $x = 3$, $y = 2 \times 3 - 1 = 5$
etc.
Since -1 is not in the set N_0,
$\therefore C = \{(1, 1), (2, 3), (3, 5), \ldots\}$.
The pattern indicates that:
the domain of C is $\{1, 2, 3, 4, \ldots\}$ or the domain of C is N;
the range of C is $\{1, 3, 5, 7, \ldots\}$ or the set of odd natural numbers.

3. Since $x, y \in R$ and $y = x$, for all $x \in R$,
then $D = \{(x, x) \mid x \in R\}$.
Since both x and y may assume any real value,
therefore the domain of D is R or $\{x \mid x \in R\}$;
the range of D is R or $\{y \mid y \in R\}$.

4. E is the set of ordered pairs represented by (x, y) which satisfy the defining sentence $x^2 + y^2 = 25$, $x, y \in I$.
To discover whether an ordered pair does or does not belong to the relation, it must be shown that the ordered pair does or does not satisfy the equation.

For the ordered pair (3, 4): $x^2 + y^2 = 9 + 16 = 25$,	$\therefore (3, 4) \in E$.
For the ordered pair (0, 0): $x^2 + y^2 = 0 + 0 = 0$,	$\therefore (0, 0) \notin E$.
For the ordered pair (0, -5): $x^2 + y^2 = 0 + 25 = 25$,	$\therefore (0, -5) \in E$.
For the ordered pair (4, -3): $x^2 + y^2 = 16 + 9 = 25$,	$\therefore (4, -3) \in E$.
For the ordered pair (5, 0): $x^2 + y^2 = 25 + 0 = 25$,	$\therefore (5, 0) \in E$.
For the ordered pair (0, 25): $x^2 + y^2 = 0 + 625 = 625$,	$\therefore (0, 25) \notin E$.
For the ordered pair (4, 3): $x^2 + y^2 = 16 + 9 = 25$,	$\therefore (4, 3) \in E$.
For the ordered pair (6, -1): $x^2 + y^2 = 36 + 1 = 37$,	$\therefore (6, -1) \notin E$.

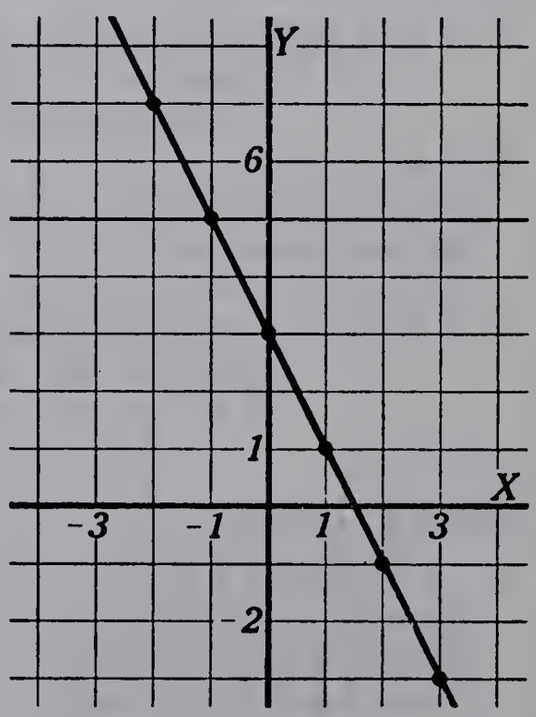
The ordered pairs (0, 0), (0, 25), (6, -1) do not belong to E.

Section 3.5 (page 52).

1. $M = \{(x, y) \mid 2x + y = 3, x, y \in R\}$
For ease in constructing a table of values the defining equation may be written in the form $y = 3 - 2x$.
$\therefore M = \{(x, y) \mid y = 3 - 2x, x, y \in R\}$
The relation may also be written
$M = \{(x, 3 - 2x) \mid x \in R\}$,
because y (or $3 - 2x$) $\in R$
for all $x \in R$.

For M:

x	-2	-1	0	1	2	3
y	7	5	3	1	-1	-3



The partial graph is drawn.
From the graph:
the domain is $\{x \mid x \in R\}$;
the range is $\{y \mid y \in R\}$.

NOTE: The equation $y = 3 - 2x$ expresses *explicitly* what y is equivalent to in terms of x. Similarly, the equation $x = \dfrac{3 - y}{2}$ expresses explicitly what x is equivalent to

in terms of y. Equations written in this manner are said to be written in *explicit form*. The equation $2x + y = 3$ or $2x + y - 3 = 0$ *implies* a relationship between x and y and some given numbers. Equations written in this manner are said to be written in *implicit form*.

The following are different forms of the same equation:

(i) $2x + y = 3$ (ii) $2x + y - 3 = 0$

(iii) $y = 3 - 2x$ (iv) $x = \dfrac{3 - y}{2}$

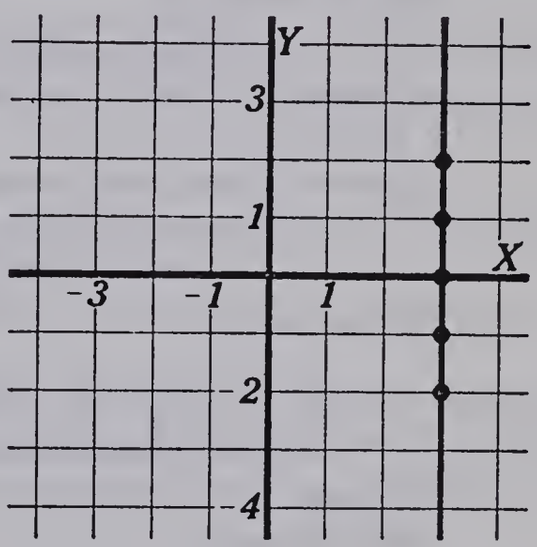
The explicit form of the equation is particularly useful if a table of values is to be constructed.

2. $N = \{(x, y) \mid x = 3, x, y \in R\}$

Since y does not appear in the equation $x = 3$, then since $y \in R$, y may have any real value, but x must always be 3.

For N:

x	3	3	3	3	3
y	−2	−1	0	1	2



The partial graph is drawn.
From the graph or otherwise:
the domain of N is $\{3\}$;
the range of N is $\{y \mid y \in R\}$.

NOTE: A table of values is really not necessary for the drawing of this graph. If x must always be 3 and y may be any real number, then all the points on the graph must be 3 units to the right of the y-axis, and the graph must be the line parallel to the y-axis and three units to the right of it.

In a similar manner, it may be argued that the graphs of the relations with the following defining sentences are as indicated:

(i) $x = -3, x, y \in R$: line parallel to y-axis, 3 units to the left of it;
(ii) $x = 0, x, y \in R$: the y-axis;
(iii) $y = 4, x, y \in R$: line parallel to x-axis, 4 units above it;
(iv) $y = -7, x, y \in R$: line parallel to x-axis, 7 units below it;
(v) $y = 0, x, y \in R$: the x-axis.

Section 3.7 (page 55).

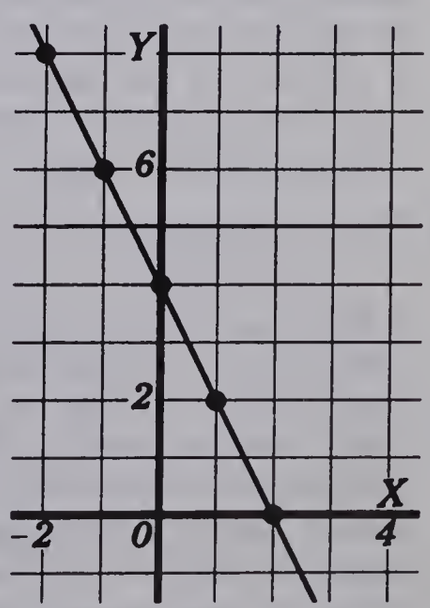
Solution.

(i) (a) ***x-intercepts.*** Let $y = 0$, then $2x = 4$
or $x = 2$.
$\therefore$ the x-intercept is 2.

y-intercepts. Let $x = 0$, then $y = 4$
$\therefore$ the y-intercept is 4.

(b) ***Table of values.***

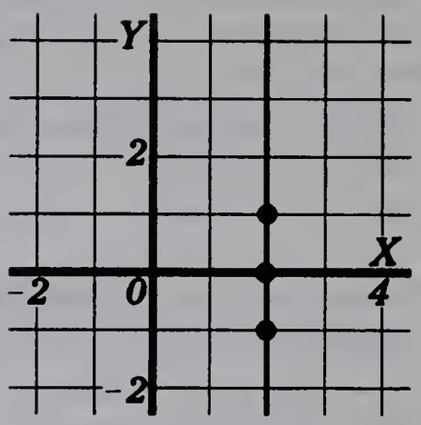
x	0	1	2	−1	−2
y (or $4 - 2x$)	4	2	0	6	8



(Since L_1 is defined by a linear equation, its graph is a straight line, and it would have sufficed to determine only two points; however a third point acts as a check on computational errors.)

(ii) (a) *x-intercepts.* Since $x = 2$ for all $(x, y) \in L_2$,
if $y = 0$, then $x = 2$.
$\therefore$ the x-intercept is 2.

y-intercepts. There is no y-intercept, since $x \neq 0$ for any $(x, y) \in L_2$.

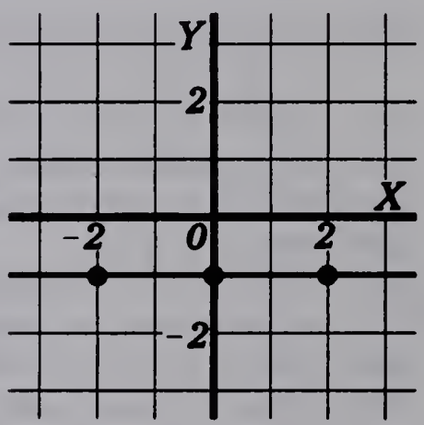


(b) *Table of values.*

x	2	2	2
y	0	1	−1

The graph is a vertical line two units to the right of the y-axis.

(iii) (a) For L_3, the defining equation is $y = -1$.
x-intercepts. There is no x-intercept, since $y \neq 0$ for any $(x, y) \in L_3$.



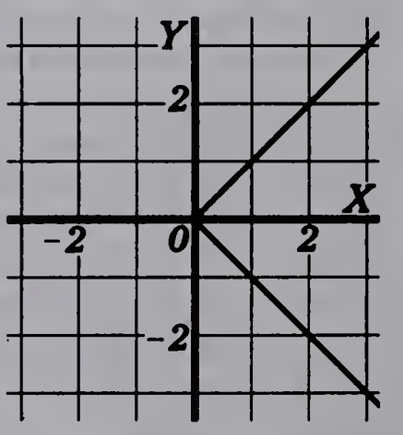
y-intercepts. Since $y = -1$ for all $(x, y) \in L_3$,
if $x = 0$, then $y = -1$.
$\therefore$ the y-intercept is -1.

(b) *Table of values.*

x	0	2	−2
y	−1	−1	−1

The graph is a horizontal line lying one unit below the x-axis.

Section **3.10** (page 64).



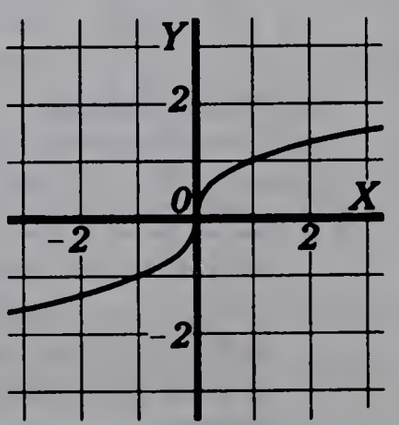
1. The defining equation is

$$|y| = x.$$

Substituting $-x$ for x, the equation becomes $|y| = -x$. Since $-x \neq x$ except when $x = 0$, the graph is not symmetric with respect to the y-axis.

Substituting $-y$ for y the equation becomes $|-y| = x$. Since $|-y| = |y|$, the equation is unchanged and the graph is symmetric with respect to the x-axis.

If $-x$ and $-y$ are substituted for x and y respectively, the equation becomes $|-y| = -x \leftrightarrow |y| = -x$. Since the equation is changed the graph is not symmetric with respect to the origin.



2. The defining equation is

$$y^3 = x.$$

Substituting $-x$ for x, the equation becomes $y^3 = -x$. The graph is not symmetric with respect to the y-axis. Substituting $-y$ for y, the equation becomes $(-y)^3 = x \leftrightarrow y^3 = -x$. The graph is not symmetric with respect to the x-axis.

Substituting $-x$ for x and $-y$ for y the equation becomes $(-y)^3 = -x \leftrightarrow y^3 = x$. The graph is symmetric with respect to the origin.

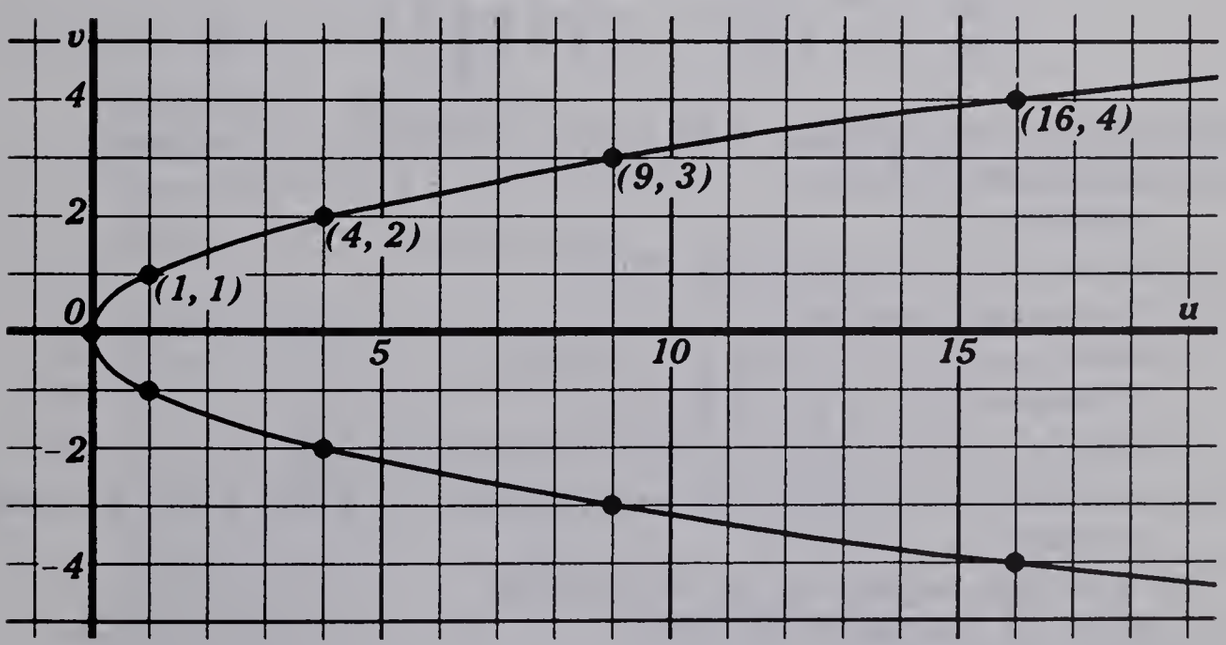
3. (i) *u-intercepts.* Let $v = 0$, then $u = 0$.
$\therefore$ the u-intercept is 0.
v-intercepts. Let $u = 0$, then $v^2 = 0$.
$\therefore$ the v-intercept is 0.

(ii) *Domain.* $v^2 = u$
$\leftrightarrow v = \pm\sqrt{u}$.
$\therefore v \in R \leftrightarrow u \geqq 0, u \in R$.
$\therefore$ the domain of A is $\{u \mid u \geqq 0, u \in R\}$.
Range. Since $u = v^2$
$\therefore u \in R \leftrightarrow v \in R$.
$\therefore$ the range of A is $\{v \mid v \in R\}$.

(iii) *Symmetry.* Substituting $-v$ for v, the equation becomes $(-v)^2 = u \leftrightarrow v^2 = u$.
$\therefore$ the graph is symmetric with respect to the u-axis.
Substituting $-u$ for u, the equation becomes $v^2 = -u$ which is not equivalent to $v^2 = u$.
$\therefore$ the graph is not symmetric with respect to the v-axis.
Substituting $-u$ for u and $-v$ for v, the equation becomes $(-v)^2 = -u$ which is not equivalent to $v^2 = u$.
$\therefore$ the graph is not symmetric with respect to the origin.

(iv) *Table of values.*
Since the graph is symmetric with respect to the u-axis, it suffices to compute the coordinates (u, v) of points of the graph having $v \geqq 0$. The part of the graph whose points have negative ordinates can then be obtained by a reflection in the u-axis.

u (or v^2)	0	1	4	9	16
v	0	1	2	3	4



Chapter 4

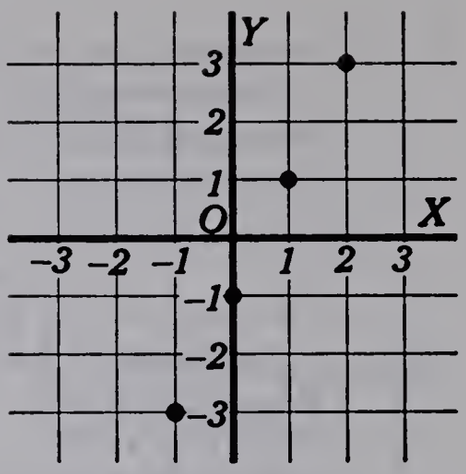
Section 4.1 (page 66).

1. (i) $R_1 = \{(-1, -3), (0, -1), (1, 1), (2, 3)\}$.
(ii) The domain is $\{-1, 0, 1, 2\}$;
and the range is $\{-3, -1, 1, 3\}$.

(iii) One element of the range corresponds to each element of the domain.

(iv) See accompanying graph.

(v) A line drawn parallel to the y-axis can meet the graph in at most one point.



2. (i) $R_2 = \{(4, 4), (1, 2), (0, 0), (1, -2), (4, -4)\}$.

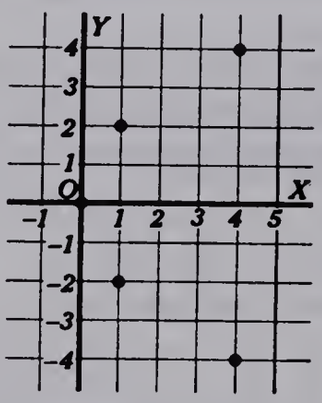
(ii) The domain is $\{0, 1, 4\}$;
and the range is $\{-4, -2, 0, 2, 4\}$.

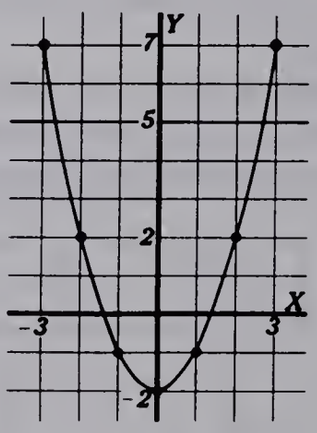
(iii) The elements 2 and -2 of the range correspond to the number 1 of the domain.

(iv) There are two elements of the range for every element of the domain, except

(v) See accompanying graph (left below).

(vi) Some lines parallel to the y-axis meets the graph in two points.





3. (i) $(-3, 7), (-2, 2), (-1, -1), (0, -2), (1, -1), (2, 2)$

(ii) x-intercepts $\pm\sqrt{2}$;
y-intercept -2

(iii) See accompanying graph (right above).

(iv) See accompanying graph.

(v) The domain is $\{x \mid -3 \leqq x \leqq 3, x \in R\}$;
the range is $\{y \mid -2 \leqq y \leqq 7, y \in R\}$.

(vi) There is a unique y in the range for each x in the domain.

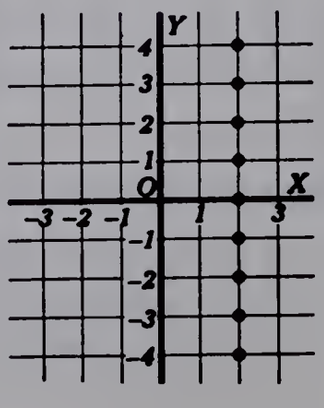
(vii) If a line drawn parallel to the y-axis intersects the graph, it does so in only one point.

4. (i) $R_4 = \{(2, -4), (2, -3), (2, -2), (2, -1), (2, 0), (2, 1), (2, 2), (2, 3), (2, 4)\}$

(ii) The domain is $\{2\}$;
the range is $\{-4, -3, -2, -1, 0, 1, 2, 3, 4\}$.

(iii) Nine elements of the range correspond to the element 2 of the domain.

(iv) If a line drawn parallel to the y-axis intersects the graph, it does so in 9 points



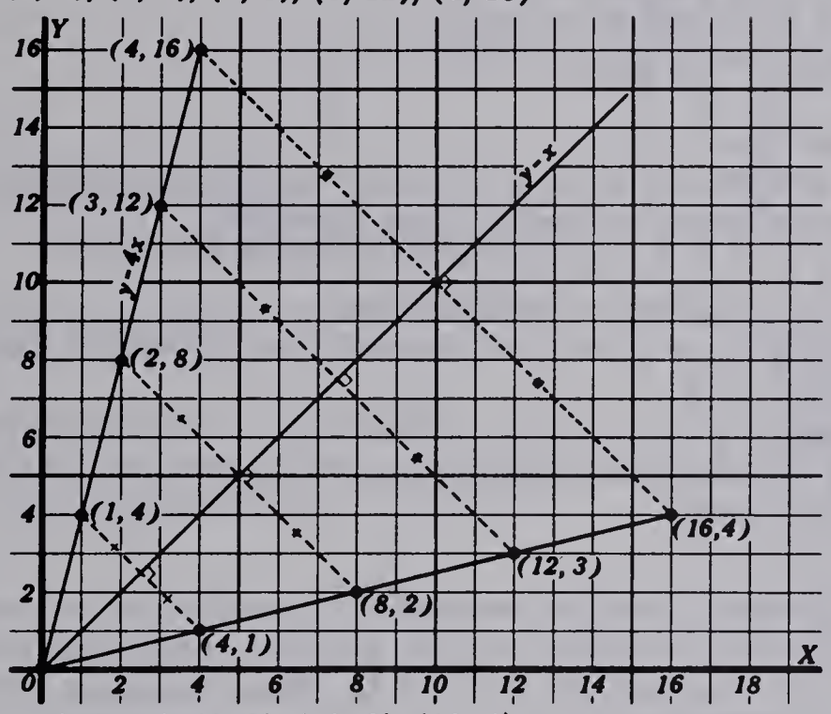
5. (i) R_1 and R_3 are functions.

(ii) If a line drawn parallel to the y-axis intersects the graph of a function, it does so in only one point.

Section **4.3** (page 73).

1. (i) (0, 0), (1, 4), (2, 8), (3, 12), (4, 16)

(ii)
(iv)
(v)
(vi)



(iii) (0, 0), (4, 1), (8, 2), (12, 3), (16, 4)

(vii) $y = \frac{x}{4}$, $0 \leqq x \leqq 16$, $x \in R$.

(viii) $f^{-1} = \left\{(x, y) \mid y = \frac{x}{4},\ 0 \leqq x \leqq 16,\ x \in R\right\}$.

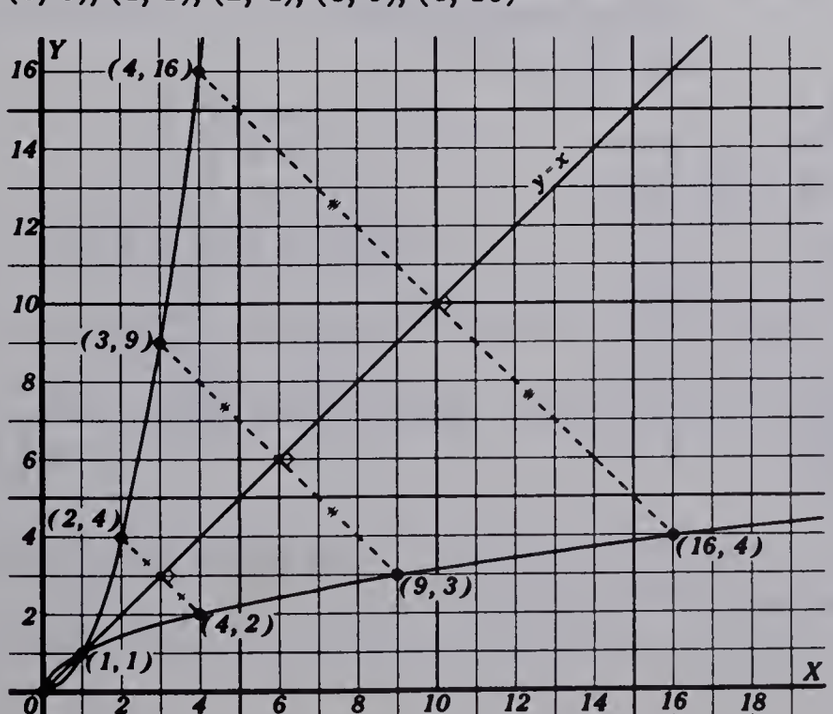
The relation f^{-1} is a function.

Domain of f^{-1} is $\{x \mid 0 \leqq x \leqq 16, x \in R\}$. Range is $\{y \mid 0 \leqq y \leqq 4, y \in R\}$.

Domain of f is $\{x \mid 0 \leqq x \leqq 4, x \in R\}$. Range is $\{y \mid 0 \leqq y \leqq 16, y \in R$

2. (i) (0, 0), (1, 1), (2, 4), (3, 9), (4, 16)

(ii)
(iv)
(v)
(vi)



(iii) $(0, 0), (1, 1), (4, 2), (9, 3), (16, 4)$

(vii) $x = y^2, y \in R, 0 \leqq y \leqq 4$ or $y = \sqrt{x}, 0 \leqq x \leqq 16, x \in R$.

(viii) $g^{-1} = \{(x, y) \mid x = y^2, 0 \leqq y \leqq 4, y \in R\}$ or
$g^{-1} = \{(x, y) \mid y = \sqrt{x}, 0 \leqq x \leqq 16, x \in R\}$.
Domain of g is $\{x \mid 0 \leqq x \leqq 4, x \in R\}$.
Range of g is $\{y \mid 0 \leqq y \leqq 16, y \in R\}$.
Domain of g^{-1} is $\{x \mid 0 \leqq x \leqq 16, x \in R\}$.
Range of g^{-1} is $\{y \mid 0 \leqq y \leqq 4, y \in R\}$.

It should be noted that
$g = \{(x, y) \mid y = x^2, 0 \leqq x \leqq 4, x \in R\}$, range is $\{y \mid 0 \leqq y \leqq 16, y \in R\}$.
The three changes required to obtain g^{-1} from g produce
$g^{-1} = \{(x, y) \mid x = y^2, 0 \leqq y \leqq 4, y \in R, 0 \leqq x \leqq 16, x \in R\}$
The function g^{-1} is completely defined by writing
(i) $g^{-1} = \{(x, y) \mid x = y^2, 0 \leqq y \leqq 4, y \in R\}$, for this ensures that $0 \leqq x \leqq 16, x \in R$.
or (ii) $g^{-1} = \{(x, y) \mid y = \sqrt{x}, 0 \leqq x \leqq 16, x \in R\}$, for this ensures that $0 \leqq y \leqq 4, y \in R$. This second form in which y is expressed in terms of x is the more usual.

We observe also that g describes the *squaring* of a real number in the domain $0 \leqq x \leqq 4, x \in R$, and g^{-1} describes the *finding of the principal square root* of a real number in the domain $0 \leqq x \leqq 16, x \in R$. These operations are inverse operations to each other.

Section 4.5 (page 82).

1. (i) Note that $f[g(-2)]$ is read "f at g at (-2)" and is the value of the function f at $g(-2)$, that is, the value of $f(x)$ when $x = g(-2)$.

$$\text{Now} \quad g(-2) = \sqrt{(-2)^2 + 4} = \sqrt{8} \text{ or } 2\sqrt{2}.$$
$$\therefore f[g(-2)] = 2\sqrt{2} + 2.$$

(ii) $f(-2) = (-2) + 2 = 0.$ $\quad \therefore g[f(-2)] = \sqrt{0^2 + 4} = 2.$

Note that $g[f(-2)] \neq f[g(-2)]$.

(iii) $g(x) = \sqrt{x^2 + 4}.$ $\quad \therefore f[g(x)] = \sqrt{x^2 + 4} + 2.$

2. (i) $f(x^2) = x^2 + \dfrac{1}{x^2}$, for $x \neq 0$. $\quad$ (ii) $f(u + 2) = u + 2 + \dfrac{1}{u + 2}$, for $u \neq -2$.

(iii) $f\left(\dfrac{1}{u}\right) = \dfrac{1}{u} + \dfrac{1}{\left(\frac{1}{u}\right)}$

$= \dfrac{1}{u} + u$

$= f(u)$, for $u \neq 0$.

(iv) $f[f(x)] = \left(x + \dfrac{1}{x}\right) + \dfrac{1}{\left(x + \frac{1}{x}\right)}$

$= x + \dfrac{1}{x} + \dfrac{x}{x^2 + 1}$ for $x \neq 0$.

Chapter 5

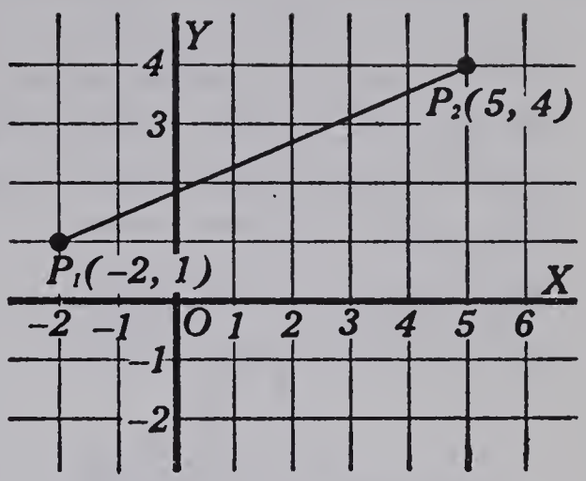
Section 5.2 (page 88).

1. (i) The slope of $P_1P_2 = \dfrac{\triangle y}{\triangle x}$, $x_2 \neq x_1$

$= \dfrac{y_2 - y_1}{x_2 - x_1}$, $\triangle x \neq 0$

$= \dfrac{4 - 1}{5 - (-2)}$

$= \dfrac{3}{7}$



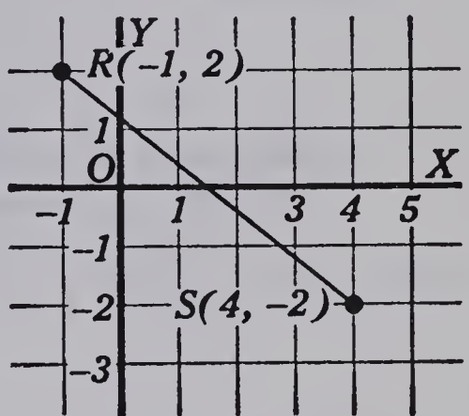
(ii) The slope of a line which rises to the right is positive.

2. (i) (a) The slope of $RS = \dfrac{-2 - 2}{4 - (-1)}$

$= \dfrac{-4}{5}$.

(b) The slope of $SR = \dfrac{2 - (-2)}{-1 - 4}$

$= \dfrac{4}{-5}$.



The slopes are equal. The slope of a line segment is independent of the order in which the end points are named.

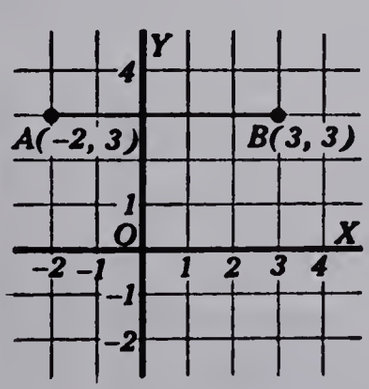
(ii) The slope of a line segment which rises to the left is negative.

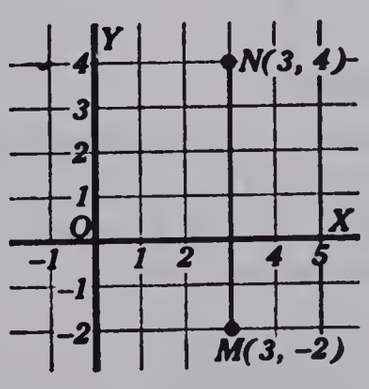
3. (i) The slope of $AB = \dfrac{3 - 3}{3 - (-2)}$ (diagram at left below.

$= \dfrac{0}{5}$

$= 0$.

(ii) The slope of a line parallel to the x-axis is 0.





4. (i) The slope of $MN = [4 - (-2)] : [3 - 3]$ (diagram at the right above)

$= 6 : 0$ (not a real number).

(ii) The slope of a line parallel to the y-axis is not defined.

5. (i) The slope of a line is independent of the order in which the end points are named.
(ii) The slope of a line which rises to the right is positive.
(iii) The slope of a line which rises to the left is negative.
(iv) The slope of a line parallel to the x-axis is zero.
(v) The slope of a line parallel to the y-axis is not defined.

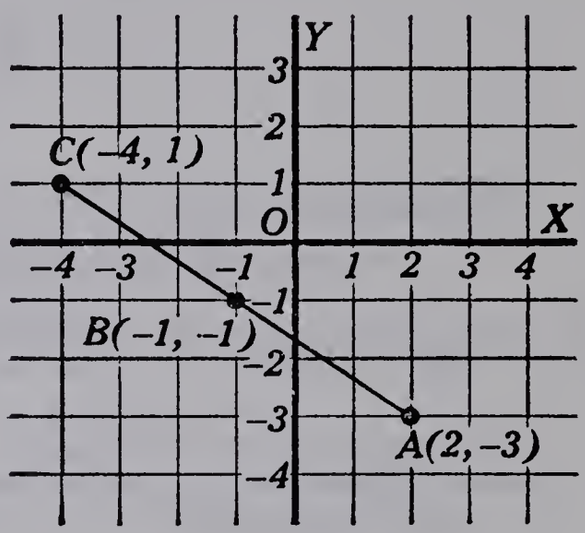
6. Slope of $AB = \dfrac{-1-(-3)}{-1-2}$

$= -\dfrac{2}{3}$.

Slope of $BC = \dfrac{1-(-1)}{-4-(-1)}$

$= -\dfrac{2}{3}$.

The slope of AB = the slope of BC.
$\therefore$ A, B, and C are collinear.



7. Slope of $PQ = \dfrac{3-2}{-1-x}$

$= \dfrac{1}{-1-x}$

Slope of $QR = \dfrac{-2-3}{3-(-1)}$

$= \dfrac{-5}{4}$.

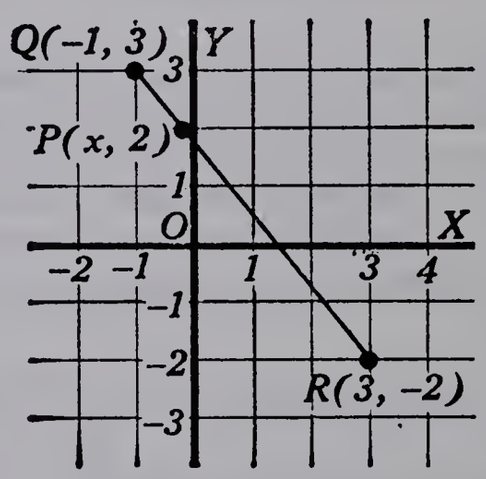
P, Q, and R are collinear.
$\therefore$ the slope of PQ = the slope of QR.

$\leftrightarrow \dfrac{1}{-1-x} = \dfrac{-5}{4}$

$\leftrightarrow 5 + 5x = 4$

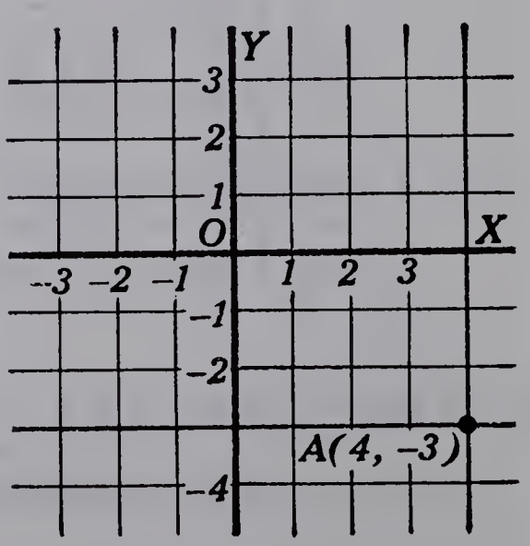
$\leftrightarrow 5x = -1$

$\leftrightarrow x = -\dfrac{1}{5}$



Section 5.3 (page 94).

1. (i) The equation of a line $\parallel$ y-axis is $y = y_1$.
$\therefore$ the equation is $y = -3$.
(ii) The equation of a line $\parallel$ x-axis is $x = x_1$.
$\therefore$ the equation is $x = 4$.



2. The slope of segment $AB = \dfrac{3-(-1)}{-2-4}$

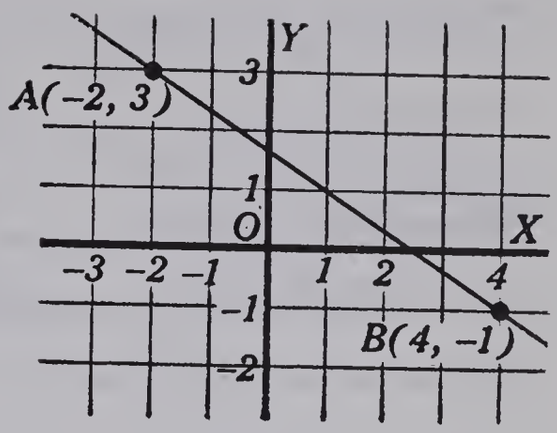
$= -\dfrac{2}{3}$.

$\therefore$ the equation is

$y - 3 = -\frac{2}{3}(x+2)$

$\leftrightarrow 3y - 9 = -2x - 4$

$\leftrightarrow 2x + 3y - 5 = 0.$

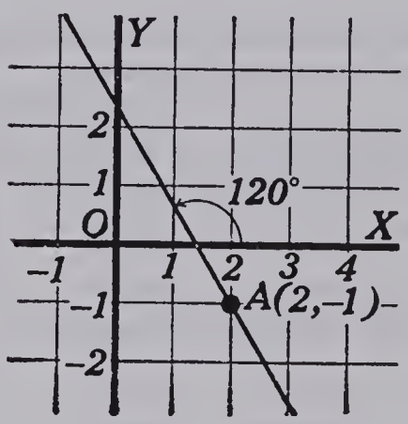


3. The slope of the line $= -\sqrt{3}$.

$\therefore$ the equation is $y + 1 = -\sqrt{3}(x-2)$

$\leftrightarrow y + 1 = -\sqrt{3}x + 2\sqrt{3}$

$\leftrightarrow \sqrt{3}x + y + 1 - 2\sqrt{3} = 0.$



Section 5.4 Discovery Exercise 5-4 (page 96).

1. Using the slope-point form of the equation of a line, the equation of line l is

$y - 1 = m(x-0)$

$\leftrightarrow y = mx + 1.$

2. The point B has coordinates $(0, b)$.

The equation of the line is $y - b = m(x-0)$

$\leftrightarrow y = mx + b.$

3. The point A has coordinates $(a, 0)$.

The equation of the line is $y - 0 = m(x-a)$

$\leftrightarrow y = m(x-a).$

Section 5.4 (page 97).

1. (i) The slope $= \sqrt{3}$.

Using the form $y = mx + b$, the equation is

$y = \sqrt{3}x - 2$

$\leftrightarrow \sqrt{3}x - y - 2 = 0.$

(ii) Using the form $y = m(x-a)$, the equation is

$y = -\frac{3}{4}(x-3)$

$\leftrightarrow 4y = -3x + 9$

$\leftrightarrow 3x + 4y - 9 = 0.$

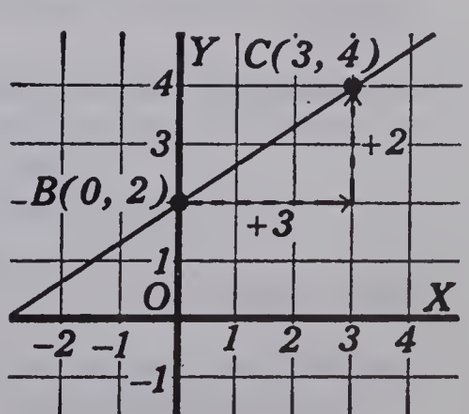
2. The equation $2x - 3y + 6 = 0$

$\leftrightarrow 3y = 2x + 6$

$\leftrightarrow y = \frac{2}{3}x + 2.$

$\leftrightarrow \therefore m = \frac{2}{3}$ and $b = 2.$

The line is on $B(0, 2)$. If we use $\triangle x = +3$ and $\triangle y = +2$, a second point $C(3, 4)$ is determined. The graph is shown at the right.



3. The equation $Ax + By + C = 0$

$$\leftrightarrow By = -Ax - C$$

$$\leftrightarrow y = -\frac{Ax}{B} - \frac{C}{B}, (B \neq 0).$$

This equation has the form $y = mx + b$. $\therefore$ the slope $= -\frac{A}{B}$.

4. The coordinates of two points on the line are (3, 0) and (0, 4).

Slope of the line is $\frac{4-0}{0-3}$ or $-\frac{4}{3}$.

The equation of the line is $y - 0 = -\frac{4}{3}(x - 3)$

$$\leftrightarrow 4x + 3y = 12.$$

5. The coordinates of two points on the line are $(a, 0)$ and $(0, b)$.

The slope of the line is $\frac{b-0}{0-a}$ or $\frac{b}{-a}$.

Hence, the equation of the line is $y - 0 = \frac{b}{-a}(x - a)$

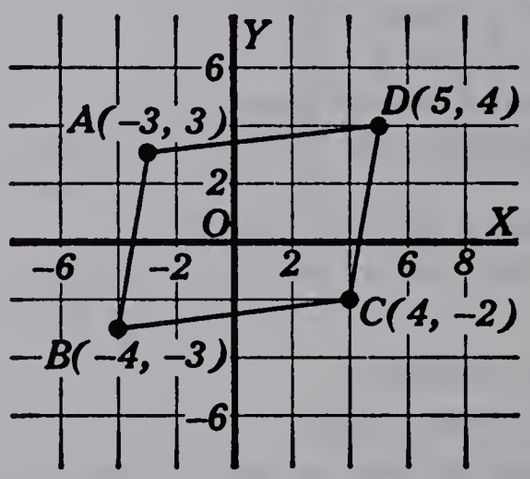
$$\leftrightarrow -ay = bx - ab$$

$$\leftrightarrow bx + ay = ab$$

$$\leftrightarrow \frac{x}{a} + \frac{y}{b} = 1.$$ (Division by ab)

Section 5.5 (page 99).

Example 1.



The slope of $AD = \frac{4-3}{5-(-3)} = \frac{1}{8}$.

The slope of $BC = \frac{-2-(-3)}{4-(-4)} = \frac{1}{8}$.

$\therefore AD \parallel BC$

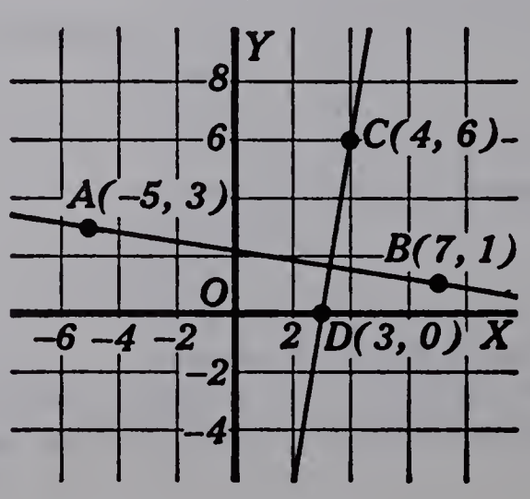
The slope of $AB = \frac{3-(-3)}{-3-(-4)} = \frac{6}{1}$.

The slope of $DC = \frac{4-(-2)}{5-4} = \frac{6}{1}$.

$\therefore AB \parallel DC$

$\therefore ABCD$ is a parallelogram.

Example 2.



Slope of segment $AB = \frac{1-3}{7-(-5)}$

$$= \frac{-2}{12} = -\frac{1}{6}.$$

Slope of segment $CD = \frac{6-0}{4-3} = \frac{6}{1}$.

These slopes are negative reciprocals.

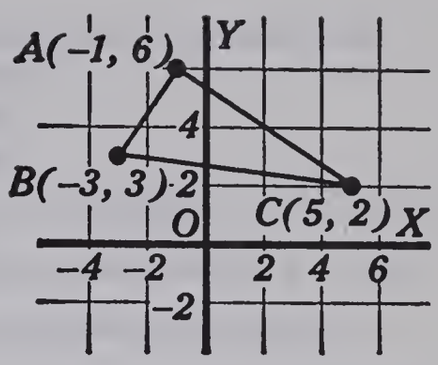
$\therefore$ the lines are perpendicular.

Example 3.

Slope of $AB = \dfrac{6-3}{-1+3} = \dfrac{3}{2}$.

Slope of $AC = \dfrac{6-2}{-1-5} = \dfrac{4}{-6} = -\dfrac{2}{3}$.

These slopes are negative reciprocals.
$\therefore AB \perp AC$, and the triangle is right-angled at A.



Section 5.7 (page 101).

1.

m	$y = mx + 2$	DESCRIPTION OF THE LINE
1	$y = x + 2$	slope 1, y-intercept 2
-2	$y = -2x + 2$	slope -2, y-intercept 2
-3	$y = -3x + 2$	slope -3, y-intercept 2
$+2$	$y = 2x + 2$	slope 2, y-intercept 2
-4	$y = -4x + 2$	slope -4, y-intercept 2
$+\frac{1}{5}$	$y = \frac{1}{5}x + 2$	slope $\frac{1}{5}$, y-intercept 2
$-\frac{1}{4}$	$y = -\frac{1}{4}x + 2$	slope $-\frac{1}{4}$, y-intercept 2
$+\frac{1}{3}$	$y = \frac{1}{3}x + 2$	slope $\frac{1}{3}$, y-intercept 2
0	$y = 2$	slope 0, y-intercept 2

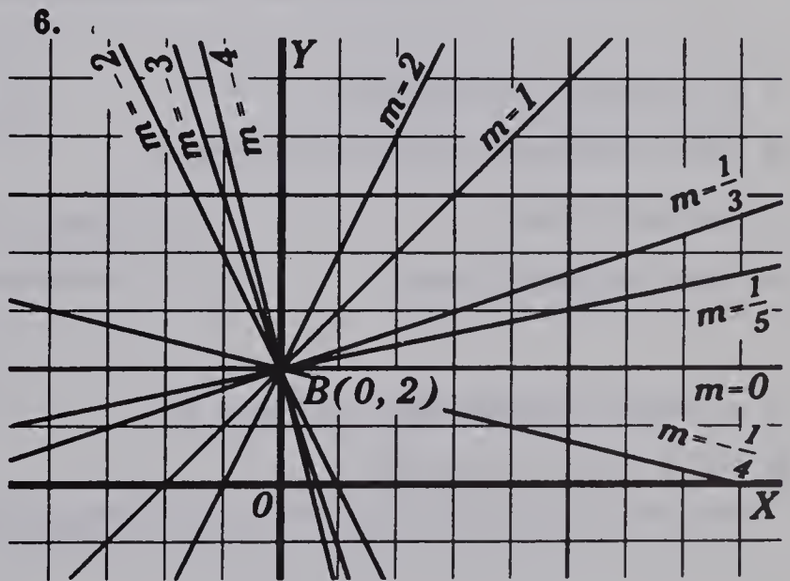
2. Each has the same y-intercept 2.

3. The slope of each line is different.

4. The same point $B(0, 2)$ or the same y-intercept.

5. They are concurrent lines.

6.



7. There are several ways of doing this problem but the following method is useful in many circumstances.

Recall that two conditions determine a straight line. These two conditions are represented in the equation $y = mx + b$ by m and b. It is necessary to determine the m and b for this line. Since we know $b = 3$ we may write:

The equation of the family of lines with y-intercept 3 is $y = mx + 3$.

Since the line is on $A(3, 2)$,

$$\therefore 2 = 3m + 3$$

$$\therefore m = -\frac{1}{3}.$$

$\therefore$ the equation of the required member of the family is

$$y = -\tfrac{1}{3}x + 3$$

or $$3y = -x + 9$$

or $$x + 3y - 9 = 0.$$

8. Recall that the slopes of two perpendicular lines are negative reciprocals. A line with slope m is perpendicular to a line with slope $-\dfrac{1}{m}$.

The equation of the family of lines with y-intercept -6 is

$$y = mx - 6.$$

Slope of line with equation $y = 5x + 4$ is 5.

Slope of perpendicular line is $-\frac{1}{5}$.

$\therefore$ the equation of the required line is

$$y = -\tfrac{1}{5}x - 6$$

or $$5y = -x - 30$$

or $$x + 5y + 30 = 0.$$

Section 5.8 (page 102).

1.

b	$y = 3x + b$	DESCRIPTION OF THE LINE
-3	$y = 3x - 3$	slope 3, y-intercept -3
-1	$y = 3x - 1$	slope 3, y-intercept -1
0	$y = 3x$	slope 3, y-intercept 0
$+2$	$y = 3x + 2$	slope 3, y-intercept 2
$+4$	$y = 3x + 4$	slope 3, y-intercept 4
$+6$	$y = 3x + 6$	slope 3, y-intercept 6

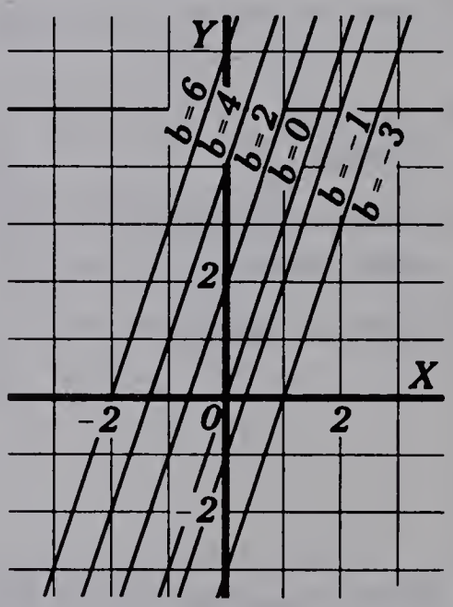
2. Each has the same slope 3.

3. The y-intercept of each line is different.

4. The same slope.

5. They are parallel lines.

6.



7. A family of parallel lines with slope 3.

8. "b" is called a parameter.

Section 5.10 (page 106).

1. (i) $AB = |-8 - (-2)| = +6$

(ii) $PQ = |0 - (-2)| = +2$

(iii) $OA = |-5 - 0| = +5$

(iv) $RS = |0 - (-2)| = +2$

(v) $MN = |-4 - (-1)| = +3$

(vi) $WZ = |r - p|$

(vii) $AB = |b_2 - b_1|$

(viii) $CD = |y_2 - y_1|$

Section 5.11 Discovery Exercise 5-9 (page 106).

1. (i) $PR = |5 - 1| = +4$ $\quad RQ = |4 - 1| = +3$

(ii) $PQ = \sqrt{4^2 + 3^2} = \sqrt{25} = 5$

2. (i) The coordinates of M are (12, 0).

(ii) $OM = |12 - 0| = +12$ $\quad AM = |5 - 0| = +5$ $\quad OA = \sqrt{12^2 + 5^2} = \sqrt{169} = +13$

3. (i) The coordinates of C are $(-3, 4)$.

(ii) $CB = |5 - (-3)| = +8$ $\quad AC = |4 - (-2)| = +6$ $\quad AB = \sqrt{8^2 + 6^2} = \sqrt{100} = +10$

4. (i) $PQ = \sqrt{|3 - (-2)|^2 + |-9 - 3|^2} = \sqrt{5^2 + 12^2} = \sqrt{169} = +13$

(ii) $AB = \sqrt{|5 - 1|^2 + |3 - 1|^2} = \sqrt{4^2 + 2^2} = \sqrt{20} = +2\sqrt{5}$

(iii) $P_1P_2 = \sqrt{|3 - (-3)|^2 + |4 - (-4)|^2} = \sqrt{6^2 + 8^2} = \sqrt{100} = +10$

(iv) $RS = \sqrt{|-8 - 7|^2 + |7 - (-1)|^2} = \sqrt{15^2 + 8^2} = \sqrt{289} = +17$

(v) $MN = \sqrt{|c - a|^2 + |d - b|^2} = \sqrt{(c - a)^2 + (d - b)^2}$

(vi) $P_1P_2 = \sqrt{|x_2 - x_1|^2 + |y_2 - y_1|^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Section 5.13 Discovery Exercise 5-11 (page 109).

1. (i) From the graph, M has coordinates (1, 2).

(ii) $P_1M = |1 - (-1)| = +2$

$MP_2 = |3 - 1| = +2$

$P_1P_2 = |3 - (-1)| = +4$

$P_1M = MP_2 = \frac{1}{2}P_1P_2$

(iii) $1 = \dfrac{3 + (-1)}{2}$; that is, the abscissa of M is one-half the sum of the abscissas of P_1 and P_2.

2. (i) The coordinates of u are $\left(\dfrac{x_1 + x_2}{2}, y_1\right)$.

(ii) $P_1M = \left|\dfrac{x_1 + x_2}{2} - x_1\right| = \left|\dfrac{x_2 - x_1}{2}\right| = \frac{1}{2}|x_2 - x_1|$

$MP_2 = \left|x_2 - \dfrac{x_1 + x_2}{2}\right| = \left|\dfrac{x_2 - x_1}{2}\right| = \frac{1}{2}|x_2 - x_1|$

$P_1P_2 = |x_2 - x_1|$

$\therefore P_1M = MP_2 = \frac{1}{2}P_1P_2$

3. (i) The coordinates of M are $(3, -1)$.

(ii) $P_1M = |-1 - 3| = +4$

$MP_2 = |-5 - (-1)| = +4$

$\therefore P_1P_2 = |(-5) - 3| = +8$

$\therefore P_1M = MP_2 = \frac{1}{2}P_1P_2$

(iii) $-1 = \dfrac{(-5) + 3}{2}$; that is, the ordinate of M is one-half the sum of the ordinates of P_1 and P_2.

4. (i) The coordinates of M are $\left(x_1, \dfrac{y_1 + y_2}{2}\right)$.

(ii) $P_1M = \left|\dfrac{y_1 + y_2}{2} - y_1\right| = \left|\dfrac{y_2 - y_1}{2}\right| = \frac{1}{2}|y_2 - y_1|$

$MP_2 = \left|y_2 - \dfrac{y_1 + y_2}{2}\right| = \left|\dfrac{y_2 - y_1}{2}\right| = \frac{1}{2}|y_2 - y_1|$

$P_1P_2 = |y_2 - y_1|$

$\therefore P_1M = MP_2 = \frac{1}{2}P_1P_2$

5. (i) The coordinates of M are $(1, 3)$.

(ii) $P_1M = \sqrt{(1 + 2)^2 + (3 - 1)^2} = \sqrt{9 + 4} = \sqrt{13}$

$MP_2 = \sqrt{(4 - 1)^2 + (5 - 3)^2} = \sqrt{9 + 4} = \sqrt{13}$

$P_1P_2 = \sqrt{(4 + 2)^2 + (5 - 1)^2} = \sqrt{36 + 16} = \sqrt{52} = 2\sqrt{13}$

$\therefore P_1M = MP_2 = \frac{1}{2}P_1P_2$

(iii) $1 = \dfrac{4 + (-2)}{2}$; hence the abscissa of M is one-half of the sum of the abscissas of P_1 and P_2.

$3 = \dfrac{5 + 1}{2}$; hence the ordinate of M is one-half the sum of the ordinates of P_1 and P_2.

6. (i) The coordinates of M are $\left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}\right)$.

(ii) $$P_1M = \sqrt{\left(\frac{x_1 + x_2}{2} - x_1\right)^2 + \left(\frac{y_1 + y_2}{2} - y_1\right)^2}$$

$$= \sqrt{\left(\frac{x_1 + x_2 - 2x_1}{2}\right)^2 + \left(\frac{y_1 + y_2 - 2y_1}{2}\right)^2}$$

$$= \sqrt{\left(\frac{x_2 - x_1}{2}\right)^2 + \left(\frac{y_2 - y_1}{2}\right)^2}$$

$$= \tfrac{1}{2}\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$MP_2 = \sqrt{\left(x_2 - \frac{x_1 + x_2}{2}\right)^2 + \left(y_2 - \frac{y_1 + y_2}{2}\right)^2}$$

$$= \sqrt{\left(\frac{2x_2 - x_1 - x_2}{2}\right)^2 + \left(\frac{2y_2 - y_1 - y_2}{2}\right)^2}$$

$$= \sqrt{\left(\frac{x_2 - x_1}{2}\right)^2 + \left(\frac{y_2 - y_1}{2}\right)^2}$$

$$= \tfrac{1}{2}\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$P_1P_2 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$\therefore P_1M = MP_2 = \tfrac{1}{2}P_1P_2$$

(iii) $\dfrac{x_1 + x_2}{2}$ = one-half the sum of the abscissas of P_1 and P_2.

$\dfrac{y_1 + y_2}{2}$ = one-half the sum of the ordinates of P_1 and P_2.

Section 5.14 (page 112).

1. (i) $\left(\dfrac{-2+4}{2}, \dfrac{-3+2}{2}\right)$ or $\left(1, -\dfrac{1}{2}\right)$

(ii) $\left(\dfrac{2+(-3)}{2}, \dfrac{-5+(-1)}{2}\right)$ or $\left(-\dfrac{1}{2}, -3\right)$

(iii) $\left(\dfrac{-4+(-6)}{2}, \dfrac{-5+(-7)}{2}\right)$ or $(-5, -6)$

(iv) $\left(\dfrac{0+1}{2}, \dfrac{-1+1}{2}\right)$ or $\left(\dfrac{1}{2}, 0\right)$

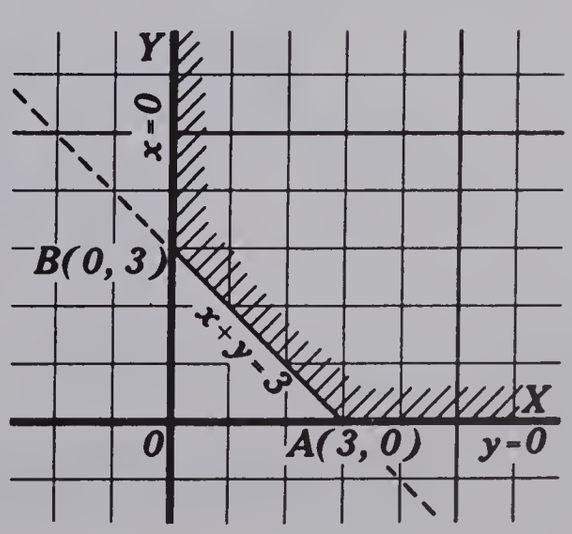
(v) $\left(\dfrac{-3+3}{2}, \dfrac{-2+2}{2}\right)$ or $(0, 0)$

(vi) $\left(\dfrac{\frac{a}{2}+\frac{c}{2}}{2}, \dfrac{\frac{b}{2}+\frac{d}{2}}{2}\right)$ or $\left(\dfrac{a+c}{4}, \dfrac{b+d}{4}\right)$

Chapter 6

Section 6.7 (page 137).

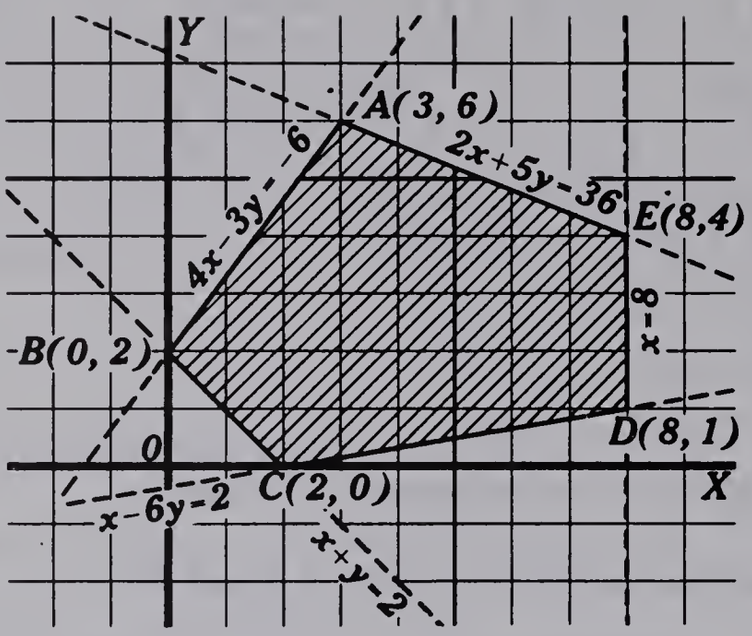
1. (i)



(ii) The polygonal convex set has two vertices $A(3, 0)$, $B(0, 3)$.

Discussion: Its boundaries are (a) the segment of the line $x + y = 3$ with end points $A(3, 0)$ and $B(0, 3)$, (b) the ray AX on $y = 0$, (c) the ray BY on $x = 0$. It has an infinite area. It does satisfy the definition given for a polygonal convex set.

2. (i) (ii) (iii)



(iv) For A

$$\begin{cases} 4x - 3y = -6 \\ 2x + 5y = 36 \end{cases}$$

For B

$$\begin{cases} 4x - 3y = -6 \\ x + y = 2 \end{cases}$$

For C

$$\begin{cases} x + y = 2 \\ x - 6y = 2 \end{cases}$$

For D

$$\begin{cases} x - 6y = 2 \\ x = 8 \end{cases}$$

For E

$$\begin{cases} x = 8 \\ 2x + 5y = 36 \end{cases}$$

Section 6.9 Discovery Exercise 6-9 (page 144).

1. (i) Slope of $AB = -\frac{1}{2}$.

(ii) Slope of $PM = 2$.

(iii) Equation of PM is $\frac{y-1}{x-1} = 2$,

or $2x - y - 1 = 0$.

(iv)
$$\begin{cases} x + 2y = 5 & (1) \\ 2x - y = 1 & (2) \end{cases}$$

$2 \times (2)$
$$\begin{cases} x + 2y = 5 & (1) \\ 4x - 2y = 2 & (3) \end{cases}$$

$(1) + (3)$ $\quad 5x = 7$

$$\therefore x = \frac{7}{5}$$

$$\therefore y = \frac{9}{5}$$

$\therefore$ the coordinates of M are $\left(\frac{7}{5}, \frac{9}{5}\right)$.

(v) Length PM

$$= \sqrt{\left(\frac{7}{5} - 1\right)^2 + \left(\frac{9}{5} + 1\right)^2}$$

$$= \sqrt{\left(\frac{2}{5}\right)^2 + \left(\frac{4}{5}\right)^2}$$

$$= \sqrt{\frac{20}{25}}$$

$$= \tfrac{2}{5}\sqrt{5}.$$

2. (i) Slope of $ED = -\dfrac{A}{B}$.

(ii) Slope of $P_1M = \dfrac{B}{A}$.

(iii) Equation of P_1M is $\dfrac{y - y_1}{x - x_1} = \dfrac{B}{A}$,

or $Ay - Ay_1 = Bx - Bx_1$,

or $Bx - Ay - Bx_1 + Ay_1 = 0$.

(iv)
$$\begin{cases} Ax + By + C = 0 & (1) \\ Bx - Ay - Bx_1 + Ay_1 = 0 & (2) \end{cases}$$

$A \times (1)$
$$\begin{cases} A^2x + ABy + AC = 0 & (3) \\ B^2x - ABy - B^2x_1 + ABy_1 = 0 & (4) \end{cases}$$
$B \times (2)$

(3) + (4) $(A^2 + B^2)x - B^2x_1 + ABy_1 + AC = 0$

$$\therefore x = \frac{B^2x_1 - ABy_1 - AC}{A^2 + B^2}, \text{ if } A^2 + B^2 \neq 0.$$

$B \times (1)$
$$\begin{cases} ABx + B^2y + BC = 0 & (5) \\ ABx - A^2y - ABx_1 + A^2y_1 = 0 & (6) \end{cases}$$
$A \times (2)$

(5) − (6) $(A^2 + B^2)y + ABx_1 - A^2y_1 + BC = 0$

$$\therefore y = \frac{-ABx_1 + A^2y_1 - BC}{A^2 + B^2}, \text{ if } A^2 + B^2 \neq 0.$$

$\therefore$ the coordinates of M are

$$\left(\frac{B^2x_1 - ABy_1 - AC}{A^2 + B^2}, \frac{-ABx_1 + A^2y_1 - BC}{A^2 + B^2}\right).$$

(v) Length P_1M_1

$$= \sqrt{\left(x_1 - \frac{B^2x_1 - ABy_1 - AC}{A^2 + B^2}\right)^2 + \left(y_1 - \frac{-ABx_1 + A^2y_1 - BC}{A^2 + B^2}\right)^2}$$

$$= \sqrt{\frac{(A^2x_1 + ABy_1 + AC)^2}{(A^2 + B^2)^2} + \frac{(ABx_1 + B^2y_1 + BC)^2}{(A^2 + B^2)^2}}$$

$$= \sqrt{\frac{A^2(Ax_1 + By_1 + C)^2}{(A^2 + B^2)^2} + \frac{B^2(Ax_1 + By_1 + C)^2}{(A^2 + B^2)^2}}$$

$$= \sqrt{\frac{(A^2 + B^2)(Ax_1 + By_1 + C)^2}{(A^2 + B^2)^2}}$$

$$= \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}$$

Section 6.9 (page 145).

1. The slope of $BC = \frac{3}{4}$.

$\therefore$ the equation of BC is $y - 0 = \frac{3}{4}(x - 4)$,

$\leftrightarrow 4y = 3x - 12$,

$\leftrightarrow 3x - 4y - 12 = 0$.

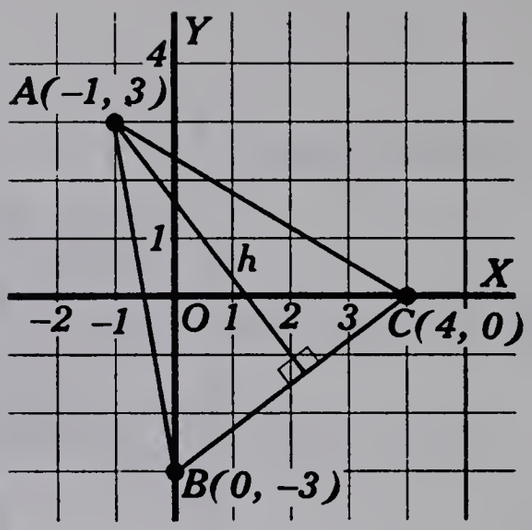
If the length of the altitude is h units,

then $h = \dfrac{|3(-1) - 4(3) - 12|}{\sqrt{3^2 + 4^2}}$.

$\therefore h = \dfrac{|-27|}{5}$

$\therefore h = 5\frac{2}{5}$

$\therefore$ the altitude is $5\frac{2}{5}$ units.



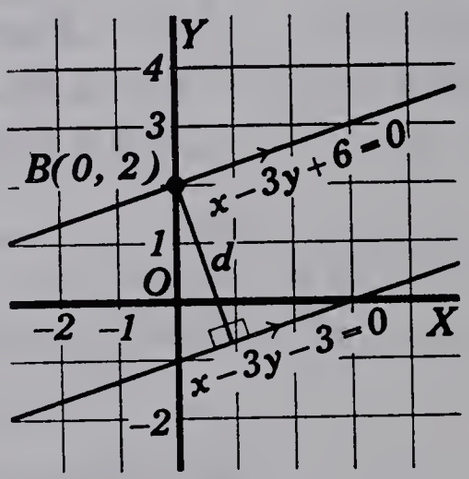
2. The y-intercept of the line with equation $x - 3y + 6 = 0$ is 2. Hence, B has coordinates $(0, 2)$. If d is the length of the perpendicular from B to the line with equation $x - 3y - 3 = 0$,

then $d = \dfrac{|0 - 3(2) - 3|}{\sqrt{1^2 + 3^2}}$

$= \dfrac{|-9|}{\sqrt{10}}$

$= \dfrac{9}{10}\sqrt{10}$.

$\therefore$ the distance between the parallel lines is $\dfrac{9}{10}\sqrt{10}$ units.



Section 6.10 Discovery Exercise 6-11 (page 147).

1. (i) Equation of P_2P_3 is $\dfrac{y - 1}{x - 4} = \dfrac{2 - 1}{-3 - 4}$,

or $-7y + 7 = x - 4$,

or $x + 7y - 11 = 0$.

(ii) Length P_1D

$= \dfrac{|(2) + 7(4) - 11|}{\sqrt{1^2 + 7^2}}$

$= \dfrac{19}{\sqrt{50}}$

$= \dfrac{19}{10}\sqrt{2}$.

(iii) Length P_2P_3

$= \sqrt{(4 + 3)^2 + (1 - 2)^2}$

$= \sqrt{50}$

$= 5\sqrt{2}$

(iv) Area $\triangle P_1P_2P_3$

$= \frac{1}{2} \times \frac{19}{10}\sqrt{2} \times 5\sqrt{2}$

$= 9.5$.

2. (i) Equation of P_2P_3 is $\dfrac{y - y_2}{x - x_2} = \dfrac{y_3 - y_2}{x_3 - x_2}$,

or $x_3y - x_2y - x_3y_2 + x_2y_2 = y_3x - x_2y_3 - y_2x + x_2y_2$,

$(y_2 - y_3)x + (x_2 - x_3)y + x_3y_2 - x_2y_3 = 0$.

(ii) Length P_1D

$$= \frac{|(y_3 - y_2)x_1 + (x_2 - x_3)y_1 + x_3y_2 - x_2y_3|}{\sqrt{(y_3 - y_2)^2 + (x_2 - x_3)^2}}.$$

(iii) Length $P_2P_3 = \sqrt{(x_3 - x_2)^2 + (y_3 - y_2)^2}$.

(iv) Area $\triangle P_1P_2P_3$

$$= \frac{1}{2}\frac{|(y_3 - y_2)x_1 + (x_2 - x_3)y_1 + x_3y_2 - x_2y_3|}{\sqrt{(y_3 - y_2)^2 + (x_2 - x_3)^2}} \times \sqrt{(x_3 - x_2)^2 + (y_3 - y_2)^2}$$

$$= \tfrac{1}{2}|x_1y_3 + x_2y_1 + x_3y_2 - x_1y_2 - x_2y_3 - x_3y_1|.$$

Chapter 7

Section 7.1 (page 154).

1. (i) $(-3)^{-2} = \dfrac{1}{(-3)^2} = \dfrac{1}{9}$ (ii) $(ab)^{-2} = a^{-2}b^{-2} = \dfrac{1}{a^2b^2}$. (iii) $\left(\dfrac{a}{b}\right)^{-3} = \dfrac{b^3}{a^3}$.

(iv) $(m^3n^2p^4)^3 = m^9n^6p^{12}$. (v) $\dfrac{9x^4y^5}{3x^5y^6} = 3x^{-1}y^{-1}$. (vi) $8 = 2^3$. (vii) $9^4 = (3^2)^4 = 3^8$.

2. (i) $3^{-4} \cdot 3^2 = 3^{-4+2} = 3^{-2}$.

(ii) $(-2)^4(-2)^{-3} = (-2)^{4+(-3)} = -2$.

(iii) $(-3)^{-2}(-3)^0 = (-3)^{-2} = \dfrac{1}{9}$.

(iv) $\left(\dfrac{2}{3}\right)^2 \times \left(\dfrac{3}{2}\right)^2 = \dfrac{2}{3} \times \dfrac{2}{3} \times \dfrac{3}{2} \times \dfrac{3}{2} = 1$.

(v) $\dfrac{2^3 \times 3^3 \times 2^4 \times 3^5}{2^5 \times 3^4} = \dfrac{2^7 \times 3^8}{2^5 \times 3^4} = 2^2 \times 3^4 = 4 \times 81 = 324$.

3. (i) $x^{27} \div x^{15} = x^{27-15} = x^{12}$.

(ii) $3^4 \div 3^{-2} = 3^{4-(-2)} = 3^6 = 729$.

(iii) $(-2)^{-6} \div (-2)^3 = (-2)^{-6-3} = (-2)^{-9} = -\frac{1}{512}$.

(iv) $a^{3x} \div a^{-4x} = a^{3x-(-4x)}$, $a \neq 0$ $= a^{7x}$.

(v) $\dfrac{4^{3a} \cdot 8^{2a}}{16^a} = \dfrac{(2^2)^{3a} \cdot (2^3)^{2a}}{(2^4)^a} = \dfrac{2^{6a} \cdot 2^{6a}}{2^{4a}} = 2^{6a+6a-4a} = 2^{8a}$.

(vi) $\dfrac{5^{4b} \cdot 25^{2b+2}}{125^{b-1}} = \dfrac{5^{4b} \cdot (5^2)^{2b+2}}{(5^3)^{b-1}} = \dfrac{5^{4b} \cdot 5^{4b+4}}{5^{3b-3}} = 5^{4b+4b+4-(3b-3)} = 5^{5b+7}$.

4. (i) $2^{3x} = 2^{2x+3}$

Since the bases are the same and the two powers are equal, therefore their exponents must be equal.

$$\therefore 3x = 2x + 3$$
$$\leftrightarrow \quad x = 3.$$

(ii) $3^{4y} = 9^{y-3}$

Before a linear equation in x and y can be written, the powers must be expressed to the same base. Write:

$$3^{4y} = (3^2)^{y-3}$$
$$\leftrightarrow 3^{4y} = 3^{2y-6}$$
$$\leftrightarrow 4y = 2y - 6$$
$$\leftrightarrow 2y = -6$$
$$\leftrightarrow \quad y = -3.$$

Section 7.9 (page 169).

1. See page 429

2. (i) For a change in x from $x = 1$ to $x = 2$, $\Delta x = 1$.

For $y = 3x$, $\Delta y = 3$, $\therefore \dfrac{\Delta y}{\Delta x} = 3$.

For $y = 3^x$, $\Delta y = 6$, $\therefore \dfrac{\Delta y}{\Delta x} = 6$.

For $y = 4^x$, $\Delta y = 12$, $\therefore \dfrac{\Delta y}{\Delta x} = 12$.

For $y = 10^x$, $\Delta y = 90$, $\therefore \dfrac{\Delta y}{\Delta x} = 90$.

(ii) For a change in x from $x = 2$ to $x = 3$, $\Delta x = 1$.

For $y = 3x$, $\Delta y = 3$, $\therefore \dfrac{\Delta y}{\Delta x} = 3$.

For $y = 3^x$, $\Delta y = 18$, $\therefore \dfrac{\Delta y}{\Delta x} = 18$.

For $y = 4^x$, $\Delta y = 48$, $\therefore \dfrac{\Delta y}{\Delta x} = 48$.

For $y = 10^x$, $\Delta y = 900$, $\therefore \dfrac{\Delta y}{\Delta x} = 900$.

3. (i) As a increases from 3 to 4 to 10, rate of exponential growth increases.

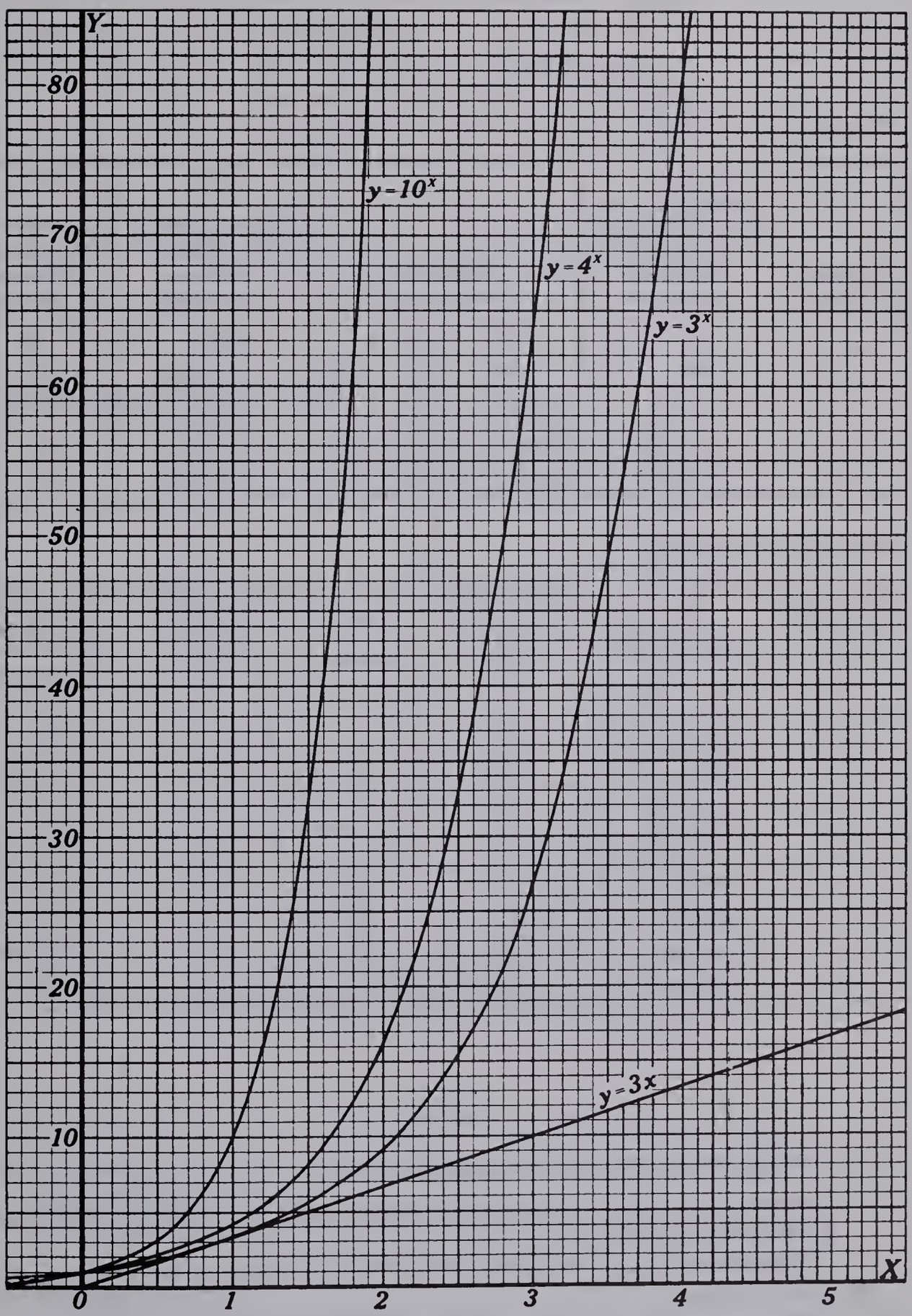
(ii) For a given base, as x changes from $x = 1$ to $x = 2$, $x = 2$ to $x = 3$, and so on, the rate at which the exponential function changes, or grows, increases.

Section 7.11 (page 176).

1. (i) (0.437, 2.735)
(ii) (1.632, 42.85)
(iii) (3 + .216, 1,644)
(iv) (−1 + .524, 0.3342)
(v) (−2 + .320, 0.02089)
(vi) (−4 + .861, 0.0007261)

2. (i) (0.497, 3.141)
(ii) (1 + .993, 98.40)
(iii) (2 + .804, 636.8)
(iv) (−1 + .115, 0.1303)
(v) (−2 + .349, 0.02234)
(vi) (−4 + .881, 0.0007603)

1.



Chapter 8

Section 8.1 (page 189).

1. (i) $\log_5 x = 3$ (ii) $\log_2 8 = 3$ (iii) $\log_{10} 1000 = 3$ (iv) $\log_q p = r$

2. (i) $x = 7^4$ (ii) $125 = 5^3$ (iii) $10^5 = 10^5$ (iv) $B = l^m$

Section 8.16 (page 213).

(1) INTEREST PERIOD	(2) PRINCIPAL AT THE BEGINNING OF INTEREST PERIOD IN DOLLARS	(3) INTEREST IN DOLLARS	(4) AMOUNT AT END OF INTEREST PERIOD IN DOLLARS	(5) AMOUNT AT 6% SIMPLE INTEREST IN DOLLARS
1	100	$100 \times .06 = 6$	106	106
2	106	$106 \times .06 = 6.36$	112.36	112
3	112.36	$112.36 \times .06 = 6.74$	119.10	118
4	119.10	$119.10 \times .06 = 7.16$	126.26	124
5	126.26	$126.26 \times .06 = 7.58$	133.84	130

Section 8.18 (page 219).

INTEREST RATE PER ANNUM %	HOW COMPOUNDED	RATE PER CONVERSION PERIOD i	NUMBER OF YEARS	NUMBER OF CONVERSION PERIODS n	COMPOUND AMOUNT OF \$1 AFTER n PERIODS
4	yearly	.04	2	2	$(1.04)^2$
4	half-yearly	.02	2	4	$(1.02)^4$
3	half-yearly	.015	$5\frac{1}{2}$	11	$(1.015)^{11}$
4	quarterly	.01	3	12	$(1.01)^{12}$
5	half-yearly	.025	4	8	$(1.025)^8$
4	quarterly	.01	$6\frac{1}{4}$	25	$(1.01)^{25}$
8	quarterly	.02	$4\frac{1}{2}$	18	$(1.02)^{18}$
6	monthly	.005	2	24	$(1.005)^{24}$
9	half-yearly	.045	5	10	$(1.045)^{10}$

Chapter 9

Section 9.3 Discovery Exercise 9-2 (page 230).

1. (a) *Symmetry.* $\because (-x)^2 = x^2$, the defining equations of q_1, q_2, and q_3 are unchanged if x is replaced by $(-x)$.

$\therefore$ the graphs of q_1, q_2, and q_3 are symmetric with respect to the y-axis.

(b) *Table of values.*

Because each graph is symmetric with respect to the y-axis, it is sufficient to compute the ordinates of points for which $x \geqq 0$, and then reflect these points in the y-axis to obtain the corresponding points for which $x < 0$.

(i)

x	0	1	2	3
y (or x^2)	0	1	4	9

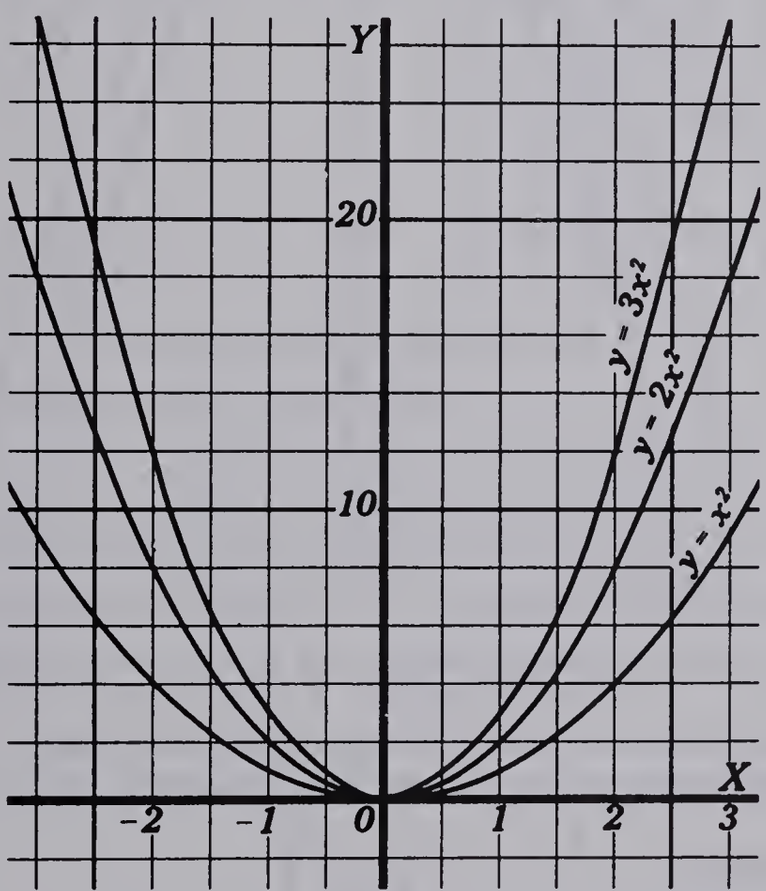
(ii)

x	0	1	2	3
y (or $2x^2$)	0	2	8	18

(iii)

x	0	1	2	3
y (or $3x^2$)	0	3	12	27

(c) The graphs of q_1, q_2, and q_3.



2. (a) *Symmetry.* $\because (-x)^2 = x^2$, the defining equations of q_4, q_5, and q_6 are unchanged if x is replaced by $(-x)$.

$\therefore$ the graphs of q_4, q_5, and q_6 are symmetric with respect to the y-axis.

(b) *Table of values.*

Because each graph is symmetric with respect to the y-axis, it is sufficient to compute the ordinates of points for which $x \geqq 0$, and then reflect these points in the y-axis to obtain the corresponding points for which $x < 0$.

(i)

x	0	1	2	3
y (or $-x^2+4$)	4	3	0	-5

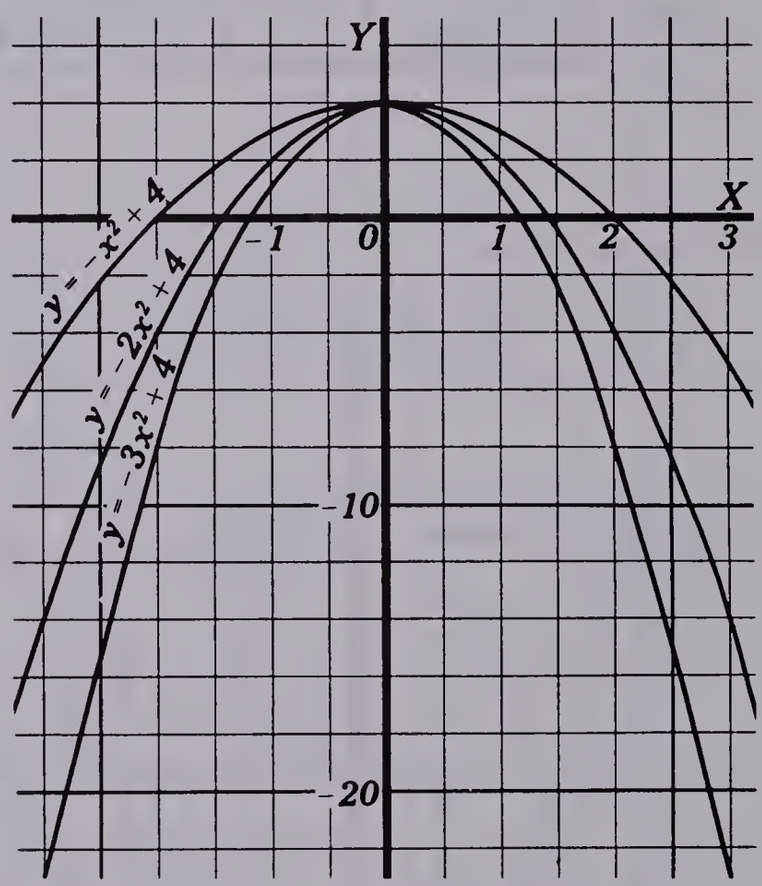
(ii)

x	0	1	2	3
y (or $-2x^2+4$)	4	2	-4	-14

(iii)

x	0	1	2	3
y (or $-3x^2+4$)	4	1	-8	-23

(c) The graphs of q_4, q_5, and q_6.



3. (a) *Symmetry.* Since the defining sentences of q_7, q_8, q_9 are all changed if x is replaced by $(-x)$,

$\therefore$ the graphs are not symmetric with respect to the y-axis.

Similarly, these graphs are not symmetric with respect to the x-axis nor with respect to the origin.

(b) *Table of values.*

(i)

x	-4	-3	-2	-1	0	1	2	3	4
y (or x^2+2x-3)	5	0	-3	-4	-3	0	5	12	21

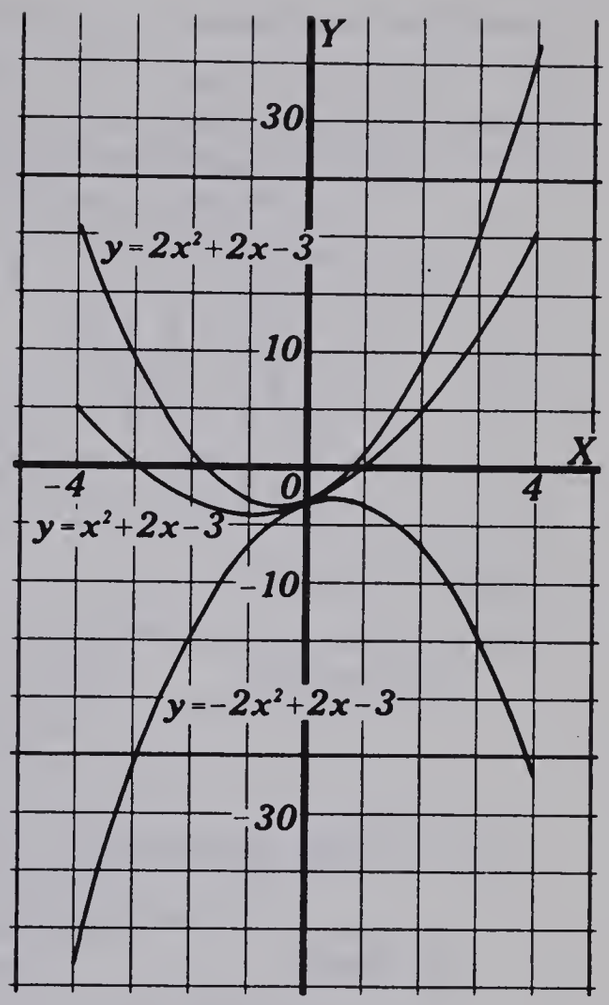
(ii)

x	-4	-3	-2	-1	0	1	2	3	4
y (or $2x^2+2x-3$)	21	9	1	-3	-3	1	9	21	37

(iii)

x	-4	-3	-2	-1	0	1	2	3	4
y (or $-2x^2+2x-3$)	-43	-27	-15	-7	-3	-3	-7	-15	-27

(c) The graphs of q_7, q_8, and q_9.



4. (i) upward;
(ii) downward;
(iii) graph decreases (or falls) more rapidly;
(iv) graph decreases (or falls) more rapidly.

Section 9.4 (page 232).

$$q = \{(x, y) \mid y = 3x^2 - 12x + 4,\ -3 \leqq x \leqq 7\}.$$

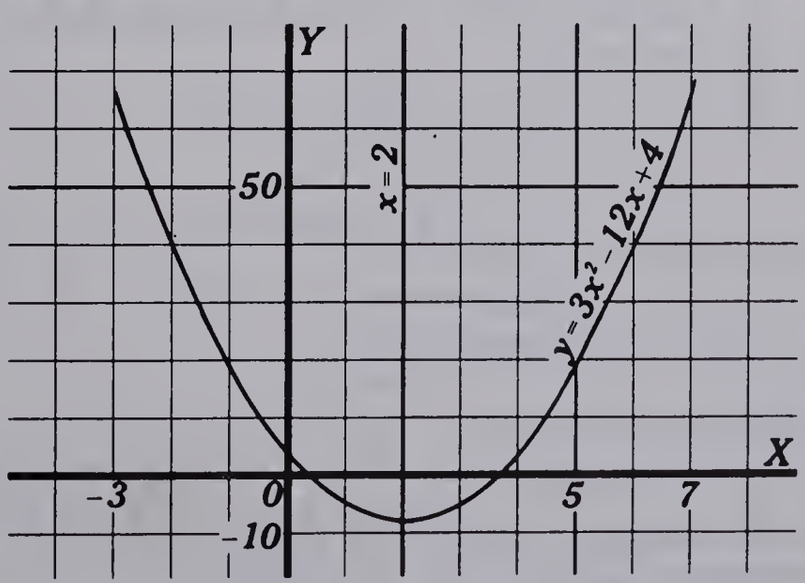
(a) To sketch the graph, make a table of values:

x	−3	−2	−1	0	1	2	3	4	5	6	7
y	67	40	19	4	−5	−8	−5	4	19	40	67

(b) Apparent vertex has coordinates (2, −8). Apparent axis of symmetry is the line with equation $x = 2$.

(c) Estimated roots of $3x^2 - 12x + 4 = 0$ are 0.3 and 3.7.

Graph of $q = \{(x, y) \mid y = 3x^2 - 12x + 4,\ -3 \leqq x \leqq 7\}$.



Section 9.5 (page 234).

1. $x^2 + 3x + 2 = 0$
$\leftrightarrow (x + 1)(x + 2) = 0$
$\leftrightarrow x + 1 = 0$ or $x + 2 = 0$
$\leftrightarrow x = -1$ or $x = -2$.

2. $2x^2 + 5x - 3 = 0$
$\leftrightarrow x^2 + \frac{5}{2}x = \frac{3}{2}$
$\leftrightarrow x^2 + \frac{5}{2}x + \frac{25}{16} = \frac{3}{2} + \frac{25}{16}$
$\leftrightarrow (x + \frac{5}{4})^2 = \frac{49}{16}$
$\leftrightarrow x + \frac{5}{4} = \pm\frac{7}{4}$
$\leftrightarrow x = \frac{7}{4} - \frac{5}{4}$ or $x = -\frac{7}{4} - \frac{5}{4}$
$\leftrightarrow x = \frac{1}{2}$ or $x = -3$.

3. $-4x^2 + 7x - 2 = 0$
$\leftrightarrow x^2 - \frac{7}{4}x = -\frac{1}{2}$
$\leftrightarrow x^2 - \frac{7}{4}x + \frac{49}{64} = -\frac{1}{2} + \frac{49}{64}$
$\leftrightarrow (x - \frac{7}{8})^2 = \frac{17}{64}$
$\leftrightarrow x - \frac{7}{8} = \pm\frac{\sqrt{17}}{8}$
$\leftrightarrow x = \frac{7}{8} + \frac{\sqrt{17}}{8}$ or $x = \frac{7}{8} - \frac{\sqrt{17}}{8}$
$\leftrightarrow x = \frac{7+\sqrt{17}}{8}$ or $x = \frac{7-\sqrt{17}}{8}$.

Section 9.7 (page 239).

1. (a) $q_1 = \{(x, y) \mid y = x^2 - 2x\}$.

(i) *Intercepts.* Let $y = 0$ in $y = x^2 - 2x$.
$\therefore x^2 - 2x = 0$
$\leftrightarrow x(x - 2) = 0$
$\leftrightarrow x = 0$ or $x = 2$.
$\therefore$ the x-intercepts are 0 and 2.
Let $x = 0$ in $y = x^2 - 2x$.
$\therefore y = 0$, so the y-intercept is 0.

(ii) *Range.*
$y = x^2 - 2x$
$= (x^2 - 2x + 1) - 1$
$= (x - 1)^2 - 1$.
$\therefore y \geqq -1$. ($\because (x - 1)^2 \geqq 0$ for all $x \in R$.)
$\therefore q_1$ has a minimum value of -1, occurring for $x = 1$.
$\therefore$ the range of q_1 is $\{y \mid y \geqq -1, y \in R\}$.

(iii) *Axis of symmetry.* The equation of the axis of symmetry is $x = 1$.

(iv) *Vertex.* The coordinates of the vertex are $(1, -1)$.

(v) *Table of values.* $\because$ the graph is symmetric with respect to the line with equation $x = 1$, only values of $x \geqq 1$ are used. Points on the graph with $x < 1$ may be obtained by reflection in the axis of symmetry.

x	1	2	3	4
y (or $x^2 - 2x$)	-1	0	3	8

(b) $q_2 = \{(x, y) \mid y = x^2 - x\}$.

(i) *Intercepts.* Let $y = 0$ in $y = x^2 - x$.
$\therefore x^2 - x = 0$
$\leftrightarrow x(x - 1) = 0$
$\leftrightarrow x = 0$ or $x = 1$.
$\therefore$ the x-intercepts are 0 and 1.
Let $x = 0$ in $y = x^2 - x$.
$\therefore y = 0$, so the y-intercept is 0.

(ii) *Range.*
$y = x^2 - x$
$= (x^2 - x + \frac{1}{4}) - \frac{1}{4}$
$= (x - \frac{1}{2})^2 - \frac{1}{4}$.
$\therefore y \geqq -\frac{1}{4}$. ($\because (x - \frac{1}{2})^2 \geqq 0$ for all $x \in R$.)
$\therefore q_2$ has a minimum value of $-\frac{1}{4}$, occurring for $x = \frac{1}{2}$.
$\therefore$ the range of q_2 is $\{y \mid y \geqq -\frac{1}{4}, y \in R\}$.

(iii) *Axis of symmetry.* The equation of the axis of symmetry is $x = \frac{1}{2}$.

(iv) *Vertex.* The coordinates of the vertex are $(\frac{1}{2}, -\frac{1}{4})$.

(v) *Table of values.* Only values of $x \geqq \frac{1}{2}$ are used in the table of values. Points on the graph with $x < \frac{1}{2}$ are obtained by reflection in the axis of symmetry.

x	$\frac{1}{2}$	1	2	3	4
y (or $x^2 - x$)	$-\frac{1}{4}$	0	2	6	12

(c) $q_3 = \{(x, y) \mid y = x^2 + 2x\}$.

(i) *Intercepts.* Let $y = 0$ in $y = x^2 + 2x$,

$\therefore x^2 + 2x = 0$

$\leftrightarrow x(x + 2) = 0$

$\leftrightarrow \quad x = 0$ or $x = -2$.

$\therefore$ the x-intercepts are 0 and -2.

Let $x = 0$ in $y = x^2 + 2x$,

$\therefore y = 0$, so the y-intercept is 0.

(ii) *Range.*

$$y = x^2 + 2x$$
$$= (x^2 + 2x + 1) - 1$$
$$= (x + 1)^2 - 1.$$

$\therefore y \geqq -1$. ($\because (x + 1)^2 \geqq 0$ for all $x \in R$.)

$\therefore q_3$ has a minimum value of -1, occurring for $x = -1$.

$\therefore$ the range of q_3 is $\{y \mid y \geqq -1, y \in R\}$.

(iii) *Axis of symmetry.* The equation of the axis of symmetry is $x = -1$.

(iv) *Vertex.* The coordinates of the vertex are $(-1, -1)$.

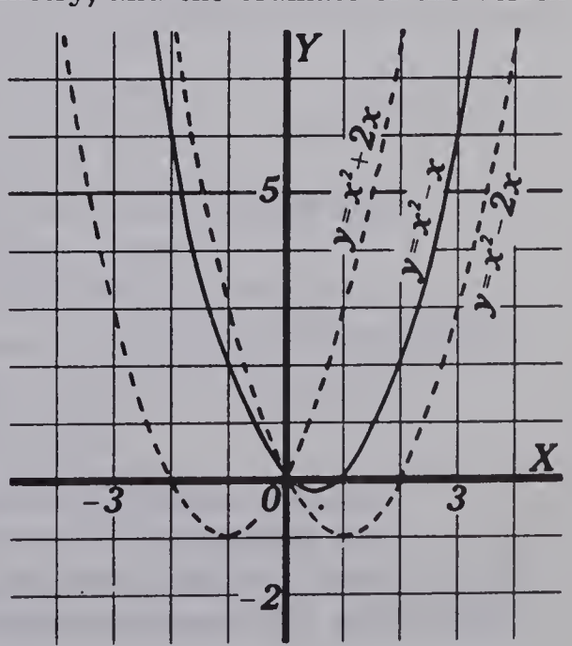
(v) *Table of values.* Only values of $x \geqq -1$ are used in the table of values. Points on the graph with $x < -1$ are obtained by reflection in the axis of symmetry.

x	-1	0	1	2	3	4
y (or $x^2 + 2x$)	-1	0	3	8	15	24

2. (i) No, the size and shape of the graph are not affected.

(ii) Yes, both the location of the axis of symmetry, and the ordinate of the vertex are affected by a change in b.

(iii) Yes, the intercepts are affected by a change in b.



3. (a) $q_4 = \{(x, y) \mid y = -2x^2 + 4x - 2\}$.

(i) *Intercepts.*

Let $y = 0$ in $y = -2x^2 + 4x - 2$,

$\therefore -2x^2 + 4x - 2 = 0$

$\leftrightarrow x^2 - 2x + 1 = 0$

$\leftrightarrow \quad (x - 1)^2 = 0$

$\leftrightarrow \quad x = 1$ or $x = 1$.

$\therefore$ the x-intercept is 1.

Let $x = 0$ in $y = -2x^2 + 4x - 2$,

$\therefore y = -2$, so the y-intercept is -2.

(ii) *Range.*
$$\begin{aligned} y &= -2x^2 + 4x - 2 \\ &= -2(x - 1)^2. \end{aligned}$$
$\therefore y \leqq 0$. ($\because -2(x - 1)^2 \leqq 0$ for all $x \in R$.)

$\therefore q_4$ has a maximum value of 0, occurring for $x = 1$.

$\therefore$ the range of q_4 is $\{y \mid y \leqq 0, y \in R\}$.

(iii) *Axis of symmetry.* The equation of the axis of symmetry is $x = 1$.

(iv) *Vertex.* The coordinates of the vertex are (1, 0).

(v) *Table of values.*

x	1	2	3	4
y (or $-2x^2 + 4x - 2$)	0	-2	-8	-18

(b) $q_5 = \{(x, y) \mid y = -2x^2 + 4x\}$.

(i) *Intercepts.* Let $y = 0$ in $y = -2x^2 + 4x$,
$$\begin{aligned} \therefore -2x^2 + 4x &= 0 \\ \leftrightarrow x(x - 2) &= 0 \\ \leftrightarrow x = 0 \text{ or } x &= 2. \end{aligned}$$
$\therefore$ the x-intercepts are 0 and 2.

Let $x = 0$ in $y = -2x^2 + 4x$.

$\therefore y = 0$, so the y-intercept is 0.

(ii) *Range.*
$$\begin{aligned} y &= -2x^2 + 4x \\ &= -2(x^2 - 2x + 1) + 2 \\ &= -2(x - 1)^2 + 2. \end{aligned}$$
$\therefore y \leqq 2$. ($\because -2(x - 1)^2 \leqq 0$ for all $x \in R$.)

$\therefore q_5$ has a maximum value of 2, occurring for $x = 1$.

$\therefore$ the range of q_5 is $\{y \mid y \leqq 2, y \in R\}$.

(iii) *Axis of symmetry.* The equation of the axis of symmetry is $x = 1$.

(iv) *Vertex.* The coordinates of the vertex are (1, 2).

(v) *Table of values.*

x	1	2	3	4
y (or $-2x^2 + 4x$)	2	0	-6	-16

(c) $q_6 = \{(x, y) \mid y = -2x^2 + 4x + 3\}$.

(i) *Intercepts.* Let $y = 0$ in $y = -2x^2 + 4x + 3$,
$$\begin{aligned} \therefore -2x^2 + 4x + 3 &= 0 \\ \leftrightarrow x^2 - 2x &= \tfrac{3}{2} \\ \leftrightarrow (x - 1)^2 &= \tfrac{3}{2} + 1 \\ \leftrightarrow (x - 1)^2 &= \tfrac{5}{2} \\ \leftrightarrow x = 1 + \sqrt{\tfrac{5}{2}} &\text{, or } x = 1 - \sqrt{\tfrac{5}{2}}, \end{aligned}$$
$\therefore$ the x-intercepts are $1 \pm \sqrt{\frac{5}{2}}$.

Let $x = 0$ in $y = -2x^2 + 4x + 3$.

$\therefore y = 3$, so the y-intercept is 3.

(ii) *Range.*
$$\begin{aligned} y &= -2x^2 + 4x + 3 \\ &= -2(x^2 - 2x + 1) + 2 + 3 \\ &= -2(x - 1)^2 + 5. \end{aligned}$$
$\therefore y \leqq 5$. ($\because -2(x - 1)^2 \leqq 0$ for all $x \in R$.)

$\therefore q_6$ has a maximum value of 5, occurring for $x = 1$.

$\therefore$ the range of q_6 is $\{y \mid y \leqq 5, y \in R\}$.

(iii) *Axis of symmetry.* The equation of the axis of symmetry is $x = 1$.

(iv) *Vertex.* The coordinates of the vertex are (1, 5).

(v) *Table of values.*

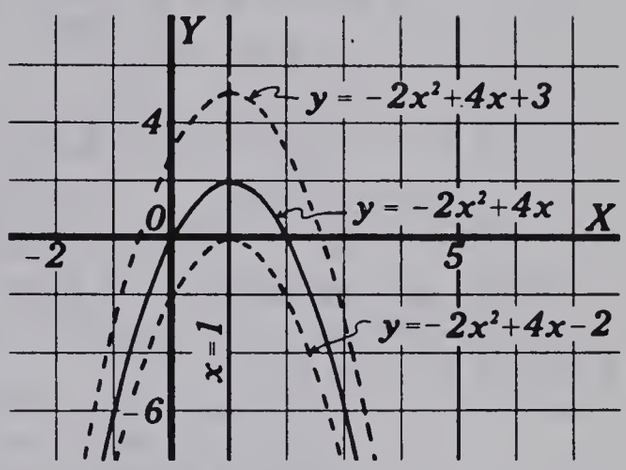
x	1	2	3	4
y (or $-2x^2 + 4x + 3$)	5	3	-3	-13

4. (i) No, the size and shape of the graph are not affected by a change in c.

(ii) The location of the axis of symmetry is not affected by a change in c, but the ordinate of the vertex is affected.

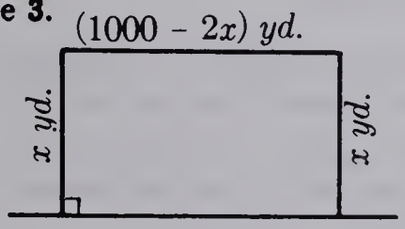
(iii) Yes, the intercepts are affected by a change in c.

(iv) c is the y-intercept of the graph.



Section 9.8 (page 241).

Example 3.



$$A(x) = x(1000 - 2x), \quad 0 < x < 500$$
$$= -2x^2 + 1000x$$
$$= -2(x^2 - 500x + 250^2) + 2(250)^2$$
$$= -2(x - 250)^2 + 2 \times 62{,}500.$$
$$\therefore A(x) \leqq 125{,}000.$$

$\therefore$ the maximum area the lot may have is 125,000 square yards. This maximum area occurs when the lot has the dimensions 250 yards (perpendicular to the street) by 500 yards (parallel to the street).

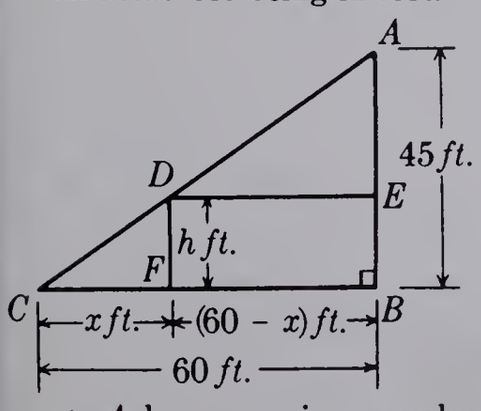
Note. The solution of Example 3 could also have been completed by noting that for each x in the domain of A, the area enclosed is exactly twice that enclosed in Example 2. From this, it follows that the maximum value of A still occurs for $x = 250$, and is twice the maximum value found in Example 2.

1. Let x denote any real number. Then the amount by which x exceeds its square is

$$A(x) = x - x^2,$$
$$\therefore A(x) = -(x^2 - x)$$
$$= -(x - \tfrac{1}{2})^2 + \tfrac{1}{4}.$$

$\therefore$ A has a maximum value of $\frac{1}{4}$, when $x = \frac{1}{2}$. Hence the real number which exceeds its square by the largest amount possible is $\frac{1}{2}$.

2. Let $DFBE$ be any inscribed rectangle. Let $CF = x$, so $FB = 60 - x$, and let $DF = h$, all measures being in feet.



Since triangles CFD and CBA are similar,

$$\therefore \frac{h}{x} = \frac{45}{60}, \text{ so } h = \tfrac{3}{4}x.$$

$\therefore$ the area of rectangle $DFBE$ is $h(60 - x)$ sq. ft.,

$$\therefore A(x) = \tfrac{3}{4}x(60 - x), \ 0 < x < 60, \ x \in R.$$
$$\therefore A(x) = -\tfrac{3}{4}x^2 + 45x$$
$$= -\tfrac{3}{4}(x^2 - 60x)$$
$$= -\tfrac{3}{4}(x - 30)^2 + \tfrac{3}{4} \cdot 900.$$

$\therefore$ A has a maximum value of 675 when $x = 30$. If $x = 30$, $h = \frac{3}{4} \times 30 = 22\frac{1}{2}$. Hence the rectangle of largest area has the dimensions 30 ft. (along BC) by $22\frac{1}{2}$ ft. (along AB).

Section 9.11 Discovery Exercise 9-9 (page 248).

1. For $x \in R$, $x^2 + 3x + c = 0$

$\leftrightarrow \quad x^2 + 3x = -c$

$\leftrightarrow \quad x^2 + 3x + \frac{9}{4} = \frac{9}{4} - c$

$\leftrightarrow \quad (x + \frac{3}{2})^2 = \dfrac{9 - 4c}{4}$

$\leftrightarrow x + \frac{3}{2} = \dfrac{\sqrt{9 - 4c}}{2}$ or $x + \frac{3}{2} = -\dfrac{\sqrt{9 - 4c}}{2}$, if $9 - 4c \geqq 0$,

$\leftrightarrow x = \dfrac{-3 + \sqrt{9 - 4c}}{2}$ or $x = \dfrac{-3 - \sqrt{9 - 4c}}{2}$.

(i) The roots are real if $9 - 4c \geqq 0$.

(ii) The roots are real and equal if $9 - 4c = 0$.

2. For $x \in R$, $x^2 + bx + c = 0$

$\leftrightarrow \quad x^2 + bx = -c$

$\leftrightarrow \quad x^2 + bx + \dfrac{b^2}{4} = \dfrac{b^2}{4} - c$

$\leftrightarrow \quad \left(x + \dfrac{b}{2}\right)^2 = \dfrac{b^2 - 4c}{4}$

$\leftrightarrow \quad x + \dfrac{b}{2} = \pm \dfrac{\sqrt{b^2 - 4c}}{2}$, if $b^2 - 4c \geqq 0$,

$\leftrightarrow x = \dfrac{-b + \sqrt{b^2 - 4c}}{2}$ or $x = \dfrac{-b - \sqrt{b^2 - 4c}}{2}$.

(i) The roots are real if $b^2 - 4c \geqq 0$.

(ii) The roots are real and equal if $b^2 - 4c = 0$.

3. For $x \in R$, $3x^2 + bx + c = 0$

$\leftrightarrow \quad x^2 + \dfrac{b}{3}x = -\dfrac{c}{3}$

$\leftrightarrow \quad x^2 + \dfrac{b}{3}x + \dfrac{b^2}{36} = \dfrac{b^2}{36} - \dfrac{c}{3}$

$\leftrightarrow \quad \left(x + \dfrac{b}{6}\right)^2 = \dfrac{b^2 - 12c}{36}$

$\leftrightarrow \quad x + \dfrac{b}{6} = \pm \dfrac{\sqrt{b^2 - 12c}}{6}$, if $b^2 - 12c \geqq 0$,

$\leftrightarrow x = \dfrac{-b + \sqrt{b^2 - 12c}}{6}$ or $x = \dfrac{-b - \sqrt{b^2 - 12c}}{6}$.

(i) The roots are real if $b^2 - 12c \geqq 0$.

(ii) The roots are real and equal if $b^2 - 12c = 0$.

4. For $x \in R$, $ax^2 + bx + c = 0$, $(a \neq 0)$

$\leftrightarrow \quad x^2 + \dfrac{b}{a}x = -\dfrac{c}{a}$

$\leftrightarrow \quad x^2 + \dfrac{b}{a}x + \dfrac{b^2}{4a^2} = \dfrac{b^2}{4a^2} - \dfrac{c}{a}$

$\leftrightarrow \quad \left(x + \dfrac{b}{2a}\right)^2 = \dfrac{b^2 - 4ac}{4a^2}$

$\leftrightarrow \quad x + \dfrac{b}{2a} = \pm \dfrac{\sqrt{b^2 - 4ac}}{2a}$, if $b^2 - 4ac \geqq 0$,

$\leftrightarrow x = \dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$ or $x = \dfrac{-b - \sqrt{b^2 - 4ac}}{2a}$.

(i) The roots are real if $b^2 - 4ac \geqq 0$.

(ii) The roots are real and equal if $b^2 - 4ac = 0$.

Section 9.13 (page 252).

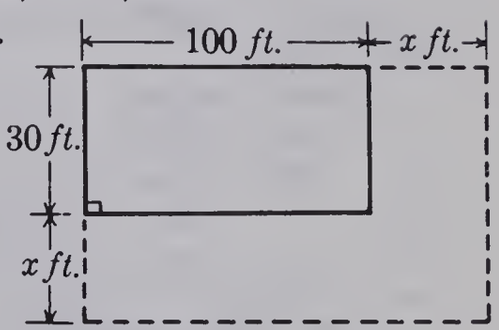
1. Represent the number of rows in the theatre by x, $x \in N$, $x > 6$.
The number of seats in a row is 6 less than the number of rows.
$\therefore$ represent the number of seats in each row by $x - 6$.
The number of seats in the theatre is $x(x - 6)$.
But the number of seats is 1015.

$$\therefore x(x - 6) = 1015$$
$$\leftrightarrow x^2 - 6x - 1015 = 0$$
$$\leftrightarrow (x - 35)(x + 29) = 0$$
$$\leftrightarrow x = 35 \text{ or } x = -29.$$

-29 is inadmissible since $x > 6$.
The number of rows in the theatre is 35.

Verification.
Number of rows $= 35$.
Number of seats per row $= 35 - 6 = 29$.
Total number of seats $= 35(29) = 1015$.

2. Represent the number of feet in the new width by $30 + x$,
and the number of feet in the new length by $100 + x$, $x \in R$, $x > 0$.
The new area of the pool is $(100 + x)(30 + x)$ sq. ft.
The original area of the pool is (30×100) sq. ft.



$$\therefore (100 + x)(30 + x) = 2(30)(100)$$
$$\leftrightarrow x^2 + 130x - 3000 = 0$$
$$\leftrightarrow (x + 150)(x - 20) = 0$$
$$\leftrightarrow x = -150 \text{ or } x = 20.$$

-150 is inadmissible since $x > 0$.
The new dimensions of the pool are 50 ft. by 120 ft.

Verification.
The new area $= (50 \times 120)$ sq. ft. $= 6000$ sq. ft.
The original area $= (30 \times 100)$ sq. ft. $= 3000$ sq. ft.
The new area is twice the original area.

Section 9.15 (page 255).

1. $3x^2 + 2x + 1 = 0$
$$\leftrightarrow x^2 + \tfrac{2}{3}x = -\tfrac{1}{3}$$
$$\leftrightarrow x^2 + \tfrac{2}{3}x + \tfrac{1}{9} = -\tfrac{1}{3} + \tfrac{1}{9}$$
$$\leftrightarrow (x + \tfrac{1}{3})^2 = -\tfrac{2}{9}$$
$$\leftrightarrow (x + \tfrac{1}{3})^2 = \tfrac{2}{9}i^2$$
$$\leftrightarrow x + \tfrac{1}{3} = \pm \frac{\sqrt{2}}{3}i$$
$$\leftrightarrow x = -\tfrac{1}{3} \pm \frac{\sqrt{2}}{3}i.$$

2. $27z^2 - 6z + 2 = 0$
$$\leftrightarrow z^2 - \tfrac{2}{9}z = -\tfrac{2}{27}$$
$$\leftrightarrow z^2 - \tfrac{2}{9}z + \tfrac{1}{81} = -\tfrac{2}{27} + \tfrac{1}{81}$$
$$\leftrightarrow (z - \tfrac{1}{9})^2 = -\tfrac{5}{81}$$
$$\leftrightarrow (z - \tfrac{1}{9})^2 = \tfrac{5}{81}i^2$$
$$\leftrightarrow z - \tfrac{1}{9} = \pm \frac{\sqrt{5}}{9}i$$
$$\leftrightarrow z = \tfrac{1}{9} \pm \frac{\sqrt{5}}{9}i.$$

3. $27z^2 - 6z - 2 = 0$
$$\leftrightarrow z^2 - \tfrac{2}{9}z = \tfrac{2}{27}$$
$$\leftrightarrow z^2 - \tfrac{2}{9}z + \tfrac{1}{81} = \tfrac{2}{27} + \tfrac{1}{81}$$
$$\leftrightarrow (z - \tfrac{1}{9})^2 = \tfrac{7}{81}$$
$$\leftrightarrow z - \tfrac{1}{9} = \pm \frac{\sqrt{7}}{9}$$
$$\leftrightarrow z = \tfrac{1}{9} \pm \frac{\sqrt{7}}{9}.$$

Section 9.16 (page 257).

1. For $y^2 - 1 \neq 0$ and $x = \dfrac{y^2+2}{y^2-1}$,

$$\frac{y^2+2}{y^2-1} + 2\left(\frac{y^2-1}{y^2+2}\right) = 3$$

$$\leftrightarrow \quad x + \frac{2}{x} = 3$$

$$\leftrightarrow \quad x^2 + 2 = 3x$$

$$\leftrightarrow \quad x^2 - 3x + 2 = 0$$

$$\leftrightarrow \quad (x-2)(x-1) = 0$$

$$\leftrightarrow \quad x = 2 \text{ or } x = 1$$

$$\leftrightarrow \quad \frac{y^2+2}{y^2-1} = 2 \text{ or } \frac{y^2+2}{y^2-1} = 1$$

$$\leftrightarrow \quad y^2 + 2 = 2y^2 - 2 \text{ or } y^2 + 2 = y^2 - 1$$

$$\leftrightarrow \quad y^2 = 4 \text{ or } 0 = -3, \text{ which is false.}$$

$$\leftrightarrow \quad y = 2 \text{ or } y = -2.$$

$\because$ $+1$ or -1 do not satisfy the equation, 2 and -2 are the roots.

2. For $y = \log x$,

$$(\log x)^2 - 5\log x + 6 = 0$$

$$\leftrightarrow \quad y^2 - 5y + 6 = 0$$

$$\leftrightarrow \quad (y-3)(y-2) = 0$$

$$\leftrightarrow \quad y = 3 \text{ or } y = 2$$

$$\leftrightarrow \quad \log x = 3 \text{ or } \log x = 2$$

$$\leftrightarrow \quad x = 10^3 \text{ or } x = 10^2.$$

3. $27(3^{2x}) - 242(3^x) - 9 = 0$

$\leftrightarrow 27(3^x)^2 - 242(3^x) - 9 = 0.$

For $y = 3^x$,

$$27(3^x)^2 - 242(3^x) - 9 = 0$$

$$\leftrightarrow \quad 27y^2 - 242y - 9 = 0$$

$$\leftrightarrow \quad (27y+1)(y-9) = 0$$

$$\leftrightarrow \quad y = -\tfrac{1}{27} \text{ or } y = 9$$

$$\leftrightarrow \quad 3^x = -\tfrac{1}{27} \text{ or } 3^x = 9$$

$$\leftrightarrow \quad 3^x = 9$$

$(3^x > 0 \text{ for all } x \in R)$

$$\leftrightarrow \quad x = 2.$$

Example 4.

For $x > 0$, $\dfrac{330}{x} - \dfrac{330}{x+5} = \tfrac{1}{2}$

$$\leftrightarrow \quad 660(x+5-x) = x(x+5)$$

$$\leftrightarrow \quad x^2 + 5x - 3300 = 0$$

$$\leftrightarrow \quad (x+60)(x-55) = 0$$

$$\leftrightarrow \quad x = -60 \text{ or } x = 55.$$

The root -60 is inadmissible since $x > 0$.

$\therefore$ the speeds of the two cars are 55 m.p.h. and 60 m.p.h.

Verification.

Time of travel by slower car is $\frac{330}{55}$ or 6 hours.

Time of travel by faster car is $\frac{330}{60}$ or $5\frac{1}{2}$ hours.

$\therefore$ the difference in times is $\frac{1}{2}$ hour.

Section 9.17 (page 260).

1.

$$r = 5 + \sqrt{1+r} \qquad (1)$$

$$\leftrightarrow \quad r - 5 = \sqrt{1+r}$$

$$\rightarrow \quad r^2 - 10r + 25 = 1 + r \qquad \text{(squaring both sides)}$$

$$\leftrightarrow \quad r^2 - 11r + 24 = 0$$

$$\leftrightarrow (r-3)(r-8) = 0$$

$$\leftrightarrow \quad r = 3 \text{ or } r = 8.$$

Verification in (1). If $r = 3$,

L.S. $= 3$.

R.S. $= 5 + \sqrt{1+3}$
$= 5 + \sqrt{4}$
$= 7.$

$\therefore$ 3 is not a root of (1).

If $r = 8$,

L.S. $= 8$.

R.S. $= 5 + \sqrt{1+8}$
$= 5 + \sqrt{9}$
$= 8.$

$\therefore$ 8 is a root of (1).

2. $\sqrt{4y-3} = 1 + \sqrt{y+1}$ (1)

$\rightarrow$ $4y - 3 = 1 + y + 1 + 2\sqrt{y+1}$ (squaring both sides)

$\leftrightarrow$ $3y - 5 = 2\sqrt{y+1}$

$\rightarrow$ $9y^2 - 30y + 25 = 4y + 4$ (squaring both sides)

$\leftrightarrow$ $9y^2 - 34y + 21 = 0$

$\leftrightarrow$ $(9y - 7)(y - 3) = 0$

$\leftrightarrow$ $y = \frac{7}{9}$ or $y = 3$.

Verification in (1). If $y = \frac{7}{9}$,

L.S. $= \sqrt{\frac{28}{9} - 3}$ $= \sqrt{\frac{1}{9}} = \frac{1}{3}$. R.S. $= 1 + \sqrt{\frac{7}{9} + 1}$ $= 1 + \sqrt{\frac{16}{9}} = \frac{7}{3}$.

$\therefore$ $\frac{7}{9}$ is not a root of (1).

If $y = 3$,

L.S. $= \sqrt{12 - 3} = 3$. R.S. $= 1 + \sqrt{3 + 1} = 3$.

$\therefore$ 3 is a root of (1).

Example 2.

$\sqrt{144 + (x+4)^2} = \sqrt{144 + x^2} + 2$

$\rightarrow$ $144 + (x+4)^2 = 144 + x^2 + 4 + 4\sqrt{144 + x^2}$ (squaring both sides)

$\leftrightarrow$ $144 + x^2 + 8x + 16 = 148 + x^2 + 4\sqrt{144 + x^2}$

$\leftrightarrow$ $12 + 8x = 4\sqrt{144 + x^2}$

$\leftrightarrow$ $3 + 2x = \sqrt{144 + x^2}$

$\rightarrow$ $9 + 12x + 4x^2 = 144 + x^2$ (squaring both sides)

$\leftrightarrow$ $x^2 + 4x - 45 = 0$

$\leftrightarrow$ $(x - 5)(x + 9) = 0$

$\leftrightarrow$ $x = 5$, since $x \in {}^+R$.

$\therefore$ the shorter side of the given triangle is 5 inches long.

Verification.

Length of hypotenuse of larger triangle is $\sqrt{81 + 144}$ inches or 15 inches.

Length of hypotenuse of original triangle is $\sqrt{25 + 144}$ inches or 13 inches.

$\therefore$ the difference in lengths is 2 inches.

Section 9.18 (page 264).

1. $C = \{(x, y) \mid x^2 - 5x + 4 < y \leqq 3,\ x, y \in R\}$.

The graph of C consists of all points whose coordinates (x, y) satisfy both of the inequations

(i) $y > x^2 - 5x + 4$,

and (ii) $y \leqq 3$.

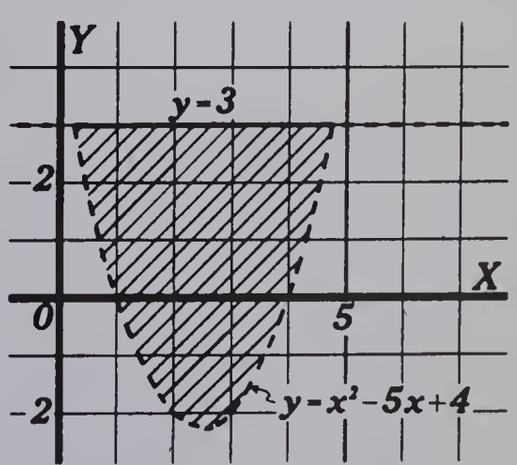
$\therefore$ the graph of C consists of all points which are

(i) above the parabola with equation $y = x^2 - 5x + 4$,

and (ii) on or below the horizontal line with equation $y = 3$.

The parabola opens upward and has y-intercept 4,

and x-intercepts $\frac{5 \pm \sqrt{25 - 16}}{2} = \frac{5 \pm 3}{2} = 4$ or 1.



$y = x^2 - 5x + 4$

$\leftrightarrow$ $y = (x^2 - 5x + \frac{25}{4}) + 4 - \frac{25}{4}$

$\leftrightarrow$ $y = (x - \frac{5}{2})^2 - \frac{9}{4}$.

$\therefore$ the coordinates of the vertex of the parabola are $(\frac{5}{2}, -\frac{9}{4})$.

2. $D = \{(r, s) \mid 2s > 2r^2 - 10r + 9,\ r, s \in R,\ 0 \leqq r \leqq 6\}$.

The graph of D consists of all points whose coordinates (r, s) satisfy both of the inequations

(i) $2s > 2r^2 - 10r + 9$, and

(ii) $0 \leqq r \leqq 6$.

Now $2s > 2r^2 - 10r + 9 \leftrightarrow s > r^2 - 5r + \frac{9}{2}$.

$\therefore$ the graph of D consists of all points which are

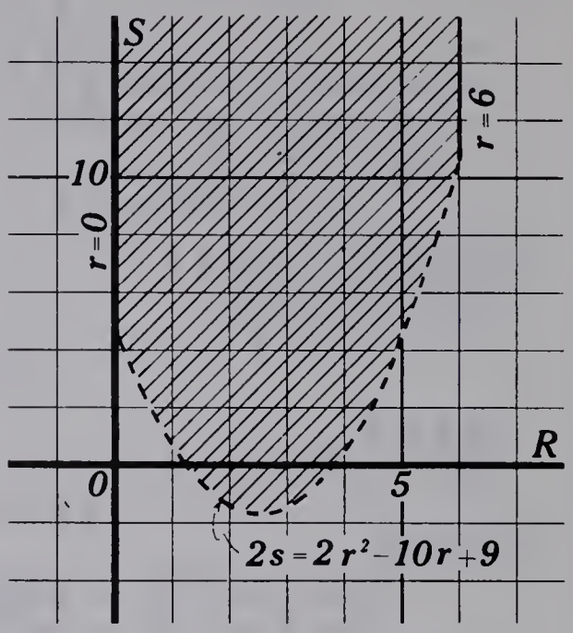
(i) above the parabola with equation $s = r^2 - 5r + \frac{9}{2}$,

and (ii) between (or on) the vertical lines with equations $r = 0$, $r = 6$.

The parabola opens upward and has s-intercept $\frac{9}{2}$,

and r-intercepts $\dfrac{5 \pm \sqrt{25 - 18}}{2} = \dfrac{5}{2} \pm \dfrac{\sqrt{7}}{2}$

$\doteq 1.2$ or 3.8.



$$s = r^2 - 5r + \tfrac{9}{2}$$
$$\leftrightarrow \quad s = (r^2 - 5r + \tfrac{25}{4}) + \tfrac{9}{2} - \tfrac{25}{4}$$
$$\leftrightarrow \quad s = (r - \tfrac{5}{2})^2 - \tfrac{7}{4}.$$

$\therefore$ the coordinates of the vertex of the parabola are $(\frac{5}{2}, -\frac{7}{4})$.

Chapter 10

Section 10.1 (page 271).

1. (i) $2x(x + 1) = x - 4 \leftrightarrow 2x^2 + x + 4 = 0$.

$D = 1 - 4(2)(4) = -31$.

$\because D < 0$,

$\therefore$ the roots are non-real.

(ii) Compare $2x^2 + x + 4 = 0$

with $ax^2 + bx + c = 0$.

$a = 2, b = 1, c = 4$.

$\therefore$ by the quadratic formula.

$$x = \frac{-1 \pm \sqrt{1 - 32}}{4}$$
$$= \frac{-1 \pm \sqrt{31i^2}}{4}$$
$$= \frac{-1 \pm \sqrt{31}i}{4}.$$

The roots are non-real.

2. $2x(x + 1) = x - k \leftrightarrow 2x^2 + x + k = 0$.

$D = 1 - 8k$,

(i) The roots are equal if and only if $1 - 8k = 0$, or $k = \frac{1}{8}$.

(ii) The roots are real and unequal if and only if $1 - 8k > 0$, or $8k < 1$ or $k < \frac{1}{8}$.

(iii) The roots are non-real if and only if $1 - 8k < 0$, or $8k > 1$, or $k > \frac{1}{8}$.

Section 10.2 Discovery Exercise 10-2 (page 272).

1.

EQUATION	a	b	c	ROOTS	SUM OF ROOTS	PRODUCT OF ROOTS
$x^2 - 3x + 2 = 0$	1	-3	2	$1, 2$	3	2
$x^2 + \frac{1}{4}x - \frac{1}{8} = 0$	1	$\frac{1}{4}$	$-\frac{1}{8}$	$-\frac{1}{2}, \frac{1}{4}$	$-\frac{1}{4}$	$-\frac{1}{8}$
$x^2 - 7x + 12 = 0$	1	-7	12	$4, 3$	7	12
$x^2 - 4x - 3 = 0$	1	-4	-3	$2 + \sqrt{7}, 2 - \sqrt{7}$	4	-3
$2x^2 + 7x + 3 = 0$	2	7	3	$-3, -\frac{1}{2}$	$-\frac{7}{2}$	$\frac{3}{2}$
$3x^2 - 4x - 4 = 0$	3	-4	-4	$2, -\frac{2}{3}$	$\frac{4}{3}$	$-\frac{4}{3}$
$5x^2 + 19x - 4 = 0$	5	19	-4	$\frac{-19 + \sqrt{441}}{10}, \frac{-19 - \sqrt{441}}{10}$	$-\frac{19}{5}$	$-\frac{4}{5}$

2. (i) Product of the roots is $\frac{c}{a}$. (ii) Sum of the roots is $-\frac{b}{a}$.

3. (i)
$$x_1 + x_2 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} + \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{-2b}{2a} = -\frac{b}{a}.$$

(ii)
$$x_1x_2 = \left(\frac{-b + \sqrt{b^2 - 4ac}}{2a}\right)\left(\frac{-b - \sqrt{b^2 - 4ac}}{2a}\right)$$
$$= \frac{b^2 - (b^2 - 4ac)}{4a^2}$$
$$= \frac{4ac}{4a^2} = \frac{c}{a}.$$

Section 10.2 (page 273).

1. (i) Sum of the roots is $-\frac{b}{a} = -\frac{3}{1} = -3$.

Product of the roots is $\frac{c}{a} = \frac{1}{1} = 1$.

(ii) Sum of the roots is $-\frac{b}{a} = -\frac{-5}{\sqrt{2}} = \frac{5}{\sqrt{2}} = \frac{5\sqrt{2}}{2}$.

Product of the roots is $\frac{c}{a} = \frac{\sqrt{2}}{\sqrt{2}} = 1$.

(iii) Sum of the roots is $-\frac{b}{a} = -\frac{-7}{3} = \frac{7}{3}$.

Product of the roots is $\frac{c}{a} = \frac{3}{3} = 1$.

2. Represent the roots by x_1, x_2.

The product of the roots is $x_1x_2 = \frac{a}{a} = 1$.

$\therefore x_1 = \frac{1}{x_2}$, so x_1, x_2 are reciprocals.

3. Represent the second root by x_1, then

$$\begin{aligned} 35x_1 &= \frac{c}{1} \\ \text{and } x_1 + 35 &= 64. \\ \therefore\ x_1 &= 29 \\ \text{and } c &= 35 \times 29 \\ &= 1015. \end{aligned}$$

Section 10.5 Discovery Exercise 10-6 (page 282).

1. $x^3 - 4x^2 - 7x + 10 = (x - 1)(x^2 - 3x - 10)$. (by the Factor Theorem)
For $x \in R$, $x^3 - 4x^2 - 7x + 10 = 0$
$\leftrightarrow \quad (x - 1)(x^2 - 3x - 10) = 0$
$\leftrightarrow \quad (x - 1)(x - 5)(x + 2) = 0$
$\leftrightarrow \quad x = 1$ or $x = 5$ or $x = -2$.
$\therefore\ x^3 - 4x^2 - 7x + 10 = 0$ has three real roots 1, 5, -2.

2. $2x^3 - 9x^2 - 8x + 15 = (x - 1)(2x^2 - 7x - 15)$. (Factor Theorem)
For $x \in R$, $2x^3 - 9x^2 - 8x + 15 = 0$
$\leftrightarrow \quad (x - 1)(2x^2 - 7x - 15) = 0$
$\leftrightarrow \quad (x - 1)(2x + 3)(x - 5) = 0$
$\leftrightarrow \quad x = 1$ or $x = -\frac{3}{2}$ or $x = 5$.
$\therefore\ 2x^3 - 9x^2 - 8x + 15 = 0$ has three real roots 1, $-\frac{3}{2}$, 5.

3. $x^3 + 3x^2 + 3x + 2 = (x + 2)(x^2 + x + 1)$. (Factor Theorem)
For $x \in R$, $x^3 + 3x^2 + 3x + 2 = 0$
$\leftrightarrow \quad (x + 2)(x^2 + x + 1) = 0$
$\leftrightarrow \quad x + 2 = 0$ or $x^2 + x + 1 = 0$
$\leftrightarrow x = -2$, ($\because x^2 + x + 1 = 0$ has discriminant $D < 0$ and therefore no real roots).
$\therefore\ x^3 + 3x^2 + 3x + 2 = 0$ has one real root, -2.

4. For $x \in R$, $x^4 + x^2 - 2 = 0$
$\leftrightarrow \quad (x^2 + 2)(x^2 - 1) = 0$
$\leftrightarrow \quad x^2 + 2 = 0$ or $x^2 - 1 = 0$
$\leftrightarrow \quad x = -1$ or $x = 1$.
$\therefore\ x^4 + x^2 - 2 = 0$ has two real roots, -1, 1.

5. $x^4 + x^3 - 7x^2 - x + 6 = (x - 1)(x + 1)(x^2 + x - 6)$ (Factor Theorem)
$= (x - 1)(x + 1)(x + 3)(x - 2)$.
For $x \in R$, $x^4 + x^3 - 7x^2 - x + 6 = 0$
$\leftrightarrow \quad (x - 1)(x + 1)(x + 3)(x - 2) = 0$
$\leftrightarrow \quad x = 1$ or $x = -1$ or $x = -3$ or $x = 2$.
$\therefore\ x^4 + x^3 - 7x^2 - x + 6 = 0$ has four real roots, 1, -1, -3, 2.

6. $x^5 - x^4 + 2x^3 - 2x^2 + 3x - 3 = x^4(x - 1) + 2x^2(x - 1) + 3(x - 1)$
$= (x^4 + 2x^2 + 3)(x - 1)$.
For $x \in R$, $x^5 - x^4 + 2x^3 - 2x^2 + 3x - 3 = 0$
$\leftrightarrow \quad (x^4 + 2x^2 + 3)(x - 1) = 0$
$\leftrightarrow \quad x^4 + 2x^2 + 3 = 0$ or $x - 1 = 0$
$\leftrightarrow \quad x - 1 = 0$ ($\because x^4 + 2x^2 + 3 \geqq 3$ for all $x \in R$).
$\leftrightarrow \quad x = 1$.
$\therefore\ x^5 - x^4 + 2x^3 - 2x^2 + 3x - 3 = 0$ has one real root, 1.

7. $x^5 - 2x^4 - 8x^3 - x^2 + 2x + 8 = x^3(x^2 - 2x - 8) - (x^2 - 2x - 8)$
$= (x^3 - 1)(x^2 - 2x - 8)$
$= (x - 1)(x^2 + x + 1)(x + 2)(x - 4)$.

For $x \in R$, $x^5 - 2x^4 - 8x^3 - x^2 + 2x + 8 = 0$
$\leftrightarrow \quad (x - 1)(x + 2)(x - 4)(x^2 + x + 1) = 0$
$\leftrightarrow \quad (x - 1) = 0$ or $x + 2 = 0$ or $x - 4 = 0$ or $x^2 + x + 1 = 0$
$\leftrightarrow \quad x = 1$ or $x = -2$ or $x = 4$.
$\therefore$ $x^5 - 2x^4 - 8x^3 - x^2 + 2x + 8 = 0$ has three real roots, 1, -2, 4.

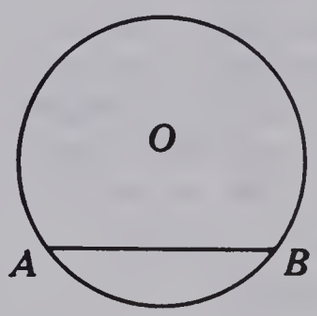
8. $x^5 - 3x^4 - 3x^3 + 12x^2 - 4x = (x^4 - 3x^3 - 3x^2 + 12x - 4)x$
$= (x - 2)(x + 2)(x^2 - 3x + 1)x$ (Factor Theorem)
For $x \in R$, $x^5 - 3x^4 - 3x^3 + 12x^2 - 4x = 0$
$\leftrightarrow \quad (x - 2)(x + 2)(x^2 - 3x + 1)x = 0$
$\leftrightarrow \quad x - 2 = 0$ or $x + 2 = 0$ or $x^2 - 3x + 1 = 0$ or $x = 0$
$\leftrightarrow \quad x = 2$ or $x = -2$ or $x = \frac{1}{2}(3 \pm \sqrt{5})$ or $x = 0$.
$\therefore$ $x^5 - 3x^4 - 3x^2 + 12x^2 - 4x = 0$ has the five real roots
$0, 2, -2, \frac{1}{2}(3 + \sqrt{5}), \frac{1}{2}(3 - \sqrt{5})$.

9. (i) three real roots; (ii) four real roots; (iii) five real roots.

Chapter 11

Section 11.2 (page 288).

1.



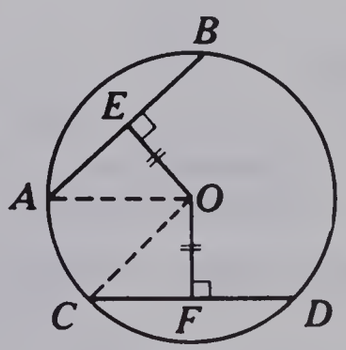
Hypothesis: AB is a chord of a circle with centre O.

Conclusion: O is in the right bisector of AB.

Proof:

STATEMENTS	AUTHORITIES
1. $OA = OB$	1. Definition of a circle
2. O is in the right bisector of AB.	2. Right Bisector Theorem.

4.



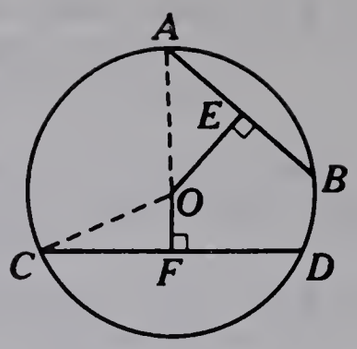
Hypothesis: AB and CD are chords of a circle, centre O.
$OE \perp AB$, $OF \perp CD$, $OE = OF$.

Conclusion: $AB = CD$.

Proof:

STATEMENTS	AUTHORITIES
1. In $\triangle$'s OAE and OCF $\left\{ \begin{array}{c} OA = OC \\ OE = OF \\ \angle OEA = \angle OFC \end{array} \right.$	1. Definition 2. Hypothesis 3. Definition
4. $\triangle OAE \cong \triangle OCF$	4. Right Triangle Congruence Th.
5. $AE = CF$	5. Definition
6. $AE = \frac{1}{2}AB$	6. Chord Property
7. $CF = \frac{1}{2}CD$	7. Chord Property
8. $\frac{1}{2}AB = \frac{1}{2}CD$	8. Replacement
9. $AB = CD$	9. Multiplication

6.



Hypothesis: AB and CD are chords of a circle, centre O.
$OE \perp AB$, $OF \perp CD$, $OF < OE$.

Conclusion: $CD > AB$.

Proof:

STATEMENTS	AUTHORITIES
1. $OC^2 = OF^2 + CF^2$	1. Pythagorean Theorem
2. $OA^2 = OE^2 + AE^2$	2. Pythagorean Theorem
3. $OC = OA$	3. Definition
4. $OC^2 = OA^2$	4. Multiplication
5. $OF^2 + CF^2 = OE^2 + AE^2$	5. Replacement
6. $OF < OE$	6. Hypothesis
7. $OF^2 < OE^2$	7. Multiplication
8. $CF^2 > AE^2$	8. Replacement 5
9. $CF > AE$	9. Square Root
10. $CF = \frac{1}{2}CD$, $AE = \frac{1}{2}AB$	10. Chord Property
11. $\frac{1}{2}CD > \frac{1}{2}AB$	11. Replacement
12. $CD > AB$	12. Multiplication

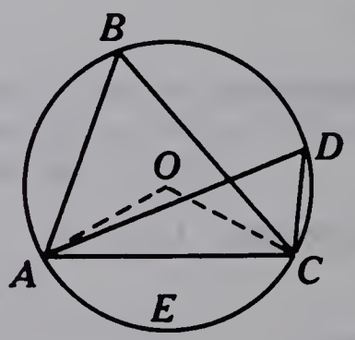
Section 11.5 (page 293).

1. $y = 2(40)$,
$\therefore y = 80$.
$x = \frac{1}{2}y$,
$\therefore x = 40$.

2. $x = 2(70)$,
$= 140$.
$z = 360 - 140$
$= 220$.
$y = \frac{1}{2}z$
$= 110$.

3. $x = \frac{1}{2}(180)$
$= 90$.

Corollary 1.



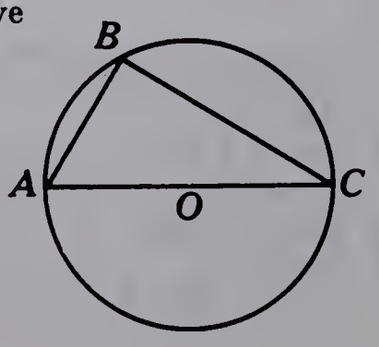
Hypothesis: $\angle ABC$ and $\angle ADC$ are inscribed angles of circle with centre O, and are subtended by the same arc AEC.

Conclusion: $\angle ABC = \angle ADC$.

Proof:

STATEMENTS	AUTHORITIES
1. $\angle ABC = \frac{1}{2}\angle AOC$	1. Inscribed Angle Theorem
2. $\angle ADC = \frac{1}{2}\angle AOC$	2. Inscribed Angle Theorem
3. $\angle ABC = \angle ADC$	3. Transitive

Corollary 2.



Hypothesis: $\angle ABC$ is inscribed in a semi-circle with diameter AOC.

Conclusion: $\angle ABC = 90°$.

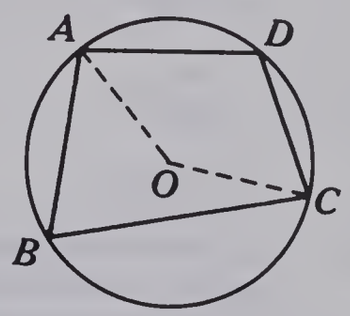
Proof:

STATEMENTS	AUTHORITIES
1. $\angle ABC = \frac{1}{2}\angle AOC$	1. Inscribed Angle Theorem
2. $\angle AOC = 180°$	2. Definition
3. $\angle ABC = 90°$	3. Division
4. $\angle ABC$ is a right angle.	4. Definition

Corollary 3.

Hypothesis: Quadrilateral $ABCD$ inscribed in a circle, centre O.

Conclusion: $\angle B + \angle D = 180°$ and $\angle BAD + \angle DCB = 180°$.



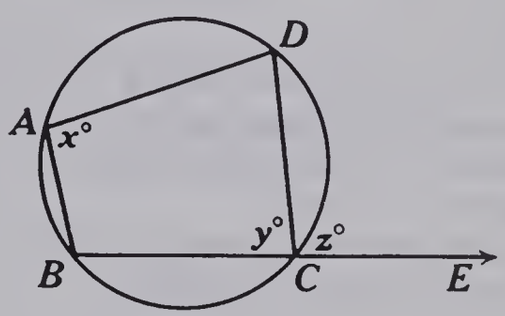
Proof:

STATEMENTS	AUTHORITIES
1. $\angle B = \frac{1}{2}\angle AOC$	1. Inscribed Angle Theorem
2. $\angle D = \frac{1}{2}$ reflex $\angle AOC$	2. Inscribed Angle Theorem
3. $\angle B + \angle D = \frac{1}{2}\angle AOC + \frac{1}{2}$ reflex $\angle AOC$	3. Addition
4. $= \frac{1}{2}(\angle AOC +$ reflex $\angle AOC)$	4. Distributive
5. $\angle B + \angle D = \frac{1}{2}(360°)$ $= 180°$	5. Completion
6. In a similar manner $\angle BAD + \angle DCB = 180°$.	

Corollary 4.

Hypothesis: $\angle DCE$ is an exterior angle of inscribed quadrilateral $ABCD$.

Conclusion: $\angle DCE = \angle A$ (or $z = x$).



Proof:

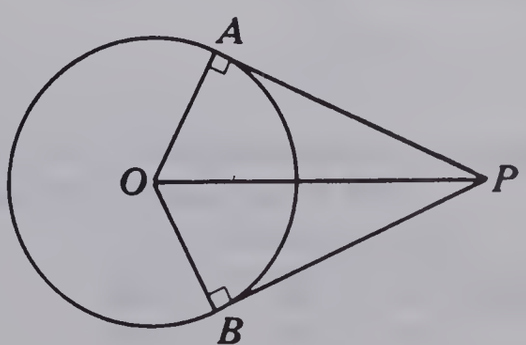
STATEMENTS	AUTHORITIES
1. $x + y = 180°$	1. Corollary 3
2. $y + z = 180°$	2. Definition
3. $x + y = y + z$	3. Addition
4. $x = z$	4. Subtraction

Section 11.8 (page 301).

1.

Hypothesis: PA and PB are tangent segments from P to the circle with centre O.

Conclusion: (i) $PA = PB$.
(ii) $\angle APO = \angle BPO$.
(iii) $\angle AOP = \angle BOP$.



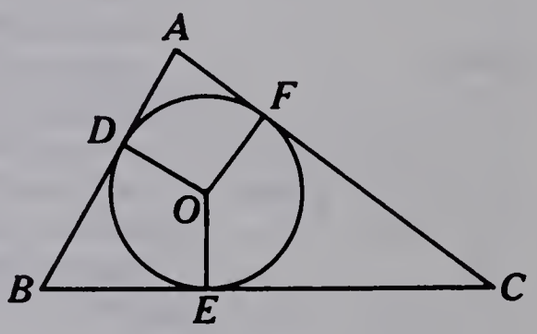
Proof:

STATEMENTS	AUTHORITIES
1. In $\triangle$'s PAO and PBO $\left\{ \begin{array}{c} PO = PO \\ OA = OB \\ \angle OAP = \angle OBP = 90° \end{array} \right.$	1. Reflexive 2. Radii 3. Tangent Property
4. $\triangle PAO \cong \triangle PBO$	4. Right Triangle Congruence Theorem
5. $PA = PB$	5. Definition
6. $\angle APO = \angle BPO$	6. Definition
7. $\angle AOP = \angle BOP$	7. Definition

3.

Hypothesis: Circle with centre O is inscribed circle of $\triangle ABC$. AB, BC, and CA are tangent to the circle at D, E, and F respectively.

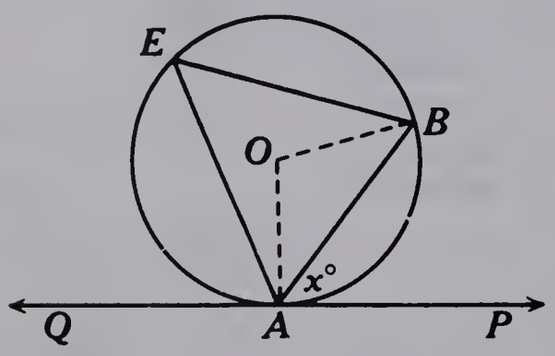
Conclusion: O is the point of intersection of the bisectors of two of the angles of $\triangle ABC$.



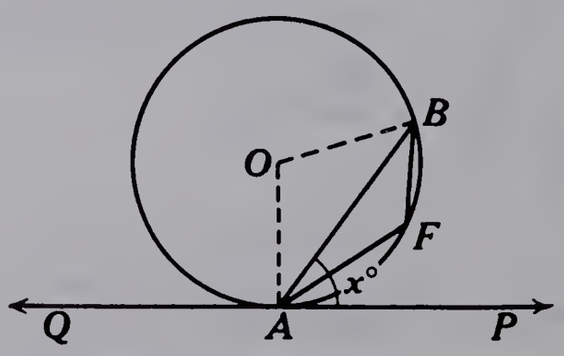
Proof:

STATEMENTS	AUTHORITIES
1. $OD \perp AB$	1. Tangent Property
2. $OE \perp BC$	2. Tangent Property
3. $OD = OE$	3. Radii
4. O is a point on the bisector of $\angle ABC$	4. Angle Bisector Theorem
5. Similarly O is a point on the bisector of $\angle BCA$ and on the bisector of $\angle CAB$.	5. Angle Bisector Theorem and Tangent Property
6. O is the point of intersection of the bisectors of two of the angles.	6. Two intersecting lines determine one and only one point.

5.



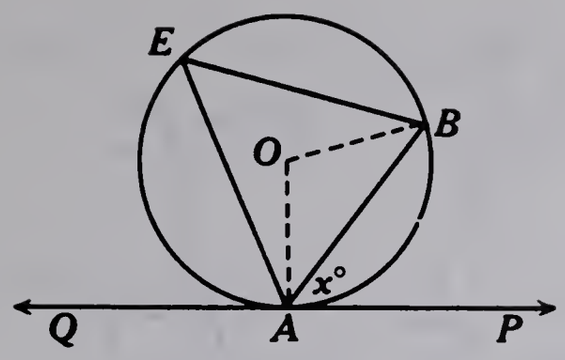
(*a*)



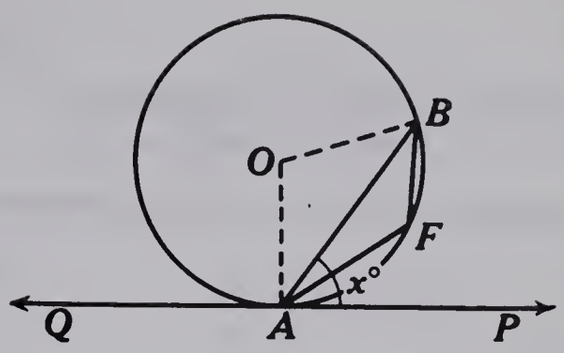
(*b*)

Hypothesis: PAQ is the tangent at A of the circle with centre O. AB is any chord. $\angle AEB$ is any angle in the major segment and $\angle AFB$ is any angle in the minor segment.

Conclusion: (a) $\angle BAP = \angle AEB$.
(b) $\angle BAQ = \angle AFB$.



(a)



(b)

Proof:

STATEMENTS	AUTHORITIES
1. $\angle OAP = 90°$	1. Tangent Property
2. $\angle OAB = (90 - x)°$	2. Definition
3. $\angle OBA = (90 - x)°$	3. Isosceles Triangle Theorem
(a) 4. $\angle AOB = [180 - 2(90 - x)]°$ $= (2x)°$	4. Triangle Angle Sum Theorem Number Postulates
5. $\angle BAP = \frac{\angle AOB}{2}$	5. Replacement and Number Postulates
6. $\angle AEB = \frac{\angle AOB}{2}$	6. Inscribed Angle Theorem
7. $\angle BAP = \angle AEB$	7. Replacement
(b) 8. Reflex $\angle AOB = (360 - 2x)°$	8. Definition
9. $= [2(180 - x)]°$	9. Number Postulate (D)
10. $= 2\angle BAQ$	10. Replacement
11. $\angle BAQ = \frac{\text{reflex } \angle AOB}{2}$	11. Number Postulate
12. $\angle BFA = \frac{1}{2}$ reflex $\angle AOB$	12. Inscribed Angle Theorem
13. $\angle BAQ = \angle AFB$	13. Replacement

Section 11.9 (page 309).

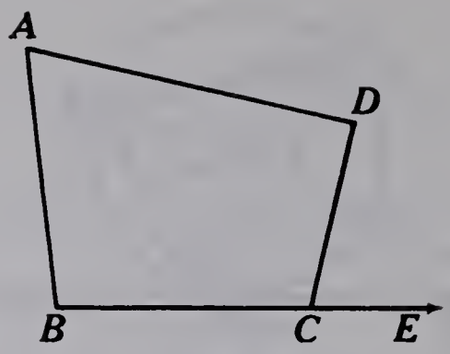
1. Proof:

STATEMENTS	AUTHORITIES
1. $\angle ADB = 90°$	1. Angle inscribed in a semicircle
2. $\angle ACB = 90°$	2. Definition, hypothesis
3. $\angle ADB = \angle ACB$	3. Replacement
4. A, D, C, B are concyclic points	4. Basic Concyclic Point Theorem

2. Proof:

STATEMENTS	AUTHORITIES
1. $\angle B + \angle D = 180°$	1. Hypothesis
2. $\angle B + \angle E = 180°$	2. Cyclic quadrilateral
3. $\angle B + \angle D = \angle B + \angle E$	3. Replacement
4. $\angle D = \angle E$	4. Subtraction
5. A, E, D, C are concyclic points.	5. Basic Concyclic Point Theorem

3.



Hypothesis: $ABCD$ is a quadrilateral in which $\angle DCE = \angle A$.

Conclusion: $ABCD$ is a cyclic quadrilateral.

Proof:

STATEMENTS	AUTHORITIES
1. $\angle DCE = \angle A$	1. Hypothesis
2. $\angle DCE + \angle DCB = 180°$	2. Definition
3. $\angle A + \angle DCB = 180°$	3. Replacement
4. $ABCD$ is a cyclic quadrilateral	4. Pair of interior opposite angles supplementary.

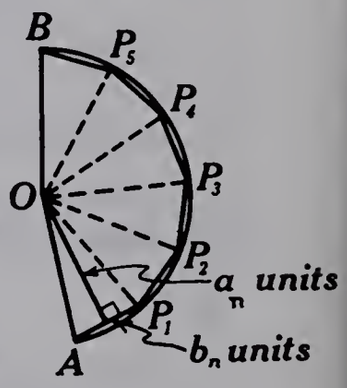
Section **11.14** (page 316).

1. Consider the polygonal figure $OAP_1P_2 \ldots P_{n-1}B$, in which

(i) $OA = OB = r$ units;

(ii) $AP_1 = P_1P_2 = \ldots = P_{n-1}B = b_n$ units;

(iii) each congruent triangle OAP_1, OP_1P_2, etc. has altitude a_n units.



Let A_n square units be the area of polygon $OAP_1P_2 \ldots P_{n-1}B$;

A_s square units be the area of sector OAB;

l units be the length of the arc AB;

then, as n the number of triangles is increased indefinitely

$$A_n = \tfrac{1}{2}a_nb_n$$

a_n approaches r,

nb_n approaches l,

$\therefore$ A_n approaches $\tfrac{1}{2}rl$.

A_n approaches A_s, ($\because$ A_s is the limit of A_n)

$\therefore$ $A_s = \tfrac{1}{2}rl$.

$\therefore$ the area of a sector is one half the product of its radius by the length of its arc.

2. From question 1. $A_s = \tfrac{1}{2}rl$.

But $l = \dfrac{\pi mr}{180}$.

$$\therefore A_s = \tfrac{1}{2}\,\frac{r\pi mr}{180}$$

$$= \frac{m\pi r^2}{360}.$$

Section **11.20** (page 338).

Example 2. *Verification.*

L.S. (1) $= 2\left(\dfrac{2 + 2\sqrt{31}}{5}\right) - \left(\dfrac{-1 + 4\sqrt{31}}{5}\right)$ R.S. (1) = 1.

$$= \frac{4 + 4\sqrt{31}}{5} - \frac{-1 + 4\sqrt{31}}{5}$$

$$= 1.$$

L.S. (1) $= 2\left(\frac{2 - 2\sqrt{31}}{5}\right) - \left(\frac{-1 - 4\sqrt{31}}{5}\right)$ R.S. (1) = 1.

$= \frac{4 - 4\sqrt{31}}{5} - \frac{-1 - 4\sqrt{31}}{5}$

$= 1.$

L.S. (2) $= \left(\frac{2 + 2\sqrt{31}}{5}\right)^2 + \left(\frac{-1 + 4\sqrt{31}}{5}\right)^2$ R.S. (2) = 25.

$= \frac{128 + 8\sqrt{31}}{25} + \frac{497 - 8\sqrt{31}}{25}$

$= 25.$

L.S. (2) $= \left(\frac{2 - 2\sqrt{31}}{5}\right)^2 + \left(\frac{-1 - 4\sqrt{31}}{5}\right)^2$ R.S. (2) = 25.

$= \frac{128 - 8\sqrt{31}}{25} + \frac{497 + 8\sqrt{31}}{25}$

$= 25.$

Section 11.22 Discovery Exercise 11-16 (page 342).

1. (i) Slope of OP is $\frac{3}{4}$.
(ii) Tangent on P is perpendicular to OP.
$\therefore$ slope of tangent on P is $-\frac{4}{3}$.
(iii) Tangent has slope $-\frac{4}{3}$ and is on $P(4, 3)$.
$\therefore$ equation of tangent is $y - 3 = -\frac{4}{3}(x - 4)$
or $4x + 3y = 25$.

2. Each problem is solved in the same manner as 1.

GIVEN POINT OF THE CIRCLE	EQUATION OF THE CIRCLE	EQUATION OF THE TANGENT AT THE GIVEN POINT
$Q(-4, 3)$	$x^2 + y^2 = 25$	$-4x + 3y = 25$
$R(3, -4)$	$x^2 + y^2 = 25$	$3x - 4y = 25$
$S(\sqrt{2}, 3)$	$x^2 + y^2 = 11$	$\sqrt{2}x + 3y = 11$
$T(-\sqrt{2}, \sqrt{5})$	$x^2 + y^2 = 7$	$-\sqrt{2}x + \sqrt{5}y = 7$
$V(\frac{1}{3}, -\sqrt{5})$	$9x^2 + 9y^2 = 46$	$\frac{1}{3}x - \sqrt{5}y = \frac{46}{9}$ or $3x - 9\sqrt{5}y = 46$

3. The equation of the tangent is $x_1x + y_1y = r^2$

Chapter 12

Section 12.12 (page 369).

1. (i) sin 37° = .6018 (ii) sec 80° = 5.7588
(iii) tan 2° = .0349 (iv) csc 29° = 2.0627

2. (i) Since tan 63° = 1.9626
and tan 64° = 2.0503,
and since 2.0000 is closer to 1.9626 than to 2.0503
$\therefore \theta° = 63°$ (to the nearest degree).
(ii) Since cos 36° = .8090
and cos 37° = .7986,
and since 0.8018 is closer to .7986 than to .8090,
$\therefore \alpha° = 37°$ (to the nearest degree).

3. As θ increases in measurement from 0 to $\frac{\pi}{2}$, the trigonometric ratios $\sin\theta$, $\tan\theta$, and $\sec\theta$ increase, while $\cos\theta$, $\csc\theta$, and $\cot\theta$ decrease.

Section 12.13 (page 373).

1. (i) $AB = \sqrt{2^2 + 5^2}$ units
$= \sqrt{4 + 25}$ units
$= \sqrt{29}$ units

$$\sin A = \frac{2}{\sqrt{29}}, \quad \sin B = \frac{5}{\sqrt{29}}$$
$$\cos A = \frac{5}{\sqrt{29}}, \quad \cos B = \frac{2}{\sqrt{29}}$$
$$\tan A = \frac{2}{5}, \quad \tan B = \frac{5}{2}$$
$$\csc A = \frac{\sqrt{29}}{2}, \quad \csc B = \frac{\sqrt{29}}{5}$$
$$\sec A = \frac{\sqrt{29}}{5}, \quad \sec B = \frac{\sqrt{29}}{2}$$
$$\cot A = \frac{5}{2}, \quad \cot B = \frac{2}{5}$$

(ii) $\sin A = \cos B$, $\cos A = \sin B$
$\tan A = \cot B$, $\cot A = \tan B$
$\sec A = \csc B$, $\csc A = \sec B$

2. (i) $LM : KM : KL = \frac{1}{2} : 1 : \frac{\sqrt{3}}{2}$
$= 1 : 2 : \sqrt{3}$

(ii) $\sin K = \frac{1}{2}$, $\sin M = \frac{\sqrt{3}}{2}$
$\cos K = \frac{\sqrt{3}}{2}$, $\cos M = \frac{1}{2}$
$\tan K = \frac{1}{\sqrt{3}}$, $\tan M = \sqrt{3}$
$\csc K = 2$, $\csc M = \frac{2}{\sqrt{3}}$
$\sec K = \frac{2}{\sqrt{3}}$, $\sec M = 2$
$\cot K = \sqrt{3}$, $\cot M = \frac{1}{\sqrt{3}}$

3. If A and B are two complementary angles,
then $\sin A = \cos B$, $\cos A = \sin B$
$\tan A = \cot B$, $\cot A = \tan B$
$\sec A = \csc B$, $\csc A = \sec B$

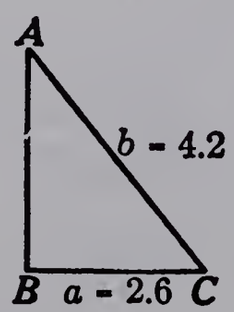
4. In $\triangle RST$,

$$\sin R = \frac{r}{s}, \quad \cos R = \frac{t}{s}, \quad \tan R = \frac{r}{t}, \quad \csc R = \frac{s}{r}, \quad \sec R = \frac{s}{t}, \quad \cot R = \frac{t}{r},$$
$$\sin T = \frac{t}{s}, \quad \cos T = \frac{r}{s}, \quad \tan T = \frac{t}{r}, \quad \csc T = \frac{s}{t}, \quad \sec T = \frac{s}{r}, \quad \cot T = \frac{r}{t}.$$

$\therefore \sin R = \cos T$, $\cos R = \sin T$
$\tan R = \cot T$, $\cot R = \tan T$
$\sec R = \csc T$, $\csc R = \sec T$

Section 12.14 (page 376).

1. (i)



(ii) $c^2 = b^2 - a^2$
$= (4.2)^2 - (2.6)^2$
$= 17.64 - 6.76$
$= 10.88$
$c = \sqrt{10.88}$
$= 3.3$ (to one decimal place)

$$\begin{array}{r|l} & 3.29 \\ \hline & 10.88\,|\,00 \\ & 9 \\ \hline 62 & 1.88 \\ & 1.24 \\ \hline 649 & 64\,|\,00 \\ & 58\,|\,41 \end{array}$$

(iii) $\sin A = \frac{a}{b} = \frac{2.6}{4.2} = .6190$ (to 4 decimal places)

(iv) $\therefore \angle A = 38°$ (to the nearest degree)

(v) $\angle C = 180° - (38 + 90)° = 52°$

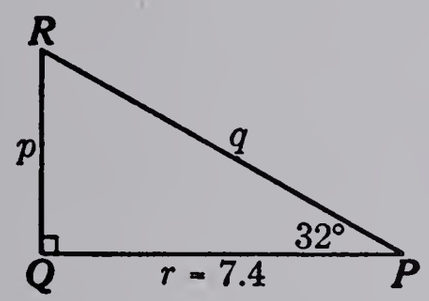
(vi) $\tan C = \tan 52° = 1.2799$

(vii) $c = a \tan C = (2.6)(1.2799) = 3.3$ (to one decimal place)

(viii) Since $\tan A = \frac{a}{c}$,

c could be calculated using the relation $c = \frac{a}{\tan A}$.

2. (i)



(ii) $\tan P = \tan 32°$
$= .6249$

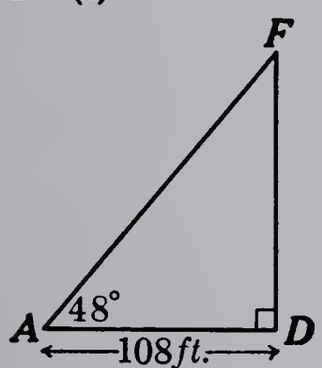
(iii) $p = r \tan P$
$= (7.4)(.6249)$
$= 4.6$ (to one decimal place)

(iv) $q = r \sec P$
$= (7.4)(\sec 32°)$
$= (7.4)(1.1792)$
$= 8.7$ (to one decimal place)

(v) $\angle R = 180° - (90 + 32)°$
$= 58°$

Section 12.15 (page 379).

1. (i)



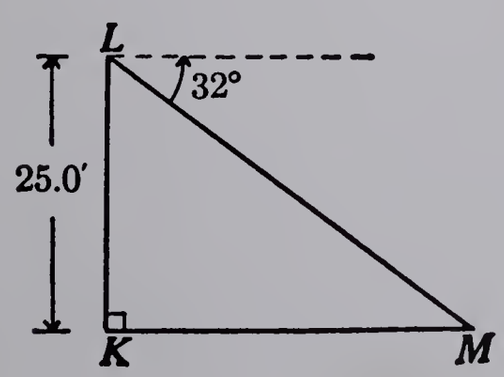
(ii) We are asked to find the height of the flagpole. Represent the height of the flagpole by h ft.

(iii) $\tan A = \frac{h}{AD}$ $\therefore \tan 48° = \frac{h}{AD}$

(iv) $h = AD \cdot \tan 48°$
$= (108)(1.1106)$
$= 119.94$ (to two decimal places)

(v) The height of the flagpole is 120 ft., to the nearest foot.

2. (i)



(ii) Represent the distance of the ship from the base of the lighthouse by x ft.

(iii) In $\triangle KLM$, $\angle K = 90°$
$\angle M = 32°$

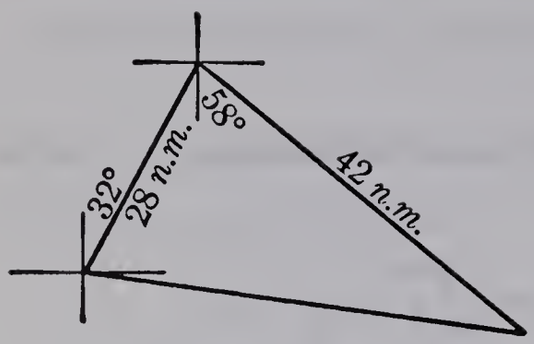
(iv) $\cot M = \frac{KM}{KL}$

$\therefore \cot 32° = \frac{x}{250}$

(v) $\therefore x = (250)(1.6003)$
$= 400.075$ (to 4 decimal places)

(vi) $\therefore$ the distance of the ship from the base of the lighthouse is 400.1 ft. (to one decimal place).

3. (i)



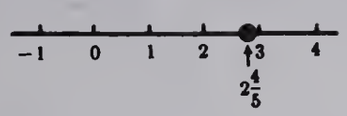
(ii) $\angle PQR = 32° + 58°$
$= 90°$

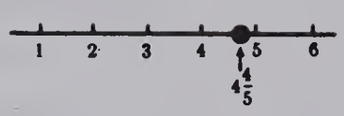
(iii) Represent the distance of the ship from the starting point by x nautical miles.
$\therefore x^2 = (28)^2 + (42)^2$
$= 784 + 1{,}764$
$= 2{,}548$
$x = 50.4$ (to one decimal place)

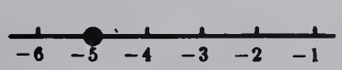
(iv) $\therefore$ the distance of the ship from the starting point is 50 n. miles (to the nearest n. mile).

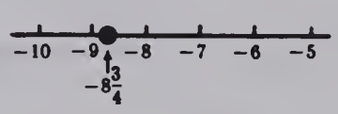
ANSWERS TO EXERCISES

Exercise 1-1 (page 6).

1. $\{2\frac{4}{5}\}$ 

2. $\{4\frac{4}{5}\}$ 

3. $\{-5\}$ 

4. $\{-8\frac{3}{4}\}$ 

5. $\{-13\}$ **6.** $\{-2\}$ **7.** $\{13\frac{5}{11}\}$ **8.** $-\frac{1}{2}$ **9.** $\{x \mid x \in R\}$ **10.** No real solution

11. $\frac{3}{a}$ **12.** 0 **13.** $\frac{b}{a}$ **14.** No solution. **15.** No solution. **16.** R (all real numbers).

17. $\frac{6}{a+b+c}$ if $a+b+c \neq 0$, no solution if $a+b+c=0$

18. $\frac{p}{5m}$ if $m \neq 0$, no solution if $m=0$, $p \neq 0$, R if $p=0$, $m=0$

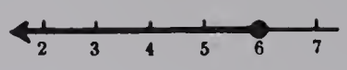
19. $\frac{1}{a+b}$ if $a-b \neq 0$, $a+b \neq 0$; no solution if $a=-b$; R if $a=b$

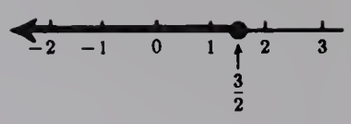
20. $\frac{1}{c+d}$ if $c+d \neq 0$; R if $c+d=0$

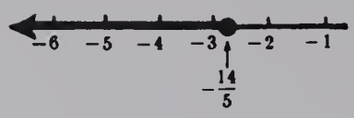
Exercise 1-2 (page 10). **1.** $(x-1)(x-2)(2x+1)$ **2.** $(a-1)(a^2+a+2)$
3. $(x-1)(x^2+x+1)$ **4.** $(x+1)(x^2-x+1)$ **5.** $(x+2)(x-1)^2$
6. $(p-1)(p-7)(p+2)$ **7.** $(x+3)(2x-1)^2$ **8.** $(x+2)(3x-1)(2x+1)$
10. $x=-18$, $y=19$, a^2+5a-1

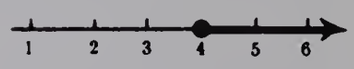
Exercise 1-3 (page 11). **1.** 1 **2.** 1 **3.** -3 **4.** $\frac{3}{2}$ **5.** $\frac{3}{2}$ **6.** $\frac{1}{3}$ **7.** 2, -1 **8.** 3, 2
9. 0, 3 **10.** 1, -1 **11.** 1, -3, 2 **12.** 0, 1, -3 **13.** $\frac{1}{2}$, -2, 3
14. 1, $\frac{1}{3}$, $\frac{3}{2}$ **15.** 3, $-\frac{4}{5}$ **16.** 4 **17.** 1 **18.** 1, -2, 3 **19.** 2, $\frac{15}{4}$
20. $\frac{ab}{a-b}$, if $a-b \neq 0$; no solution if $a=b \neq 0$; R if $a=b=0$

Exercise 1-4 (page 13).

1. $\{x \mid x \leqq 6\}$ 

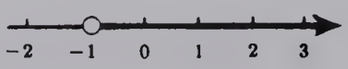
2. $\{x \mid x \leqq \frac{3}{2}\}$ 

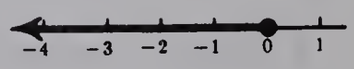
3. $\{x \mid x \leqq -1\frac{4}{5}\}$ 

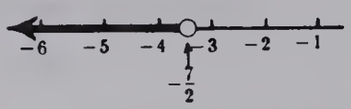
4. $\{x \mid x \geqq 4\}$ 

5. $\{x \mid x \leqq 2\}$

6. $\{x \mid 0 < x \leqq \frac{1}{2}\}$

7. $\{x \mid x > -1\}$ 

8. $\{x \mid x \leqq 0\}$ 

9. $\{x \mid x < -\frac{7}{2}\}$ 

10. $\{x \mid x \geqq 1\}$

Exercise 1-5 (page 14). **1.** $2a^2 + 6a$ **2.** $6x^2y - 10xy^2$ **3.** $10x^2 - 5xy$
4. $2a^2 - ab - b^2$ **5.** $p^2 + 2pq + q^2$ **6.** $x^2 + 2xy + y^2$ **7.** $10x^2 + 11xy - 6y^2$
8. $x^3 + y^3$ **9.** $x^3 + 3x^2y + 3xy^2 + y^3$ **10.** $p^3 + 3p^2q + 3pq^2 + q^3$ **11.** $x^3 - y^3$
12. $x^3 - 3x^2y + 3xy^2 - y^3$ **13.** $4a^2 - 12aq + 9q^2$ **14.** $x^2 - y^2$
15. $8a^3 - 12a^2 + 6a - 1$ **16.** $8a^3 - 27$ **17.** $a^2 + 2ab + b^2 - c^2$
18. $a^2 + b^2 + c^2 + 2ab + 2bc + 2ac$ **19.** $4p^2 + 49q^2 + r^2 + 28pq + 14qr + 4pr$
20. $x^2 + y^2 + z^2 + 2xy + 2xz + 2yz$ **21.** $a^2 + 4b^2 + c^2 - 4ab + 4bc - 2ac$
22. $4a^2 + 9b^2 + c^2 - 12ab + 4ac - 6bc$ **23.** $9y^2$ **24.** $7x^2 - 13y^2$ **25.** $a^2 - 1$
26. $9b^2 - 4$ **27.** $16x^2 - 25y^2$ **28.** $49x^2 - 4y^2$ **29.** $a^2 - b^2 + 4b - 4$
30. $m^2 - 6m + 9 - n^2$ **31.** $4x - 5$ **32.** $\dfrac{x + y}{xy}$ **33.** $\dfrac{ad + bc}{bd}$ **34.** $\dfrac{3x - 2y}{x^2 - xy}$
35. $\dfrac{6p + 2q}{x - y}$ **36.** $\dfrac{-2ab}{a^2 - b^2}$ **37.** $\dfrac{5}{2}$ **38.** $\dfrac{ax + bx + cx + ay + by + cy}{x - y}$

Exercise 1-6 (page 15). **1.** $x(ax + b)$ **2.** $(a + b)(x - y)$ **3.** $(3 - p)(p - 2q)$
4. $(a + b)(m + n)$ **5.** $(3p - 2q)(x - 5y)$ **6.** $(x - 3y)(a + 2b)$ **7.** $(a + b)^2$
8. $(2x + 1)^2$ **9.** $(3x + 2y)^2$ **10.** $(x - y + 1)^2$ **11.** $(x^2 + 2)^2$ **12.** $a(a - 1)^2$
13. $(a - b)(a + b)$ **14.** $(2x - 3y)(2x + 3y)$ **15.** $(a + b - c)(a + b + c)$
16. $(a - b - c)(a + b + c)$ **17.** $(3a - b - 2c)(3a + b + 2c)$

18. $(2x - p + 3q)(2x + p - 3q)$ **19.** $(x + 1)(x + 2)$ **20.** $(3x + 1)(2x + 1)$
21. $(x - 4)(x + 2)$ **22.** $(4a - 5b)(3a - 2b)$ **23.** $(x - y - 2)(x - y + 1)$
24. $(a + b + 4)(a + b - 2)$ **25.** $(x^2 + x + 1)(x^2 - x + 1)$
26. $(a^2 + 2a + 3)(a^2 - 2a + 3)$ **27.** $(a^2 - 5a + 3)(a^2 + 5a + 3)$
28. $(2x^2 + 2x + 1)(2x^2 - 2x + 1)$ **29.** $(3a^2 - 4ab + 4b^2)(3a^2 + 4ab + 4b^2)$
30. $(x^2y^2 - 3xy - 3)(x^2y^2 + 3xy - 3)$ **31.** $(x - 2)(x^2 + x + 1)$
32. $(x - 3)(3x - 1)(x - 4)$ **33.** $(2a - b)(5a - b)(a + b)$
34. $(x - 1)(x - 7)(x + 2)$ **35.** $(x + y)(2x + y)(x + y)$
36. $(x - 1)(2x^2 + 17x + 1)$ **37.** 5 **38.** $(2a - b)(4a^2 + 2ab + b^2)$
39. $(3a + 5b)(9a^2 - 15ab + 25b^2)$ **40.** $4(a + 3)(a^2 - 3a + 9)$
41. $(x^2 + y^2)(x^4 - x^2y^2 + y^4)$ **42.** $y(3x^2 - 3xy + y^2)$ **43.** $(4 - a)(7a^2 - 20a + 16)$
44. $(2a - 5b)^2$ **45.** $(a - 2y)(a - 2y - 4)$ **46.** $(x^2 - 4xy - y^2)(x^2 + 4xy - y^2)$
47. $(a - 1)(a^2 - 10a - 3)$ **48.** $xy(x - y)^2(x + y)$ **49.** $4(3x - 5)(9x^2 + 15x + 25)$
50. $(x - y)(x + y)(x^2 + xy + y^2)(x^2 - xy + y^2)$
51. $\left(x - y + \frac{1}{x} + \frac{1}{y}\right)\left(x + y + \frac{1}{x} - \frac{1}{y}\right)$ **52.** $(2x^2 - 5x - 1)(2x^2 + 5x - 1)$
53. $(a - 5)(a^2 - a - 1)$ **54.** $(a - 1)(a + 1)(x + 1)(x^2 - x + 1)$
55. $(p - 2b)(p + b)(p - 3b)(p + 2b)$ **56.** $(3p - 5q)(5p + 12q)$
57. $(x - 1)(8x^3 + 7x^2 + 8x + 8)$ **58.** $(a - 3b)(a + 3b)(a^2 + 2b^2)$
59. $(5x + 4z)(3y - 5x + 4z)$ **60.** $(x - 3)(x - 2)(x - 4)(x - 1)$ **61.** $(a + b + c)^2$
62. $\frac{2x - x^2}{x - 1}$ **63.** $\frac{2x + 5}{x(x - 1)}$ **64.** $\frac{y + 2}{2(y - 5)}$ **65.** $\frac{9a - 15}{(a - 3)(a + 3)(a - 1)}$
66. $x + y$ **67.** $\frac{x - y}{x + y}$ **68.** $\frac{a^2}{a^2 - 1}$

Exercise 2-3 (page 26).

71. $|x + 3|$ **72.** $|x - 3|$ **73.** $|2x + 1|$
74. $|3x + 2|$ **75.** $|2x - 5|$ **76.** $\left|\frac{x}{2} - \frac{1}{4}\right|$
77. $|5 + 3x|$ **78.** $|7x - 1|$ **79.** $|8x + 3|$
80. $\sqrt{a^2b}$ **81.** $\sqrt{9c}$ **82.** $\sqrt{27x^2y}$
83. $\sqrt{3(a - b)^2}$ **84.** $\sqrt{2(a + b)^2}$ **85.** $\sqrt{(x + 3)^2y}$
86. $\sqrt{(a - b)^2(a + b)}$ **87.** $\sqrt{5(2x + 4)^2}$ **88.** $\sqrt{z(3x + y)^2}$
89. $3\sqrt{11}$ **90.** $|a|\sqrt{3}$ **91.** $|x - 1|\sqrt{2}$
92. $|x + 8|\sqrt{3}$ **93.** $|x + 1|\sqrt{5}$ **94.** $|b - 4|\sqrt{a}$
95. $\left|\frac{a - b}{a + b}\right|$ **96.** $2|a + ab|\sqrt{3}$ **97.** $|x + 1|\sqrt{3}$
98. $|xy|\sqrt{6}$ **99.** $x^2\sqrt{x}$ **100.** $12\sqrt{3}$
101. 24 **102.** $|b|\sqrt{3}$ **103.** 60
104. 12 **105.** 6 **106.** $2\sqrt{10}$
107. $450\sqrt{2}$ **108.** $84\sqrt{2}$ **109.** 1,000
110. $\frac{2\sqrt{2}}{9}$ **111.** $\frac{4\sqrt{5}}{3}$ **112.** $\sqrt{2}$

Exercise 2-4 (page 29).

16. $2\sqrt{2}$ 17. $11\sqrt{2}$ 18. $22\sqrt{2}$
19. $10\sqrt{3}$ 20. $-4\sqrt{5}$ 21. $4\sqrt{3}$
22. $4\sqrt{7}$ 23. $x(\sqrt{2}-1)$ 24. $17\sqrt{2}$
25. $\frac{6}{\sqrt{5}}$ 26. $\frac{19}{12}\sqrt{5}$ 27. $-\sqrt{10}$
28. $\frac{5}{2}\sqrt{2}$ 29. $3\sqrt{b}$ 30. $3\,|a|\sqrt{2}$
31. $2\sqrt{6}$ 32. $12\sqrt{6}-9\sqrt{2}$ 33. $4\sqrt{2}$
34. $14\sqrt{2}$ 35. $25\sqrt{2}$ 36. $4\sqrt{2}$
37. $9\sqrt{6}$ 38. $2\sqrt{5}+4\sqrt{2}$ 39. $38\sqrt{2}$ 40. $1+\frac{4\sqrt{3}}{3}$

Exercise 2-5 (page 30).

29. $\sqrt{2}$ 30. $\frac{\sqrt{15}}{3}$ 31. $\frac{\sqrt{6}}{3}$ 32. $\sqrt{b}$ 33. $\frac{\sqrt{6}}{5}$ 34. $8\sqrt{x}$
35. $\frac{2\sqrt{5}}{5}$ 36. $\frac{\sqrt{77}}{7}$ 37. $\sqrt{2}$ 38. $\frac{\sqrt{15}}{2}$ 39. $\frac{\sqrt{15}}{50}$ 40. $\frac{\sqrt{70}}{14}$
41. $\frac{3\sqrt{2}}{2}$ 42. $\frac{\sqrt{2}}{2}$ 43. $\sqrt{10}$ 44. $\frac{6\sqrt{2}}{5}$ 45. 0.71 46. 0.89
47. 0.28 48. 2.24 49. 1.06

Exercise 2-6 (page 32).

1. $2+2\sqrt{2}$ 2. $6-2\sqrt{3}$ 3. $\sqrt{5}+10$ 4. $5\sqrt{2}+2$
5. $2\sqrt{6}+2\sqrt{15}$ 6. $2\sqrt{10}-12$ 7. $2\sqrt{15}+5$ 8. 1
9. 1 10. $11+6\sqrt{2}$ 11. $5-2\sqrt{6}$ 12. -2
13. $7+4\sqrt{3}$ 14. $8+4\sqrt{3}$ 15. 3 16. $\frac{1}{3}$
17. $\frac{12+2\sqrt{11}}{4}$ 18. $\frac{7-4\sqrt{3}}{4}$ 19. -22 20. $10\sqrt{6}$
21. $66-24\sqrt{6}$ 22. $9x+y-6\sqrt{xy}$ 23. $93-30\sqrt{6}$ 24. $2\sqrt{7}-28$
25. $x-1$ 26. 44 27. $x+y-2\sqrt{xy}$ 28. $a+b+2\sqrt{ab}$
29. $\sqrt{7}$ 30. $\frac{7}{y}$ 31. $19+5\sqrt{10}$ 32. $6x-3y+7\sqrt{xy}$
33. $74-10\sqrt{35}$ 34. $\sqrt{ac}+\sqrt{ad}+\sqrt{bc}+\sqrt{bd}$
35. $7\sqrt{55}-3$ 36. $22+10\sqrt{5}$

Exercise 2-7 (page 34).

1. $\sqrt{2}-1$ 2. $\frac{3+\sqrt{3}}{2}$ 3. $6\sqrt{5}-12$ 4. $\frac{15+5\sqrt{7}}{2}$
5. $\frac{2x+2\sqrt{y}}{x^2-y}$ 6. $\frac{2\sqrt{5}+\sqrt{30}}{-2}$ 7. $\frac{\sqrt{35}+3\sqrt{7}}{-4}$ 8. $\frac{2\sqrt{21}-4\sqrt{3}}{3}$
9. $\frac{2\sqrt{2}+3}{-1}$ 10. $3\sqrt{3}-3\sqrt{2}$ 11. $\frac{5-\sqrt{10}}{3}$ 12. $2\sqrt{6}+\sqrt{21}$

13. $\dfrac{5+2\sqrt{6}}{-1}$ **14.** $\dfrac{21-8\sqrt{5}}{11}$ **15.** $4+\sqrt{15}$ **16.** $\dfrac{11-4\sqrt{6}}{-5}$

17. -1 **18.** $\dfrac{15-2\sqrt{10}+2\sqrt{14}-3\sqrt{35}}{2}$

19. $\dfrac{187-40\sqrt{21}}{37}$ **20.** $\dfrac{12\sqrt{6}-35}{19}$

21. $\dfrac{7\sqrt{15}-27}{2}$ **22.** $\dfrac{x^2+y^2z+2xy\sqrt{z}}{x^2-y^2z}$

23. $\dfrac{a^2+7-4\sqrt{a^2+3}}{a^2-1}$ **24.** $\dfrac{a+b-2\sqrt{ab}}{a-b}$

25. $x+\sqrt{x^2-1}$ **26.** $y\sqrt{a}-y\sqrt{b}\ (a\neq b)$

27. $\dfrac{(a-3\sqrt{b})\sqrt{b+a}}{b+a}$ **28.** $\dfrac{2+3\sqrt{2}+2\sqrt{3}+\sqrt{6}}{2}$

Exercise 2-8 (page 36).

1. 18 **2.** no roots **3.** 22 **4.** $\frac{55}{9}$ **5.** 5 **6.** 4

7. no roots **8.** 11 **9.** no roots **10.** 2 **11.** 9 **12.** 1

Exercise 2-9 (page 37).

1. 8 **2.** no roots **3.** no roots **4.** 1 **5.** 0 **6.** 4

7. 6 **8.** 9 **9.** 12 **10.** 2 **11.** $\frac{6}{5}$ **12.** 4

13. 6 **14.** $\dfrac{ab}{a+b}$

Exercise 3-1 (page 39).

1. (Ted, George), (Ted, Andy), (Ted, Michael), (Ted, Martin), (George, Andy), (George, Michael), (George, Martin), (Andy, Michael), (Andy, Martin), (Michael, Martin)

2. (Aluminum, Diamond), (Aluminum, Iron), (Aluminum, Copper), (Aluminum, Gold), (Diamond, Iron), (Diamond, Copper), (Diamond, Gold), (Iron, Copper), (Iron, Gold), (Copper, Gold)

3. $R_1 = \{(-1, -3), (1, 3), (0, 0)\}$
$R_2 = \{(-3, -3), (-1, -1), (0, 0), (1, 1), (3, 3), (5, 5)\}$
$R_3 = \{(0, 0), (-1, 1), (1, 1)\}$
$R_4 = \{(-3, -1), (-1, 1), (1, 3), (3, 5)\}$
$R_5 = \{(-3, 5), (-1, 5), (0, 5), (1, 5), (3, 5), (-3, 3), (-1, 3), (0, 3), (1, 3), (-3, 1), (-1, 1), (0, 1), (-3, 0), (-1, 0), (-3, -1)\}$

4. $P_1 = \{(3, 0), (3, \frac{1}{3}), (3, \frac{1}{2}), (3, 1), (3, 2), (2, 0), (2, \frac{1}{3}), (2, \frac{1}{2}), (2, 1), (1, 0), (1, \frac{1}{3}), (1, \frac{1}{2}), (\frac{1}{2}, 0), (\frac{1}{2}, \frac{1}{3}), (\frac{1}{3}, 0)\}$

$P_2 = \{(\frac{1}{3}, 3), (3, \frac{1}{3}), (\frac{1}{2}, 2), (2, \frac{1}{2}), (1, 1)\}$ $P_3 = \{(0, 1), (\frac{1}{2}, 2), (1, 3)\}$

Exercise 3-2 (page 41).

6. $\{(2, 2), (2, 3), (2, 4), (3, 2), (3, 3), (3, 4), (4, 2), (4, 3), (4, 4)\}$

7. $\{(5, 4), (5, 6), (5, 8), (7, 4), (7, 6), (7, 8)\}$

8. (i) $B \times A = \{(-1, 1), (-1, 3), (-1, 5), (-2, 1), (-2, 3), (-2, 5)\}$
$A \times A = \{(1, 1), (1, 3), (1, 5), (3, 1), (3, 3), (3, 5), (5, 1), (5, 3), (5, 5)\}$
(ii) 6 elements in $A \times B$, 6 in $B \times A$, 9 in $A \times A$

9. mn, mn, m^2, n^2

10. Infinite

11. (i) 20; 20; 25 (ii) 0, 2, 4, 6, 8; 1, 3, 5, 9 (iii) 4

12. $\{(-2, -2), (-2, 0), (-2, 2), (0, -2), (0, 0), (0, 2), (2, -2), (2, 0), (2, 2)\}$

13. $\{(1, 2), (2, 3), (3, 4), (4, 5), (5, 6), (6, 7), (7, 8), (8, 9)\}$

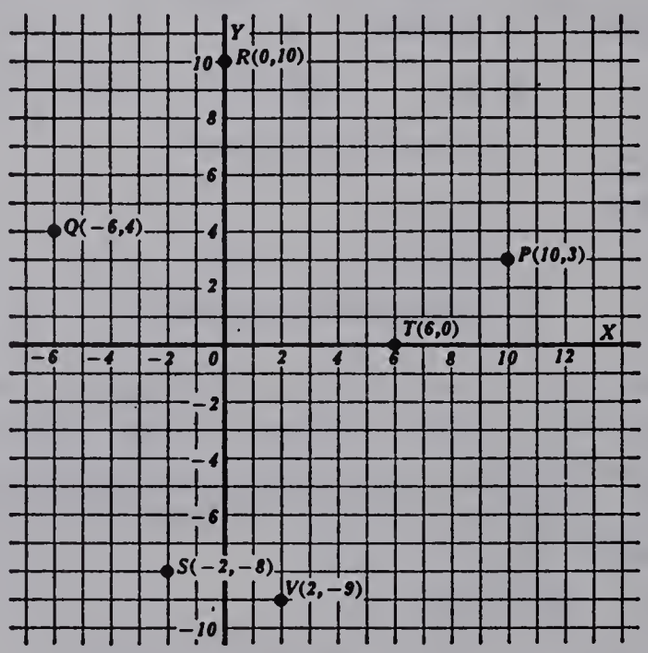
14. $\{(1, 2), (2, 4), (3, 6), (4, 8), \ldots\}$ **15.** $\{(1, 1)\}$

16. (Ham., Mont.), (Ham., Ott.), (Ham., Tor.), (Mont., Ham.), (Mont., Ott.), (Mont., Tor.), (Ott., Ham.), (Ott., Mont.), (Ott., Tor.), (Tor., Ham.), (Tor., Mont.), (Tor., Ott.)

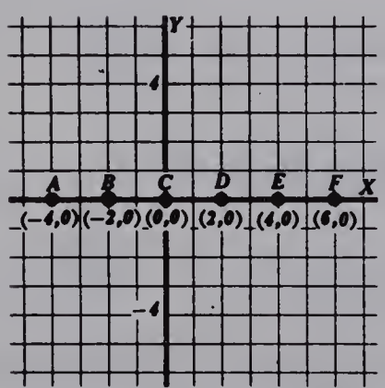
Exercise 3-3 (page 46).

2. $A(2, 3)$, $B(-3, 3)$, $C(-2, -1)$, $D(4, -3)$, $E(3, 0)$, $F(0, 2)$, $G(-4, -3)$, $H(0, -3)$, $I(4, 2)$, $J(-5, 3)$, $K(-1, 0)$, $L(3, -2)$, $M(1\frac{1}{2}, 1\frac{1}{2})$, $N(-2\frac{1}{2}, 1)$, $Q(-4, -1\frac{1}{2})$

3.

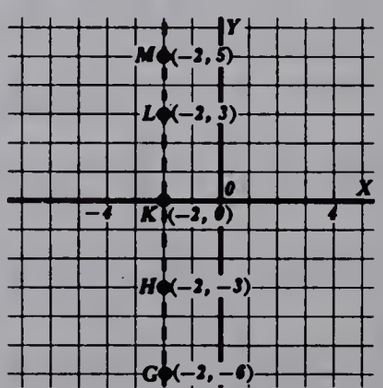


4. (i)



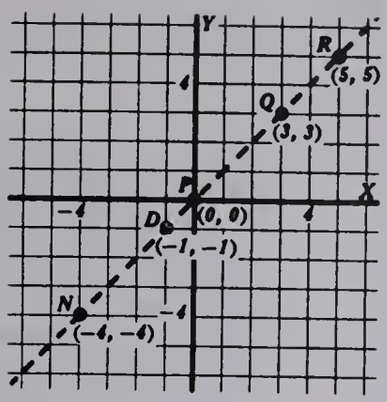
(ii) the x-axis

5. (i)



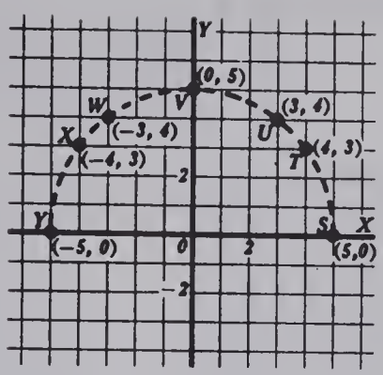
(ii) A line parallel to the y-axis 2 units to the left of it.

6. (i)



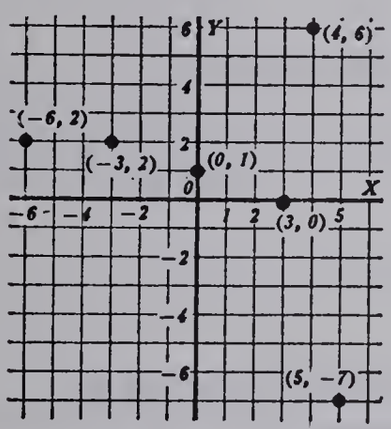
(ii) A line which bisects the angle YOX

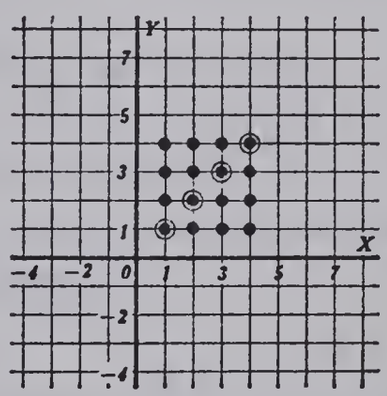
7. (i)



(ii) A semicircle, centre (0,0) and radius 5

8.





9.

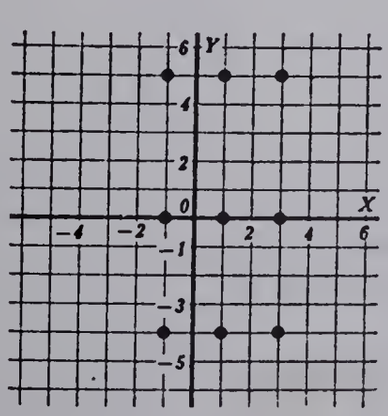
(i) $A \times A = \{(1, 1), (1, 2), (1, 3), (1, 4),$
$(2, 1), (2, 2), (2, 3), (2, 4),$
$(3, 1), (3, 2), (3, 3), (3, 4),$
$(4, 1), (4, 2), (4, 3), (4, 4)\}$

(ii) See accompanying graph (right above).

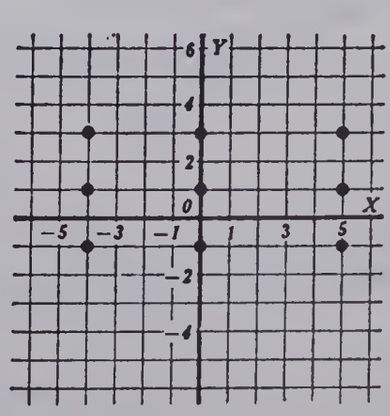
(iii) All the points above those circled in (ii);
all the points below those circled in (ii).

10. (i) $M \times N = \{(-1, -4), (-1, 0), (-1, 5), (1, -4), (1, 0), (1, 5), (3, -4),$
$(3, 0), (3, 5)\}$
$N \times M = \{(-4, -1), (-4, 1), (-4, 3), (0, -1), (0, 1), (0, 3), (5, -1),$
$(5, 1), (5, 3)\}$

(ii)

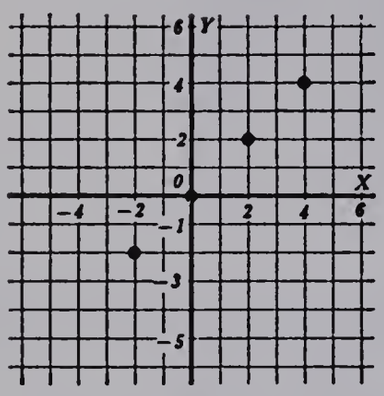


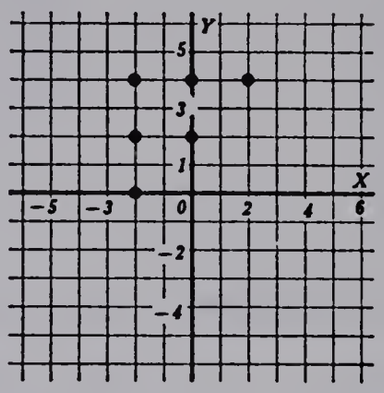
Graph of $M \times N$

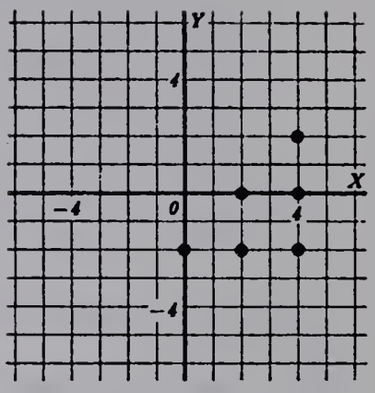


Graph of $N \times M$

11. (i) $D \times D = \{(-2, -2), (-2, 0), (-2, 2), (-2, 4),$
$(0, -2), (0, 0), (0, 2), (0, 4),$
$(2, -2), (2, 0), (2, 2), (2, 4),$
$(4, -2), (4, 0), (4, 2), (4, 4)\}$







(ii) $D_1 = \{(-2, -2), (0, 0), (2, 2), (4, 4)\}$; graph left above.

(iii) $D_2 = \{(-2, 0), (-2, 2), (-2, 4), (0, 2), (0, 4), (2, 4)\}$; graph centre above.

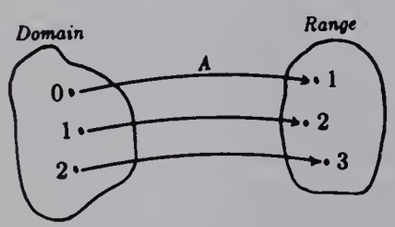
(iv) $D_3 = \{(4, 2), (4, 0), (4, -2), (2, 0), (2, -2), (0, -2)\}$; graph right above.

(v) Each is a subset of $D \times D$.

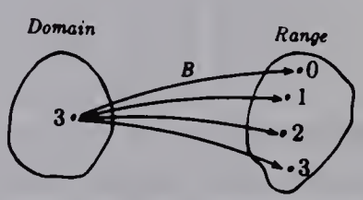
Exercise 3-4 (page 49).

3. (2, 6), (18, 3)

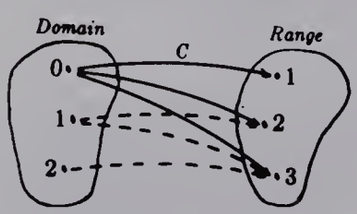
4. (i) $\{(0, 1), (1, 2), (2, 3)\}$
Domain is $\{0, 1, 2\}$.
Range is $\{1, 2, 3\}$.



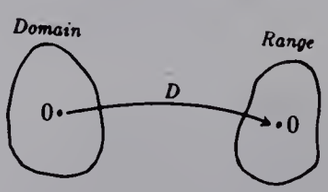
(ii) $\{(3, 0), (3, 1), (3, 2), (3, 3)\}$
Domain is $\{3\}$.
Range is $\{0, 1, 2, 3\}$.



(iii) $\{(0, 1), (1, 2), (0, 2),$
$(1, 3), (0, 3), (2, 3)\}$
Domain is $\{0, 1, 2\}$.
Range is $\{1, 2, 3\}$.

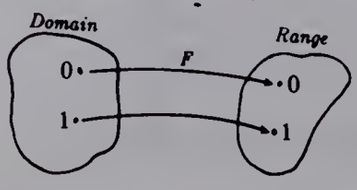


(iv) $\{(0, 0)\}$
Domain is $\{0\}$.
Range is $\{0\}$.



(v) $\emptyset$
Domain is $\emptyset$.
Range is $\emptyset$.

(vi) $\{(0, 0), (1, 1)\}$
Domain is $\{0, 1\}$.
Range is $\{0, 1\}$.



5. $A \times B = \{(-1, 2), (0, 2), (1, 2), (-1, 0), (0, 0), (1, 0), (-1, -2), (0, -2), (1, 2)\}$
$\{(-1, -2), (0, 0), (1, 2)\}$
Domain is $\{-1, 0, 1\}$.
Range is $\{-2, 0, 2\}$.

6. (i) $\{x \mid x \in R\}$; $\{y \mid y \in R, y \geqq 0\}$ (ii) $\{-2\}$; $\{y \mid y \in R\}$
(iii) $\{x \mid x \in R\}$; $\{3\}$ (iv) $\{0\}$; $\{y \mid y \in R\}$

Exercise 3-5 (page 53).

4. $\{(-1, -1), (0, 0), (1, 1), (2, 2), (3, 3), (4, 4)\}$
Domain is $\{x \mid x \in U\}$. Range is $\{y \mid y \in U\}$.

5. $\{(0, 0), (1, 2), (2, 4)\}$
Domain is $\{0, 1, 2\}$. Range is $\{0, 2, 4\}$.

6. $\{(-1, 0), (-1, 1), (-1, 2), (-1, 3), (-1, 4), (0, 1), (0, 2), (0, 3), (0, 4), (1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 2), (3, 4), (1, 1), (2, 2), (3, 3), (4, 3), (4, 4)\}$
Domain is $\{x \mid x \in U\}$. Range is $\{y \mid y \in U\}$.

7. $\{(-1, 4), (0, 4), (1, 4), (2, 4), (3, 4), (4, 4)\}$
Domain is $\{x \mid x \in U\}$. Range is $\{4\}$.

8. $\{(-1, 0), (0, 1), (1, 2), (2, 3), (3, 4)\}$
Domain is $\{-1, 0, 1, 2, 3\}$. Range is $\{0, 1, 2, 3, 4\}$.

9. $\{(-1, -1), (-1, 0), (-1, 1), (-1, 2), (-1, 3), (-1, 4)\}$
Domain is $\{-1\}$. Range is $\{y \mid y \in U\}$.

10. $(-3, -3), (-3, -2), (-3, -1), (-3, 0)$, etc.
Domain is $\{-3\}$. Range is $\{y \mid y \in I\}$.

11. $(0, 2), (1, 2), (2, 2), (3, 2)$, etc.
Domain is $\{x \mid x \in I\}$. Range is $\{2\}$.

12. $(1, 0), (2, 1), (3, 2), (4, 3)$, etc.
Domain is $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Range is $\{0, 1, 2, 3, 4, 5, 6, 7, 8\}$.

13. $(0, 0), (1, 1), (4, 2), (9, 3)$, etc.
Domain is $\{0, 1, 4, 9, 16, \ldots\}$. Range is $\{y \mid y \in I\}$.

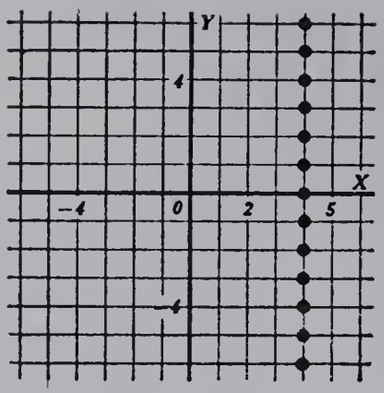
14. $(-2, 4), (-1, 2), (0, 0), (1, 2)$, etc.
Domain is $\{-3, -2, -1, 0, 1, 2\}$. Range is $\{0, 2, 4, 6\}$.

15. $(-2, 6), (0, 3), (2, 0), (4, -3)$, etc.
Domain is $\{\ldots, -4, -2, 0, 2, 4, \ldots\}$.
Range is $\{\ldots, -6, -3, 0, 3, 6, \ldots\}$.

16. $(-2, 0), (-1, 0), (0, 0), (1, 0)$, etc.
Domain is $\{x \mid x \in I\}$. Range is $\{0\}$.

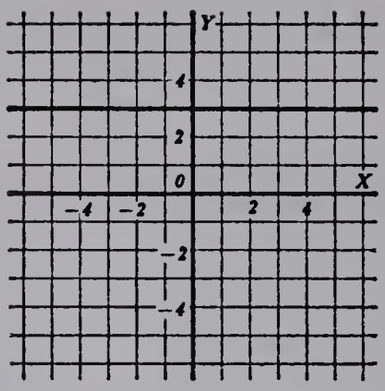
17. $(0, -1), (0, 0), (0, 1), (0, 2)$, etc.
Domain is $\{0\}$. Range is $\{y \mid y \in I\}$.

18.



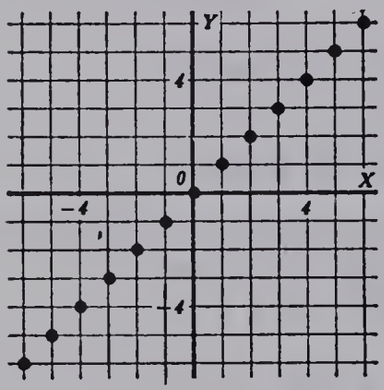
Domain is $\{4\}$.
Range is $\{y \mid y \in I\}$.

19.



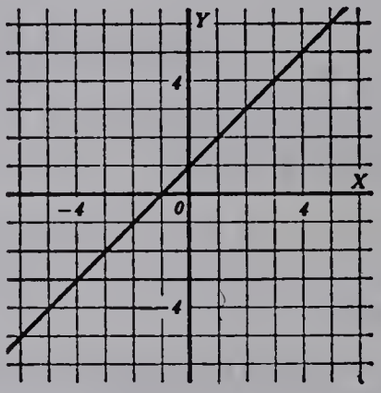
Domain is $\{x \mid x \in R\}$.
Range is $\{3\}$.

20.



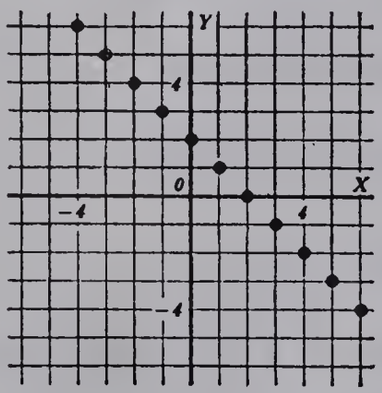
Domain is $\{x \mid x \in I\}$.
Range is $\{y \mid y \in I\}$.

21.



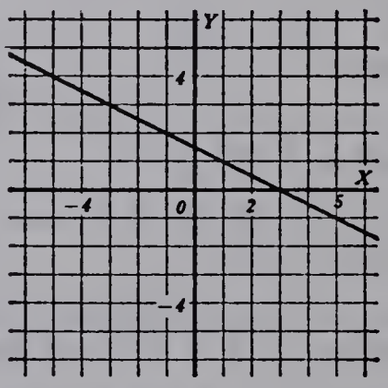
Domain is $\{x \mid x \in R\}$.
Range is $\{y \mid y \in R\}$.

22.



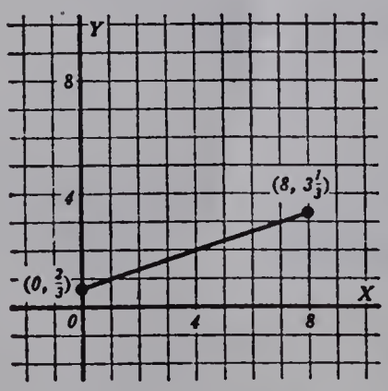
Domain is $\{x \mid x \in I\}$.
Range is $\{y \mid y \in I\}$.

23.



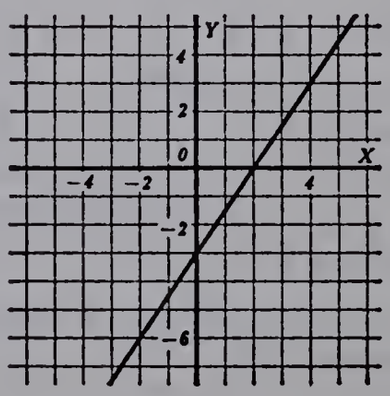
Domain is $\{x \mid x \in R\}$.
Range is $\{y \mid y \in R\}$.

24.



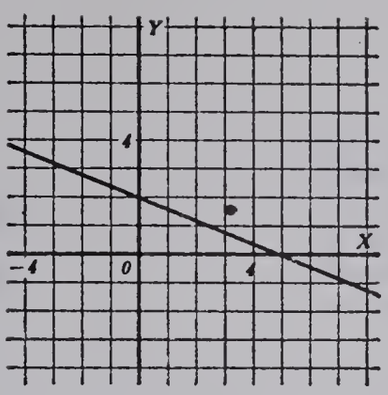
Domain is $\{x \mid 0 \leqq x \leqq 8, x \in R\}$.
Range is $\{y \mid \frac{2}{3} \leqq y \leqq 3\frac{1}{3}, y \in R\}$.

25.



Domain is $\{x \mid x \in R\}$.
Range is $\{y \mid y \in R\}$.

26.



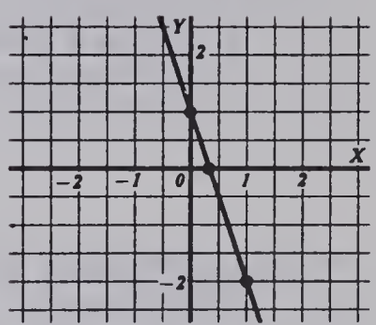
Domain is $\{x \mid x \in R\}$.
Range is $\{y \mid y \in R\}$.

Exercise 3-6 (page 55).

1. (a) x-intercept is $\frac{1}{3}$
y-intercept is 1

(b)

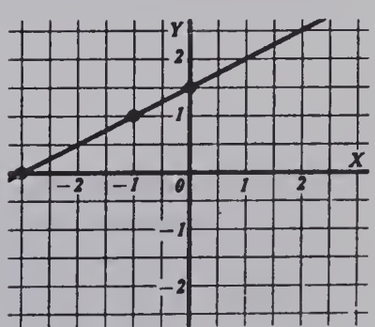
x	0	$\frac{1}{3}$	1
y	1	0	-2



2. (a) x-intercept is -3
y-intercept is $\frac{3}{2}$

(b)

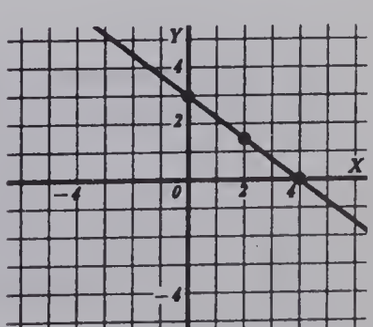
x	0	-1	-3
y	$\frac{3}{2}$	1	0



3. (a) x-intercept is 4
y-intercept is 3

(b)

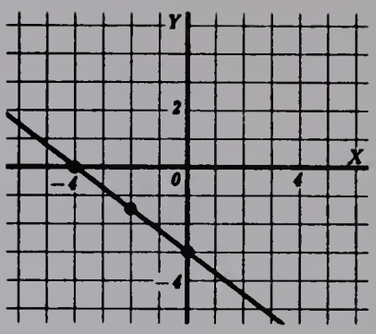
x	0	2	4
y	3	$\frac{3}{2}$	0



4. (a) x-intercept is -4
y-intercept is -3

(b)

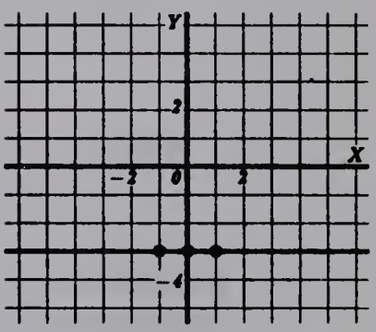
x	0	-2	-4
y	-3	$-\frac{3}{2}$	0



5. (a) No x-intercept
y-intercept is -3

(b)

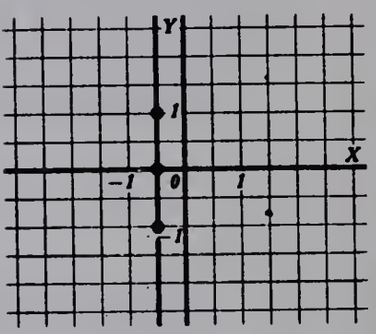
x	-1	0	1
y	-3	-3	-3



6. (a) x-intercept is $-\frac{1}{2}$
No y-intercept

(b)

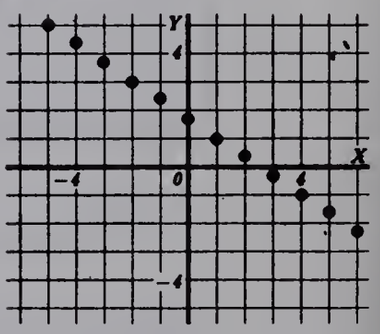
x	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$
y	-1	0	1



7. (a) No x-intercept
y-intercept is $\frac{5}{3}$

(b)

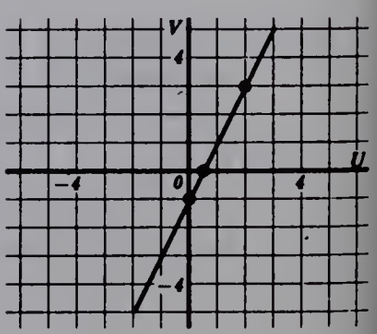
x	-1	0	1
y	$\frac{7}{3}$	$\frac{5}{3}$	1



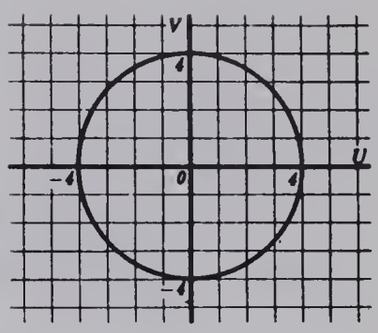
8. (a) u-intercept is $\frac{1}{2}$
v-intercept is -1

(b)

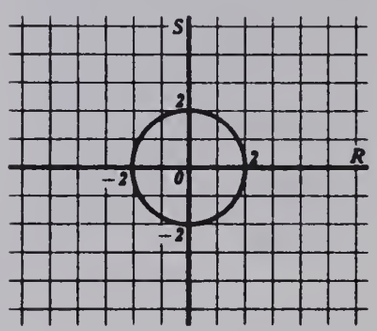
u	0	$\frac{1}{2}$	2
v	-1	0	3



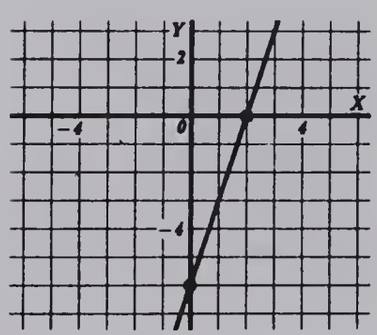
Exercise **3-7** (page 59).



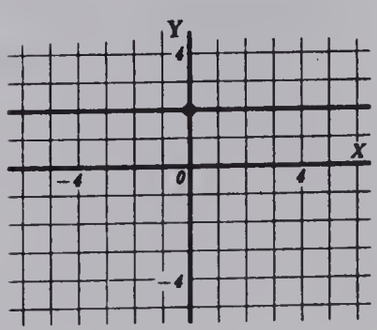
3. u-intercepts ± 4
v-intercepts ± 4
Domain $\{u \mid -4 \leqq u \leqq 4\}$
Range $\{v \mid -4 \leqq v \leqq 4\}$



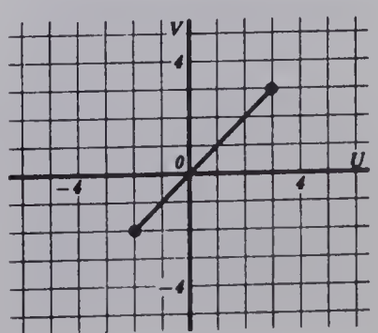
4. r-intercepts ± 2
s-intercepts ± 2
Domain $\{r \mid -2 \leqq r \leqq 2\}$
Range $\{s \mid -2 \leqq s \leqq 2\}$



5. x-intercept 2
y-intercept -6
Domain R
Range R

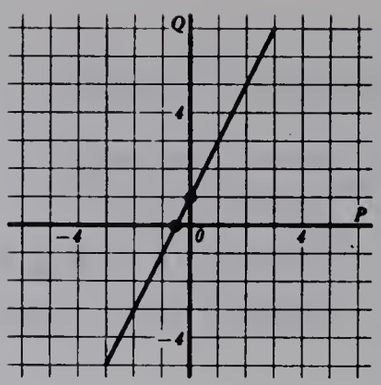


6. No x-intercept
y-intercept 2
Domain R
Range $\{2\}$

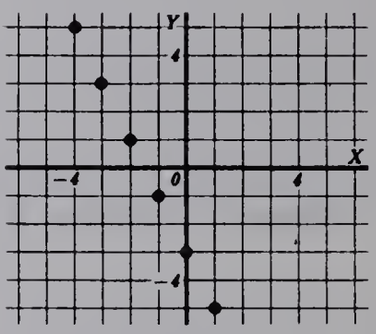


7. u-intercept 0
v-intercept 0
Domain $\{u \mid -2 \leqq u \leqq 3\}$
Range $\{v \mid -2 \leqq v \leqq 3\}$

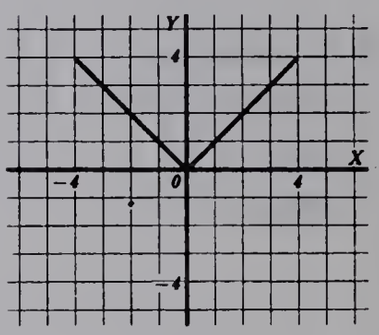
8. p-intercept $-\frac{1}{2}$
q-intercept 1
Domain $\{p \mid -3 \leqq p \leqq 3\}$
Range $\{q \mid -5 \leqq q \leqq 7\}$



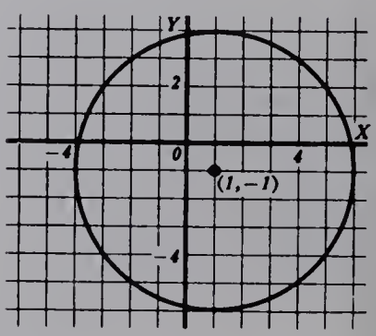
9. No x-intercept
y-intercept -3
Domain $\{x \mid x \in I\}$
Range $\{\ldots, -3, -1, 1, 3, \ldots\}$
set of all odd integers



10. x-intercept 0
y-intercept 0
Domain $\{x \mid -4 \leqq x \leqq 4\}$
Range $\{y \mid 0 \leqq y \leqq 4\}$

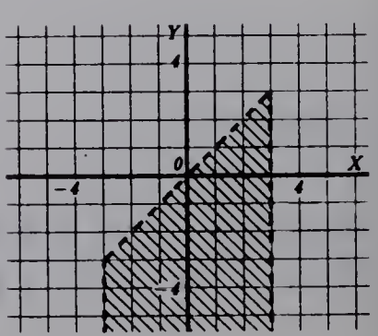


11. x-intercepts $1 \pm 2\sqrt{6}$
y-intercepts $-1 \pm 2\sqrt{6}$
Domain $\{x \mid -4 \leqq x \leqq 6\}$
Range $\{y \mid -6 \leqq y \leqq 4\}$

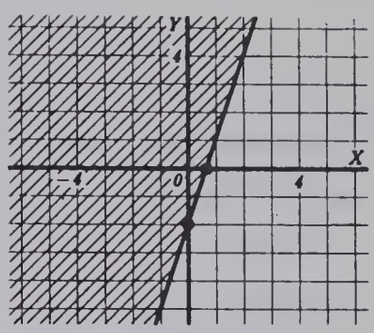


Exercise **3-8** (page 62).

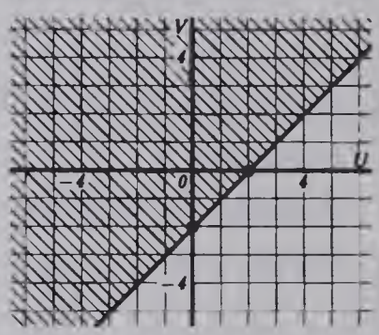
1. x-intercepts $\{x \mid 0 < x < 3\}$
y-intercepts $\{y \mid y < 0\}$
Domain $\{x \mid -3 < x < 3\}$
Range $\{y \mid y < 3\}$



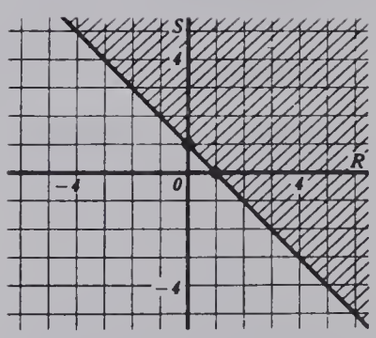
2. x-intercepts $\{x \mid x \leqq \frac{2}{3}\}$
y-intercepts $\{y \mid y \geqq -2\}$
Domain R
Range R



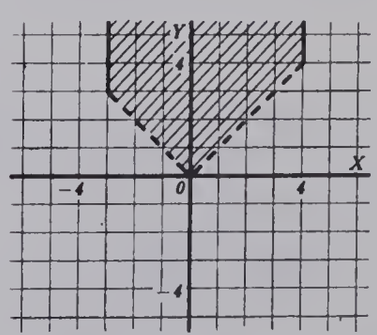
3. u-intercepts $\{u \mid u \leqq 2\}$
v-intercepts $\{v \mid v \geqq -2\}$
Domain R
Range R



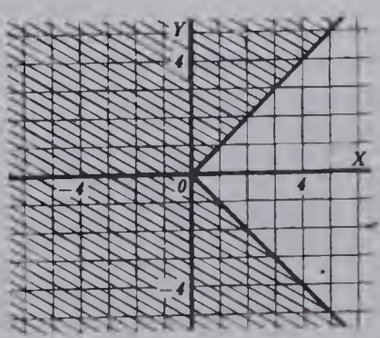
4. r-intercepts $\{r \mid r \geqq 1\}$
s-intercepts $\{s \mid s \geqq 1\}$
Domain R
Range R



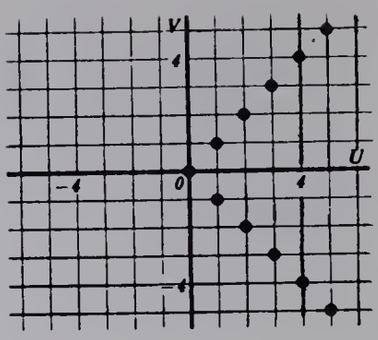
5. No x-intercepts
y-intercepts $\{y \mid y > 0\}$
Domain $\{x \mid -3 \leqq x \leqq 4\}$
Range $\{y \mid y > 0\}$



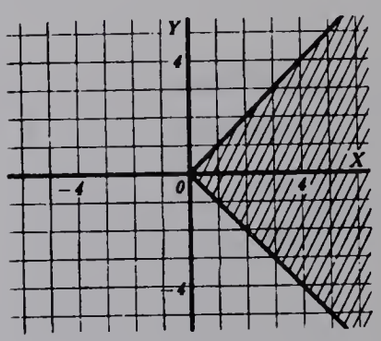
6. x-intercepts $\{x \mid x \leqq 0\}$
y-intercepts $\{y \mid y \in R\}$
Domain R
Range R



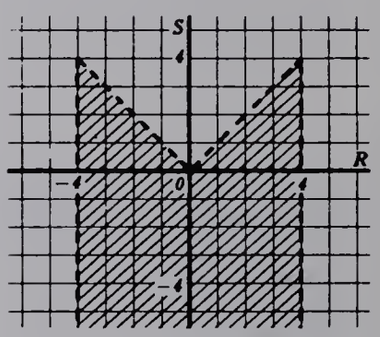
7. u-intercept 0
v-intercept 0
Domain $\{u \mid u \in I, u \geqq 0\}$
Range $\{v \mid v \in I\}$



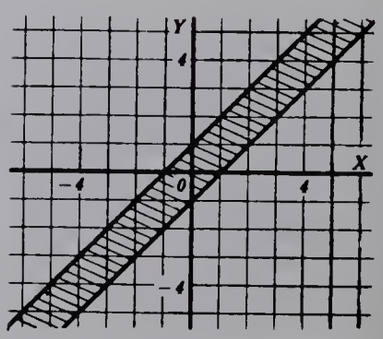
8. x-intercepts $\{x \mid x \geqq 0\}$
y-intercept 0
Domain $\{x \mid x \geqq 0\}$
Range R



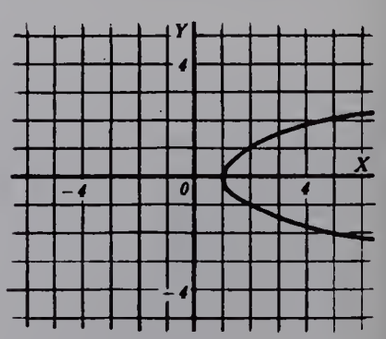
9. r-intercepts $\{r \mid -4 < r < 4, r \neq 0\}$
$\{r \mid 0 < |r| < 4\}$
s-intercepts $\{s \mid s < 0\}$
Domain $\{r \mid -4 < r < 4\}$
Range $\{s \mid s < 4\}$



10. x-intercepts $\{x \mid -1 \leqq x \leqq 1\}$
y-intercepts $\{y \mid -1 \leqq y \leqq 1\}$
Domain R
Range R



11. x-intercept 1
No y-intercept
Domain $\{x \mid x \geqq 1\}$
Range R



Exercise 3-9 (page 64).

1. Symmetry w.r.t. y-axis

2. Symmetry w.r.t. x-axis

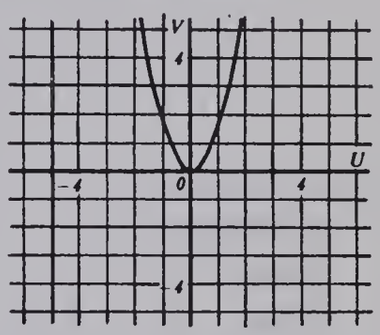
3. Symmetry w.r.t. both axes, and origin

4. Symmetry w.r.t. both axes, and origin

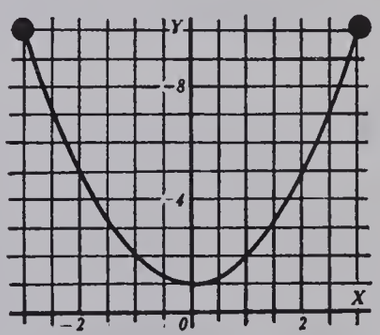
5. Symmetry w.r.t. origin

6. Symmetry w.r.t. x-axis

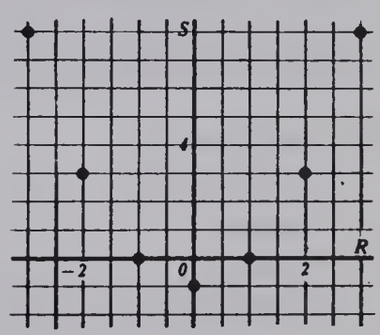
7. u-intercept 0, v-intercept 0
Domain R
Range $\{v \mid v \geqq 0\}$
Symmetry w.r.t. v-axis



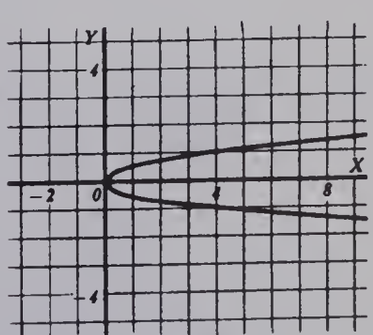
8. No x-intercept
y-intercept 1
Domain $\{x \mid -3 \leqq x \leqq 3\}$
Range $\{y \mid 1 \leqq y \leqq 10\}$
Symmetry w.r.t. y-axis



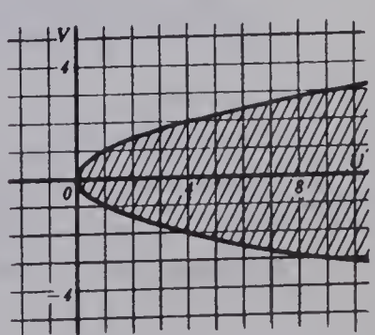
9. r-intercept ± 1
s-intercept -1
Domain $\{-3, -2, -1, 0, 1, 2, 3\}$
Range $\{-1, 0, 3, 8\}$
Symmetry w.r.t. s-axis



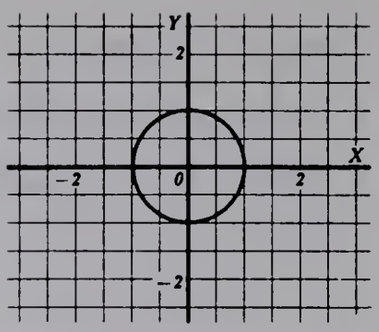
10. x-intercept 0
y-intercept 0
Domain $\{x \mid x \geqq 0\}$
Range R
Symmetry w.r.t. x-axis



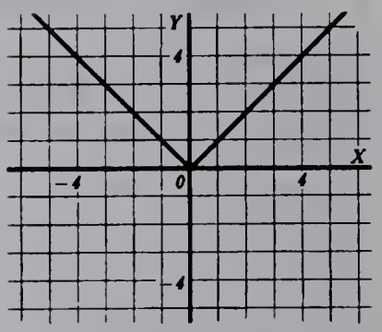
11. u-intercepts $\{u \mid u \geqq 0\}$
v-intercept 0
Domain $\{u \mid u \geqq 0\}$
Range R
Symmetry w.r.t. u-axis



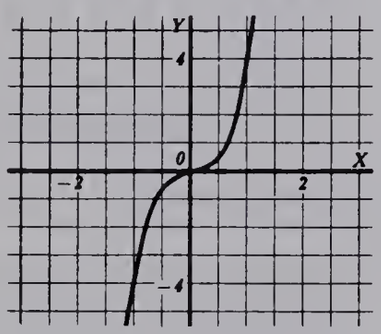
12. x-intercepts ± 1
y-intercepts ± 1
Domain $\{x \mid -1 \leqq x \leqq 1\}$
Range $\{y \mid -1 \leqq y \leqq 1\}$
Symmetry w.r.t. both axes and origin



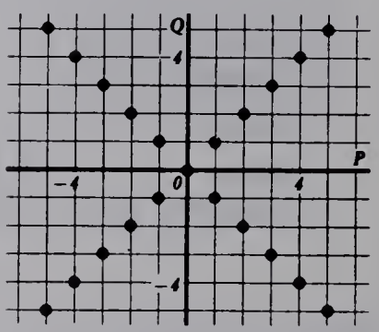
13. x-intercept 0
y-intercept 0
Domain R
Range $\{y \mid y \geqq 0\}$
Symmetry w.r.t. y-axis



14.

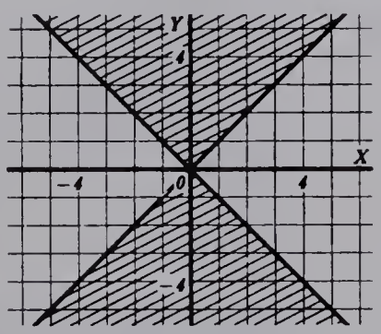


x-intercept 0
y-intercept 0
Domain R
Range R
Symmetry w.r.t. origin

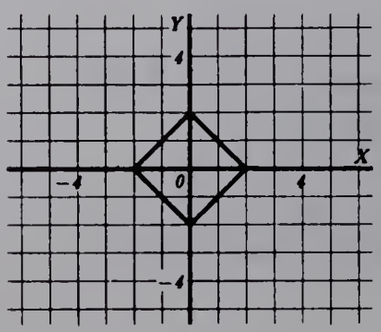


15. p-intercept 0
q-intercept 0
Domain I
Range I
Symmetry w.r.t. both axes, and origin

16.



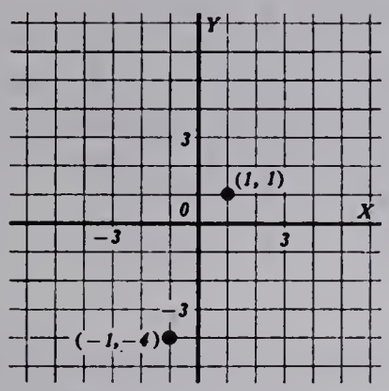
x-intercept 0
y-intercepts $\{y \mid y \in R\}$
Domain R
Range R
Symmetry w.r.t. both axes, and origin



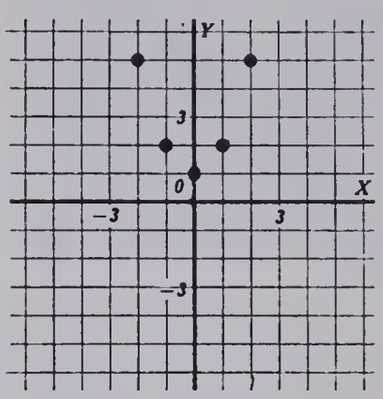
17. x-intercepts ± 2
y-intercepts ± 2
Domain $\{x \mid -2 \leqq x \leqq 2\}$
Range $\{y \mid -2 \leqq y \leqq 2\}$
Symmetry w.r.t. both axes, and origin

Exercise 4-1 (page 70).

20.



21. $F = \{(-2, 5), (-1, 2), (0, 1), (1, 2), (2, 5)\}$



22. $y = x + 1,\ -2 \leqq x \leqq 1,\ x, y \in I$

23. (i) Domain is $\{-3, -2, -1, 0, 1, 2, 3\}$. Range is $\{0, 1, 4, 9\}$. (ii) $y = x^2,\ |x| \leqq 3,\ x, y \in I$

24. (i) $\{(x, y) \mid y = 5x,\ 0 < x \leqq 25,\ x, y \in R\}$ (ii) The relation is a function.
(iii) Domain is $\{x \mid 0 < x \leqq 25,\ x \in R\}$. Range is $\{y \mid 0 < y \leqq 125,\ y \in R\}$.

25. (i) $\{(r, A) \mid A = \pi r^2,\ r > 0,\ r \in R\}$ (ii) Domain is $\{r \mid r > 0,\ r \in R\}$.
Range is $\{A \mid A > 0,\ A \in R\}$. (iii) The relation is a function.

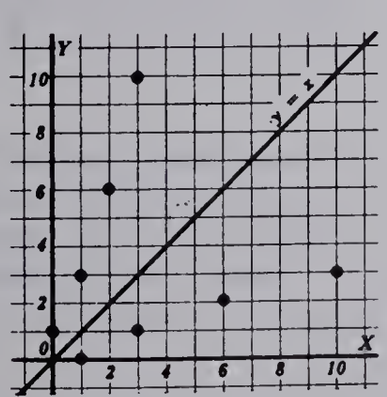
26. (i) $\{(t, d) \mid d = 600t,\ 0 \leqq t \leqq 6,\ t \in R\}$ (ii) Domain is $\{t \mid 0 \leqq t \leqq 6,\ t \in R\}$.
Range is $\{d \mid 0 \leqq d \leqq 3600,\ d \in R\}$. (iii) The relation is a function.

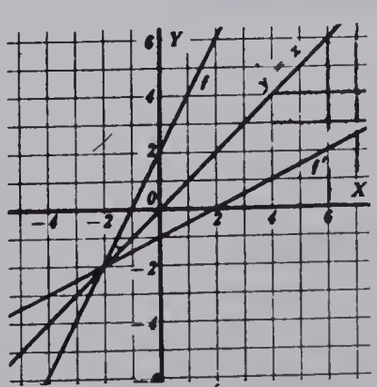
27. (i) $\{(n, d) \mid d = 25n,\ 0 \leqq n \leqq 8,\ n \in I\}$ (ii) Domain is $\{n \mid 0 \leqq n \leqq 8,\ n \in I\}$.
Range is $\{0, 25, 50, 75, 100, 125, 150, 175, 200\}$. (iii) The relation is a function.

Exercise 4-2 (page 74).

1. (i) The domain of h is $\{0, 1, 2, 3\}$. The range of h is $\{1, 3, 6, 10\}$.
(ii) The ordered pairs of h^{-1} are (1, 0), (3, 1), (6, 2), (10, 3).
(iii) The domain of h^{-1} is $\{1, 3, 6, 10\}$. The range of h^{-1} is $\{0, 1, 2, 3\}$.

(iv)

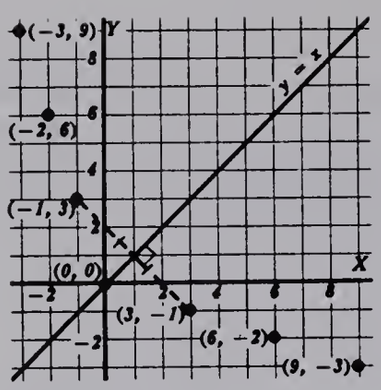




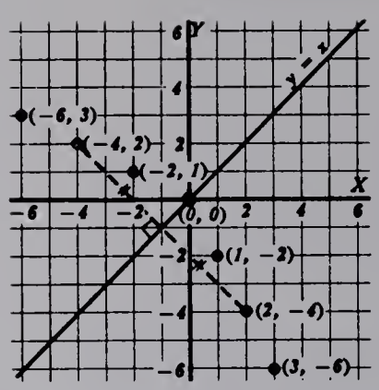
2. (i) $f^{-1} = \{(x, y) \mid x = 2y + 2,\ x, y \in R\}$ or $f^{-1} = \left\{(x, y) \mid y = \dfrac{x - 2}{2},\ x, y \in R\right\}$

(ii) See graph, right above.

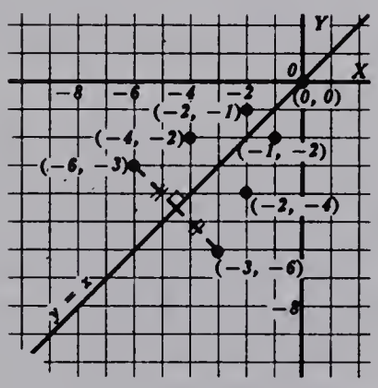
3.



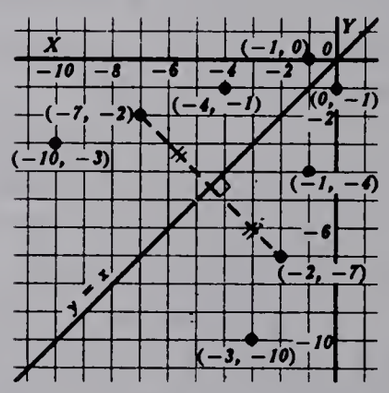
4.



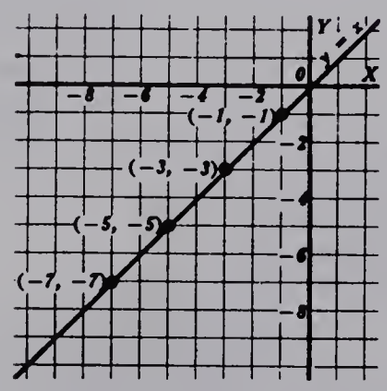
5.



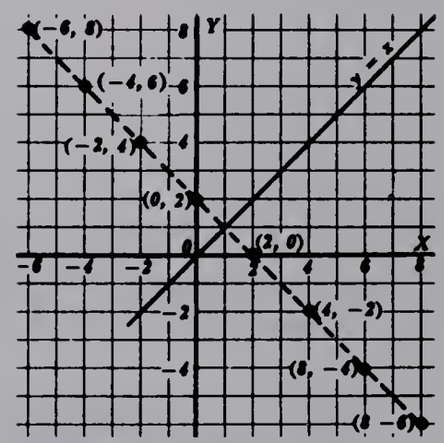
6.



7.



8.



10. $s\colon p \to 5p - 7,\ p \in R$

11. $m\colon x \to -\dfrac{x^2}{5},\ x \in R$

13. $r\colon x \to 4,\ x \in R$

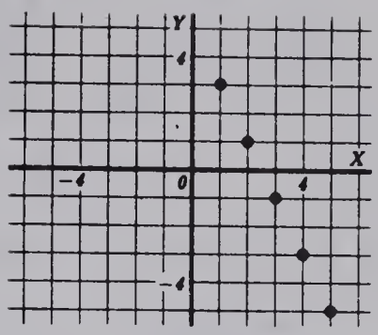
15. $n\colon t \to 10t + 15t^2,\ t \geqq 0,\ t \in R$

18. $q\colon x \to x,\ x \in \{1, 2, 3, 4\}$

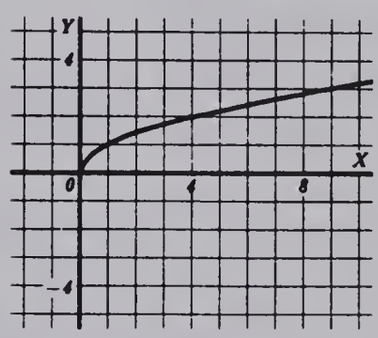
19. $g\colon x \to \frac{1}{16}x^2,\ x \in \{-8, -4, 0, 4, 8\}$

Exercise 4-3 (page 79).

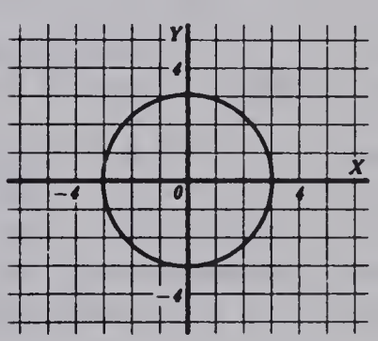
11. No x-intercept, no y-intercept
Domain $I^+ = \{1, 2, 3, \ldots\}$
Range $\{\ldots, -3, -1, 1, 3\}$
No symmetry w.r.t. axes, origin
$f = \{(x, y) \mid y = 5 - 2x, x \in {}^+I\}$



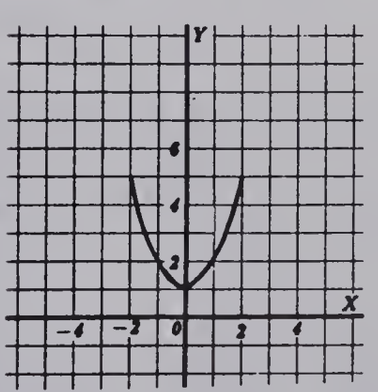
12. x-intercept 0, y-intercept 0
Domain $\{x \mid x \geqq 0\}$
Range $\{y \mid y \geqq 0\}$
No symmetry w.r.t. axes, origin
$f = \{(x, y) \mid y = \sqrt{x}, x \geqq 0\}$



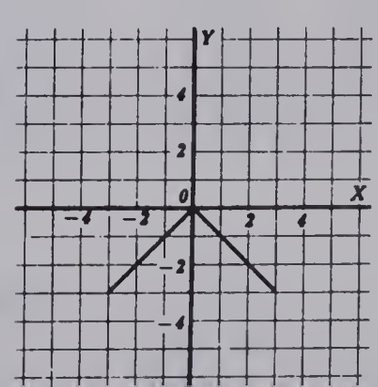
13. x-intercept ± 3, y-intercept ± 3
Domain $\{x \mid -3 \leqq x \leqq 3\}$
Range $\{y \mid -3 \leqq y \leqq 3\}$
Symmetry w.r.t. both axes, and origin



14. No x-intercepts, y-intercept 1
Domain $\{x \mid |x| \leqq 2\}$
Range $\{y \mid 1 \leqq y \leqq 5\}$
Symmetry w.r.t. y-axis
$f = \{(x, y) \mid y = 1 + x^2, |x| \leqq 2\}$



15. x-intercept 0, y-intercept 0
Domain $\{x \mid |x| \leqq 3\}$
Range $\{y \mid -3 \leqq y \leqq 0\}$
Symmetry w.r.t. y-axis
$f = \{(x, y) \mid y = -|x|, |x| \leqq 3\}$



Exercise 4-4 (page 82).

1. -4 **2.** -3 **3.** -3 **4.** -2 **5.** 0 **6.** -1

7. 0 **8.** 8 **9.** 0 **10.** 15 **11.** -4 **12.** 24 **13.** 0, 1, 2, $\sqrt{6}$, 3

14. $-5, 0, -9, 0$ **15.** $0, 4, 0, -2 + 4\sqrt{3}$ **16.** $3, |a|$ **17.** $x - 5 + 4\sqrt{x}$

18. (i) $\dfrac{u}{u^2 + 1}$ (ii) $\dfrac{2u}{4u^2 + 1}$ (iii) $\dfrac{u - 1}{u^2 - 2u + 2}$ (iv) $\dfrac{x(x^2 + 1)}{x^4 + 3x^2 + 1}$

19. Range $\{y \mid y \geqq -9\}$ **20.** Defined for all x with $x \geqq 1$, or $x \leqq -5$

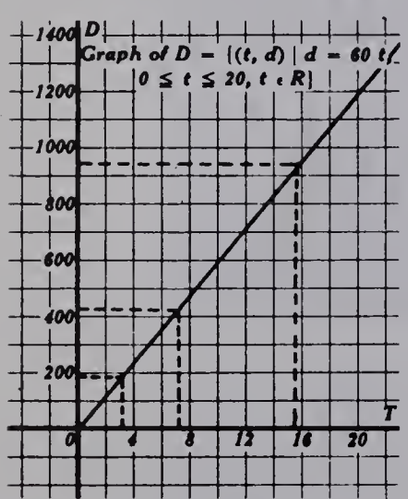
Exercise 5-1 (page 86).

3. (i) $D = \{(t, d) \mid d = 60t, 0 \leqq t \leqq 20, t \in R\}$
(ii) Domain is $\{t \mid 0 \leqq t \leqq 20, t \in R\}$ (iii) Range is $\{d \mid 0 \leqq d \leqq 1200, d \in R\}$
(iv) $D(3) = 180$, $D(13) = 780$, $D(17) = 1020$ The graph is a line segment. (v) 195, $427\frac{1}{2}$, 945

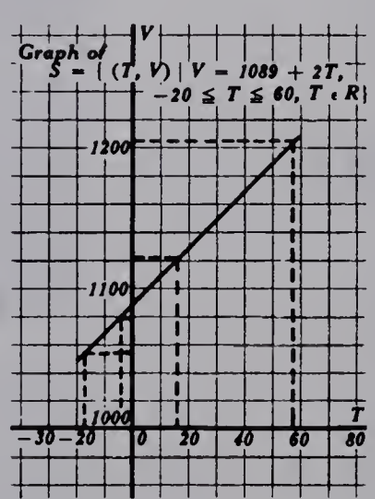
4. (i) Domain is $\{T \mid -20 \leqq T \leqq 60, T \in R\}$ (ii) Range is $\{V \mid 1049 \leqq V \leqq 1209, V \in R\}$ (iii) $S = \{(T, V) \mid 2T - V + 1089 = 0, -20 \leqq T \leqq 60, T \in R\}$, $A = 2$, $B = -1$, $C = 1089$ (iv) $S(0) = 1089$, $S(10) = 1109$, $S(50) = 1189$ (v) $S(-18) = 1053$, $S(-2) = 1085$, $S(16) = 1121$, $S(57) = 1203$. At -18°C the speed of sound is 1053 ft/sec. At -2°C the speed of sound is 1085 ft/sec. At 16°C the speed of sound is 1121 ft/sec. At 57°C the speed of sound is 1203 ft/sec.

5. (i) Domain is $\{F \mid -20 \leqq F \leqq 120, F \in R\}$ (ii) Range is $\{C \mid -28\frac{8}{9} \leqq C \leqq 48\frac{8}{9}, C \in R\}$
(iii) $T = \{(F, C) \mid 5F - 9C - 160 = 0, -20 \leqq F \leqq 120, F \in R\}$, $A = 5$, $B = -9$, $C = -160$ (iv) $T(-4) = -20$, $T(5) = -15$, $T(113) = 45$
(v) Compare your answers with the calculated values: $-23\frac{1}{3}$°, $-17\frac{7}{9}$°, $18\frac{1}{3}$°, $40\frac{5}{9}$°
(vi) Compare your answers with the calculated values: 14°, $24\frac{4}{5}$°, 86°, $107\frac{3}{5}$°

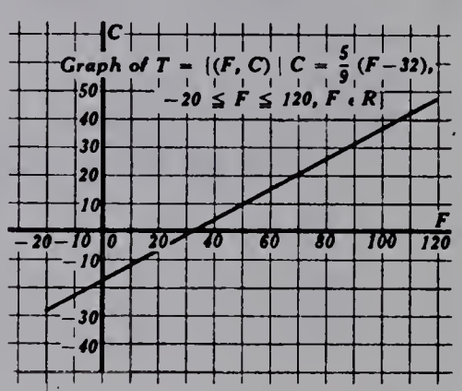
3. (iv)



4. (iv)



5. (iv)



Exercise 5-2 (page 90).

16. 1 **17.** -1 **18.** 0 **19.** not a real number

20. $\frac{1}{2}$, $a \neq 0$ **21.** not a real number **22.** $-\frac{3}{2}$ **23.** 0

24. $\dfrac{b}{a}$ **25.** Yes. **26.** No. **27.** No.

28. Yes. **29.** No. **30.** Yes. **31.** No.
32. Yes. **33.** 4 **34.** $\frac{4}{3}$ **35.** $-5\frac{1}{2}$
36. $10x + 7y - 19 = 0$ **37.** 2 **38.** Yes.
39. $6x + 7y + 9 = 0$ **40.** $\frac{3}{2}$

Exercise 5-3 (page 93).

19. (i) $4x - 5y + 43 = 0$ (ii) $x - y = 0$ (iii) $y = 4$
(iv) $x + y - 4 = 0$ (v) $x + y - 1 = 0$ (vi) $y = -4$
(vii) $x = 1$ (viii) $6x + 4y + 1 = 0$ (ix) $8x - 9y - 12 = 0$
(x) $4x + 6y + 1 = 0$
20. $y = -4$ **21.** $y = 0$ **22.** $x = 3$
23. $x = 0$ **24.** $bx - ay = 0$; no absolute term; yes

Exercise 5-5 (page 97).

2. $2x - 2\sqrt{3}y - 7 = 0$ **3.** $3x - 2y + 6 = 0$
4. $x = -3; y = -1; x = 4; y = 2$ **5.** (i) 4 (ii) $x + \sqrt{3}y - 4\sqrt{3} = 0$
6. $-\frac{1}{2}; \frac{5}{2}$ **7.** $x = -1$

Exercise 5-6 (page 99).

4. $(0, \frac{1}{2})$ **5.** Yes. **6.** $5x - 6y - 28 = 0$ **7.** $4x + 3y + 6 = 0$ **8.** 6
9. $5x + 4y + 11 = 0; x + y + 4 = 0$ **10.** $\frac{3}{4}$ **11.** $\frac{B}{A}$
12. $x - 3y + 19 = 0; 4x + 3y - 29 = 0$ **13.** $(\frac{1}{2}, 0)$

Exercise 5-7 (page 102).

5. (i) $y = mx + 4$ (ii) $y = -2x + 4$ **6.** (i) $\frac{1}{2}$
(ii) $y = \frac{1}{2}x + b$ (iii) the y-intercept of each member. (iv) $y = \frac{1}{2}x + \frac{5}{3}$ or
$3x - 6y + 10 = 0$ **7.** (i) $y = 2x + b$ (ii) $y = 2x - 1$ (iii) $y = 2x + (y_1 - 2x_1)$
8. (i) $y = -\frac{1}{3}x + b$ (ii) $y = -\frac{1}{3}x - 1$ (iii) $y = -\frac{1}{3}x + \frac{3y_1 + x_1}{3}$
9. $3x - y - 3 = 0$ **10.** $2x + y - 6 = 0$

Exercise 5-8 (page 104).

7. (i) $6x - 30y + 5 = 0$ (ii) $6x - y - 19 = 0$
(iii) $x - 3y + 2 = 0$ (iv) $3x + 5y + 1 = 0$ (v) $3x - 2y + 1 = 0$
(vi) $3x - 5y - 9 = 0$ (vii) $x - \sqrt{3}y + 3\sqrt{3} = 0$ (viii) $x + 2y - 4 = 0$
8. (i) $2x - y - 10 = 0$
(ii) $x + y + 4 = 0$ and $3x + y = 0$ (iii) $3x + y = 0$ **9.** (i) $3x - 2y + k = 0, Ax + By + k = 0, k \in R$
(ii) $2x + 3y + k = 0, Bx - Ay + k = 0, k \in R$

Exercise 5-10 (page 108).

2. (i) 5 (ii) $\sqrt{130}$ (iii) $5\sqrt{5}$ (iv) $4\sqrt{2}$ (v) 8 (vi) 2
3. ± 6 **4.** (i) 18 (ii) 28 **5.** (i) right-angled (ii) isosceles
7. -5 **8.** (i) (1, 0) (ii) (0, -1) **11.** $x^2 + y^2 + 4x - 8y - 5 = 0$
13. not collinear **14.** 7

Exercise 5-12 (page 112).

2. $\sqrt{97}, \sqrt{61}, \sqrt{10}$ **3.** (3, 2) **6.** $\left(\frac{3}{2}, -\frac{5}{2}\right)$ **7.** Yes.

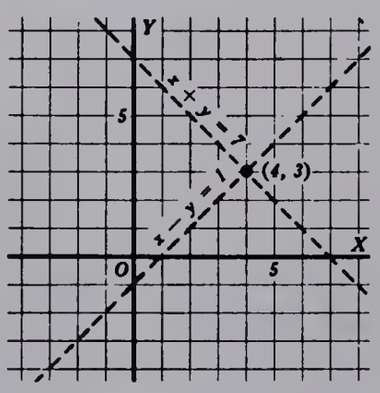
Review Exercise 5-13 (page 113).

6. $\sqrt{26}, 4\sqrt{5}, 5\sqrt{2}$ **7.** No. **8.** -15
9. $11x + 3y + 13 = 0$ **10.** $bx + ay - ab = 0$ **11.** (1, 2)
13. $(0, 1\frac{1}{4})$ **14.** $x + 2y + 1 = 0$

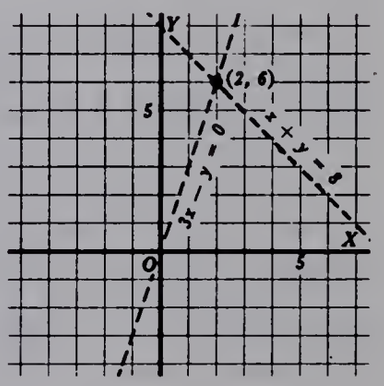
Exercise 6-1 (page 118).

1. (i) $A = \{(x, y) \mid y = -x + 7, y = x - 1\}$
(ii) $A = \{(x, y) \mid y = -x + 7\} \cap \{(x, y) \mid y = x - 1\}$
(iii) $-1, 7, 7; 1, 1, -1$

(iv)



(v) A point



(vi) $\{(4, 3)\}$

2. (i) $B = \{(x, y) \mid y = -x + 8, y = 3x\}$
(ii) $B = \{(x, y) \mid y = -x + 8\} \cap \{(x, y) \mid y = 3x\}$
(iii) $-1, 8, 8; 3, 0, 0$
(iv) See graph at the right above.
(v) A point (vi) $\{(2, 6)\}$

3. (i) $C = \{(x, y) \mid y = 2x + 4, y = -x + 1\}$
(ii) $C = \{(x, y) \mid y = 2x + 4\} \cap \{(x, y) \mid y = -x + 1\}$
(iii) $2, -2, 4; -1, 1, 1$

(iv) See graph at the left below.

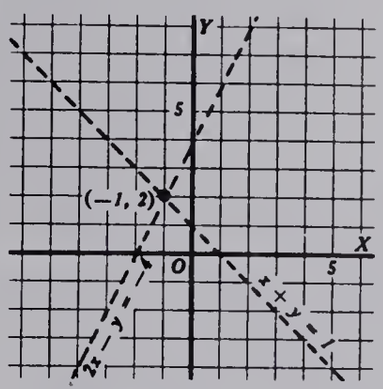
(v) A point (vi) $\{(-1, 2)\}$

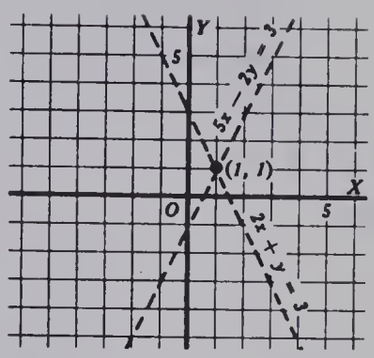
4. (i) $D = \{(x, y) \mid y = \frac{5}{2}x - \frac{3}{2}, y = -2x + 3\}$

(ii) $D = \{(x, y) \mid y = \frac{5}{2}x - \frac{3}{2}\} \cap \{(x, y) \mid y = -2x + 3\}$

(iii) $\frac{5}{2}, \frac{3}{5}, -\frac{3}{2}; -2, \frac{3}{2}, 3$

(iv) See graph at the right below.





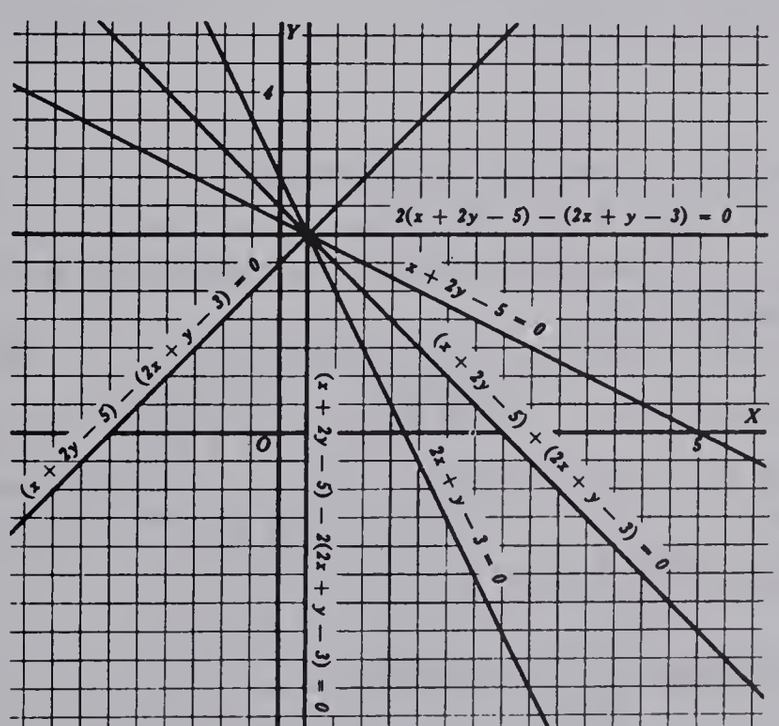
(v) A point (vi) $\{(1, 1)\}$

5. $(1, -2)$ **6.** $(3, 2)$ **7.** $(-\frac{5}{2}, 3)$ **8.** $(-3, -5)$ **9.** $\{(2, -1)\}$ **10.** $\{(2, -3)\}$

Exercise 6-2 (page 122). **6.** $(3, 2)$ **7.** $(2, 1)$ **8.** $\{(2, 1)\}$ **9.** $\{(3, 2)\}$ **10.** $(7, -3)$

11. $(2, 1)$ **12.** $(15, 11)$ **13.** $(2, 1)$ **14.** $\{(-3, -1)\}$ **15.** $\{(3, 6)\}$

16.



17. (i) $(\frac{1}{3}, \frac{7}{3})$ (ii) The set of straight lines passing through the point with coordinates $(\frac{1}{3}, \frac{7}{3})$.

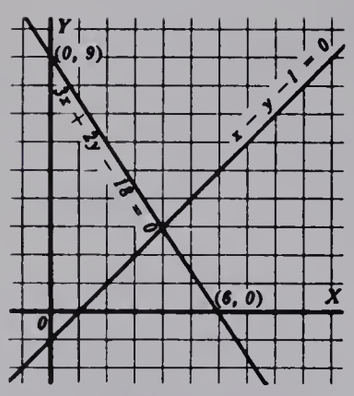
Exercise 6-3 (page 126). **1.** (i) For example

$2(3x + 2y - 18) + 1(x - y - 1) = 0$ or $7x + 3y - 37 = 0$

$-1(3x + 2y - 18) + 3(x - y - 1) = 0$ or $y - 3 = 0$

$3(3x + 2y - 18) - 2(x - y - 1) = 0$ or $7x + 8y - 52 = 0$

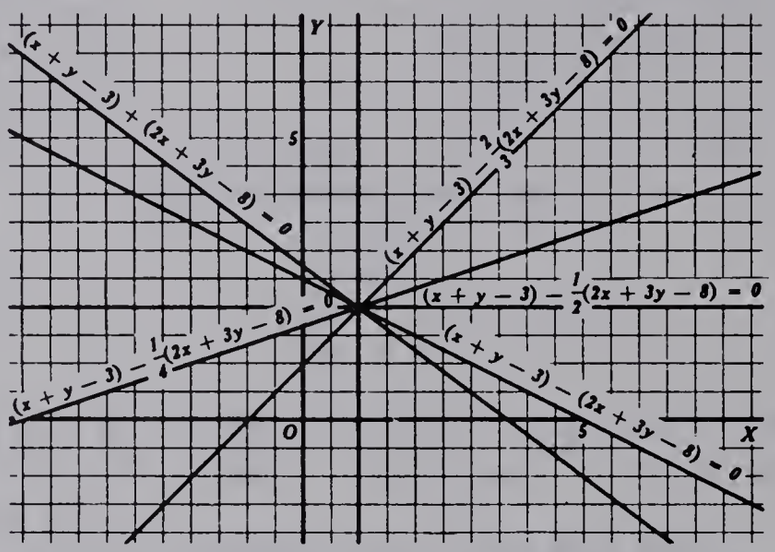
(ii)



(iii) Solve the equations (a) $x = 4$ (b) $y = 3$

2. (i) $k_1(x + y - 3) + k_2(2x + 3y - 8) = 0$

(ii)



(iii) (1, 2). The equation of the family of lines could be

$$k_1(2x + 3y - 8) + k_2(x + y - 3) = 0$$

in which case the lines defined will be different from those in the answer given but they will all lie on the point $P(1, 2)$.

3. (i) $k_1(3x + y + 2) + k_2(x + y - 7) = 0$, $k_1, k_2 \in R$ (ii) (a) $k_1 = 0$ and $k_2 \neq 0$ (b) $k_1 \neq 0$ and $k_2 = 0$ (iii) $3x + y + 2 + \dfrac{k_2}{k_1}(x + y - 7) = 0$, $k_1 \neq 0$

$3x + y + 2 + k(x + y - 7) = 0$, $k \in R$ (iv) The member defined by $x + y - 7 = 0$ or l_2 (v) $(3 + k)x + (1 + k)y + (2 - 7k) = 0$

(vi) (a) family slope $-\dfrac{3 + k}{1 + k}$, $(k \neq -1)$

(b) family x-intercept $\dfrac{7k - 2}{3 + k}$, $(k \neq -3)$

(c) family y-intercept $\dfrac{7k - 2}{1 + k}$, $(k \neq -1)$

Each involves the parameter k.

(vii) (a) $k = -\frac{5}{3}$, $4x - 2y + 41 = 0$ (b) $k = \frac{11}{4}$, $23x + 15y - 69 = 0$

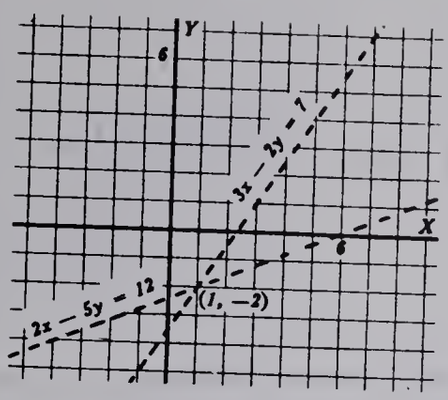
(c) $k = -\frac{2}{11}$, $31x + 9y + 36 = 0$ (viii) $k = \frac{13}{2}$, $19x + 15y - 87 = 0$

4. $6a + 5b - 21 = 0$ **5.** $3x - 4y - 13 = 0$ **6.** $22x - 11y + 11 = 0$

7. (a) $5x - 10y + 38 = 0$ (b) $23x + 12y - 92 = 0$ (c) $x + 4y - 20 = 0$

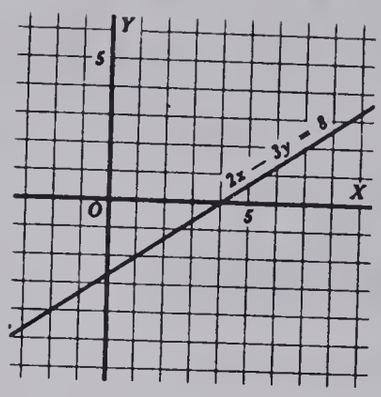
Exercise 6-4 (page 130).

1. (i)



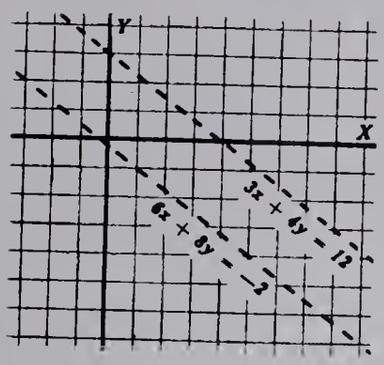
(ii) $\{(1, -2)\}$

2. (i)



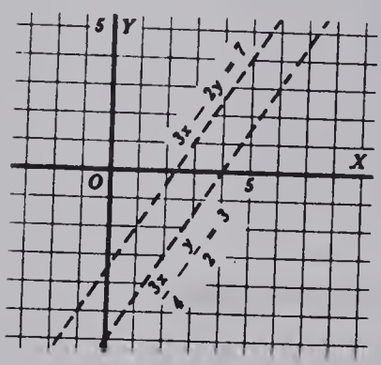
(ii) $\{(x, y) \mid 2x - 3y = 8\}$

3. (i)



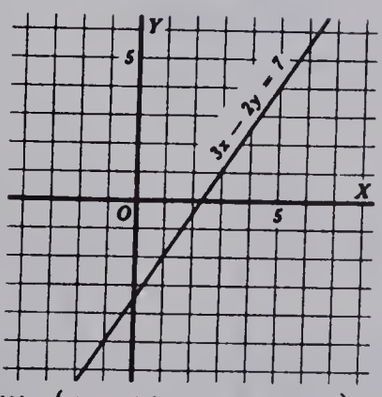
(ii) ϕ

4. (i)



(ii) ϕ

5. (i)



(ii) $\{(x, y) \mid 3x - 2y = 7\}$

Exercise 6-5 (page 131). **5.** $x = r, y = s$, if $r + s \neq 0$ **6.** $x = 0, y = 1$, if $a^2 - b^2 \neq 0$

7. $x = a - b, y = a + b$, if $a - b \neq 0$ **8.** $x = \frac{1}{a}, y = \frac{1}{b}$, if $a + b, a, b \neq 0$

9. $x = k, y = h$, if $h, k \neq 0$ **10.** $x = \frac{r}{p - q}, y = \frac{-r}{p - q}$, if $p - q \neq 0$

11. $x = \dfrac{(a+b)c}{2b+a}, y = \dfrac{c}{2b+a}$, if $2b + a \neq 0$

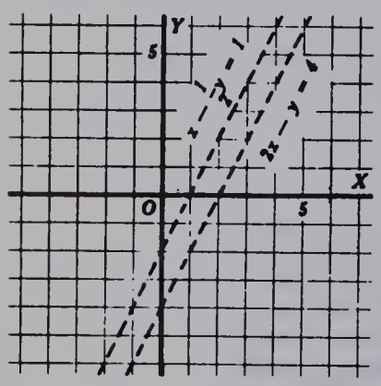
12. $x = \dfrac{(a-b)c}{a^2+b^2}, y = \dfrac{(a+b)c}{a^2+b^2}$, if $a^2 + b^2 \neq 0$

13. $x = b, y = a$, if $a^2 + b^2 \neq 0, a \neq 0$ **14.** $x = a, y = b$, if $a^2 + b^2 \neq 0$

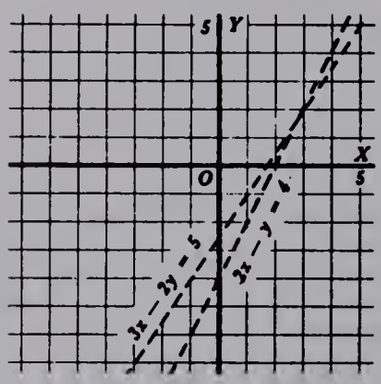
15. See Section 6.6 .

Exercise 6-6 (page 135).

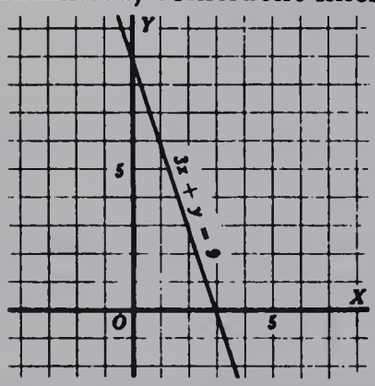
7. none, parallel lines



8. one, intersecting lines



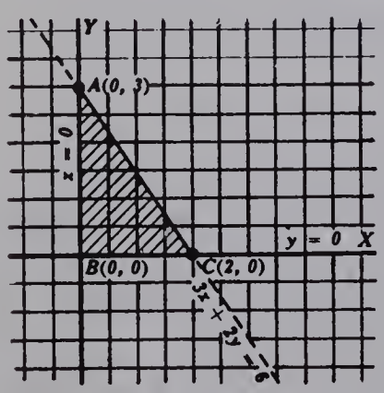
9. unlimited, coincident lines



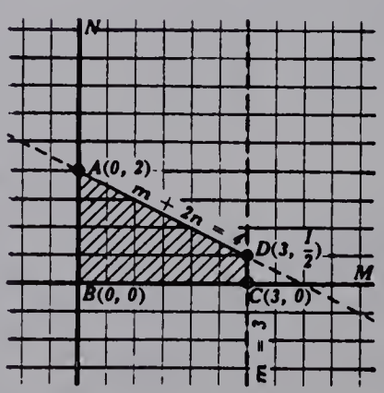
10. none, parallel lines
11. unlimited, coincident lines
12. unlimited, coincident lines
13. one, intersecting lines
14. one, $(\sqrt{2}, 1)$
15. unlimited

Exercise 6-7 (page 138).

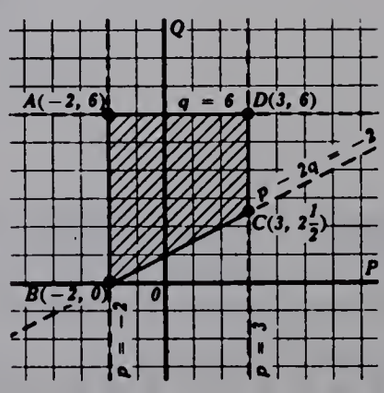
1.



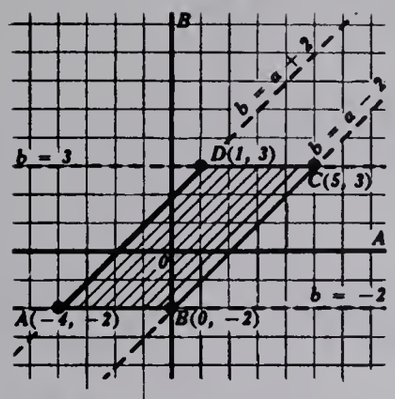
2.



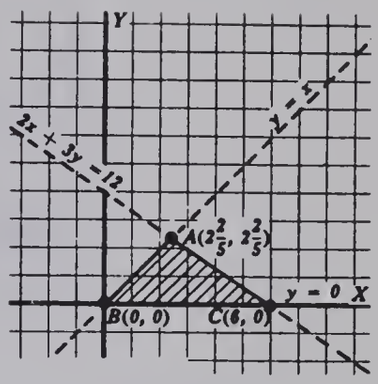
3.



4.

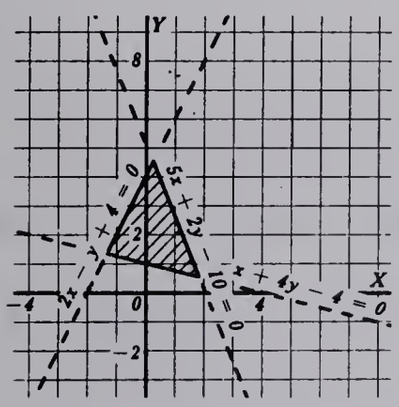


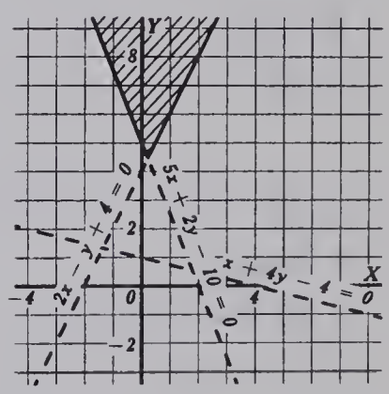
5.



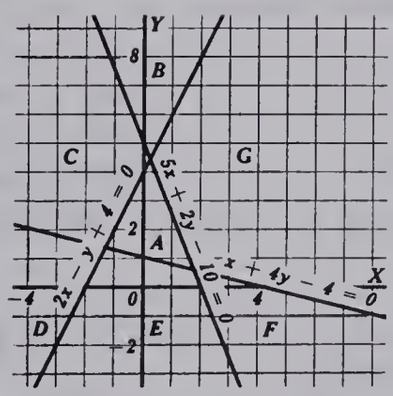
Exercise 6-8 (page 141).

1.





3. (i)



3. (ii)

$$A\begin{cases}2x - y + 4 \geqq 0\\ 5x + 2y - 10 \leqq 0\\ x + 4y - 4 \geqq 0\end{cases} \qquad B\begin{cases}2x - y + 4 \leqq 0\\ 5x + 2y - 10 \geqq 0\\ x + 4y - 4 \geqq 0\end{cases} \qquad C\begin{cases}2x - y + 4 \leqq 0\\ 5x + 2y - 10 \leqq 0\\ x + 4y - 4 \geqq 0\end{cases}$$

$$D\begin{cases}2x - y + 4 \leqq 0\\ 5x + 2y - 10 \leqq 0\\ x + 4y - 4 \leqq 0\end{cases} \qquad E\begin{cases}2x - y + 4 \geqq 0\\ 5x + 2y - 10 \leqq 0\\ x + 4y - 4 \leqq 0\end{cases} \qquad F\begin{cases}2x - y + 4 \geqq 0\\ 5x + 2y - 10 \geqq 0\\ x + 4y - 4 \leqq 0\end{cases}$$

$$G\begin{cases}2x - y + 4 \geqq 0\\ 5x + 2y - 10 \geqq 0\\ x + 4y - 4 \geqq 0\end{cases} \qquad \text{(iii)}\begin{cases}2x - y + 4 \leqq 0\\ 5x + 2y - 10 \geqq 0\\ x + 4y - 4 \leqq 0\end{cases}$$

the null set.

4. (i) (0, 3), $(\frac{6}{5}, \frac{6}{5})$, (3, 0)
(ii) (3, 1) max.; (−1, 2) min.
(ii) max. is $\frac{1}{2}$, min. is −16.
8. George 5 hours, Jim 3 hours.

(ii) $a = 0, b = 3$

5. (i) (3, 1), (1, 3), (−1, 2)
6. (i) (−7, 5), (−3, −3), $(\frac{3}{2}, -\frac{3}{2})$, (−3, 3)
7. (3, 2) a max.; (13, 2) a min.
9. I-3, II-2.

10.

	P	Q	R
w_1	8	0	4
w_2	0	8	4

or

	P	Q	R
w_1	8	4	0
w_2	0	4	8

Exercise 6-10 (page 146).

3. (i) 3 (ii) 2 (iii) 1 (iv) 0 (v) 1
(vi) $\sqrt{5}$ (vii) 8 (viii) 4 (ix) $\dfrac{|b|}{\sqrt{m^2+1}}$ (x) $\dfrac{|y_1 - mx_1|}{\sqrt{m^2+1}}$

4. ± 5 **5.** $1\frac{1}{2}$

6. (i) $\dfrac{28\sqrt{29}}{29}$ (ii) $5x + 2y = \sqrt{29} \mp 0$ (iv) 14 **7.** The lines are concurrent.

8. 3, $7\frac{1}{4}$ **9.** 2
10. $1\frac{1}{2}$

Exercise 6-12 (page 149).

1. 20 **2.** (i) 30 (ii) 27 (iii) 11 (iv) 9 (v) 0
3. collinear **4.** -7
5. (i) 14.5 (ii) 5.8 **6.** $2x + 7y - 14 = 0$ or $2x + 7y + 6 = 0$
7. (i) $x + y - 1 = 0$ **8.** 4 or 24
9. 5

Review Exercise 6-13 (page 152).

1. $(-3, 1)$ **2.** $(6, 2)$ **3.** $(7, 8)$ **4.** $\left(-\dfrac{4}{3}, \dfrac{17}{3}\right)$

5. \$1.95, \$0.39 **6.** 57 **7.** 4 m.p.h., $2\frac{1}{2}$ m.p.h. **8.** 60 m.p.h.
9. 513, 211 **10.** 500 m.p.h., 20 m.p.h. **11.** 6 bull's-eyes, 4 inners **12.** \$26, \$220

13. $x + y - 1 = 0$ **14.** $x = \dfrac{1 - q^2}{1 + q^2}$, $y = \dfrac{2pq}{1 + q^2}$, if $1 + q^2 \neq 0$

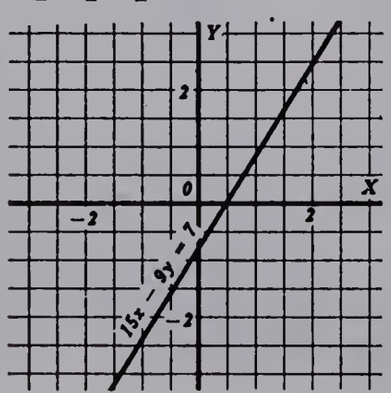
15. $x = a$, $y = b$, if $a, b \neq 0$

16. $\dfrac{1}{1}, \dfrac{1}{1}, \dfrac{5}{3}$; none; no graph

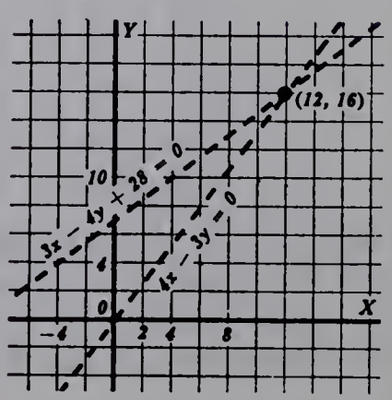
17. $\dfrac{3}{2}, \dfrac{-1}{1}, \dfrac{5}{4}$; one; a point

18. $\dfrac{1}{1}, \dfrac{1}{1}, \dfrac{1}{1}$; unlimited; a line

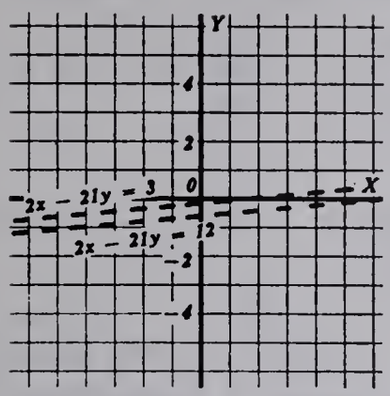
19. $\dfrac{3}{1}, \dfrac{3}{1}, \dfrac{3}{1}$; unlimited



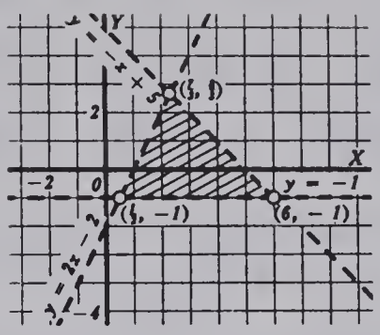
20. $\dfrac{4}{3}, \dfrac{3}{4}, \dfrac{0}{-28}$; one



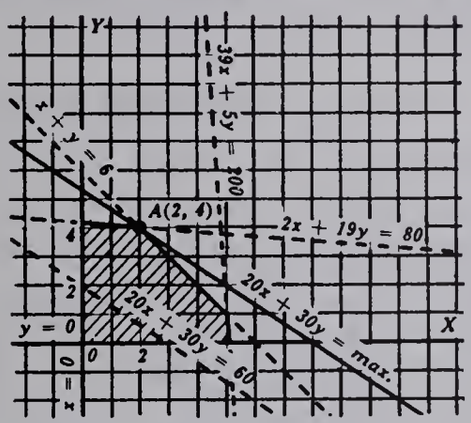
21. $\frac{1}{1}, \frac{1}{1}, \frac{1}{4}$; none.



22.



23.



Maximum value of P occurs when P-line passes through $A(2, 4)$. Hence the maximum value of $P = 20(2) + 30(4)$, or 160: that is, maximum profit occurs when 2 standard models and 4 deluxe models are manufactured.

24. $(3\frac{2}{5}, \frac{1}{5})$

25. $\frac{42\sqrt{41}}{41}$

26. $5x - 12y - 20 = 0$ or $5x - 12y + 32 = 0$

27. $\frac{|12 + k|}{10}$

28. 3 or 19

Exercise 7-1 (page 154).

44. (i) $2^2 = 4, (-2)^4 = 16, 3^{-4} = \frac{1}{81}, (-3)^{-6} = \frac{1}{729}$ (ii) $2^1 = 2, 2^3 = 8, 3^{-3} = \frac{1}{27}, 5^{-4} = \frac{1}{625}$ (iii) $(-1)^1 = -1, (-2)^3 = -8, (-3)^{-5} = -\frac{1}{243}, (-5)^{-3} = -\frac{1}{125}$ **45.** x^9 **46.** a^3b **47.** $a^8b^{12}c^{16}$ **48.** 3^{a-b+c} **49.** $\frac{c^2}{2}$ **50.** $\frac{5}{3}$ **51.** 2^3 or 8 **52.** $\frac{s^4}{r^4}$ **53.** $\frac{b^7}{a^7}$ or $\left(\frac{b}{a}\right)^7$ or $a^{-7}b^7$ **54.** $2^{10}, 3^8$ **55.** $2^{3x}, 3^{3y}$ **56.** 2^{n-4} **57.** 3^{-3} or $\frac{1}{3^3}$ or $\frac{1}{27}$ **58.** 2 **59.** 3 **60.** -4 **61.** 15 **62.** 2.80×10^2 **63.** 7.20×10^{-1} **64.** 3.49×10^5 **65.** 3.572×10^{-6} **66.** 4.96×10^{10} **67.** 4.85×10^{-9} **68.** 1×10^0 **69.** 1×10^{-1}

Exercise 7-2 (page 160).

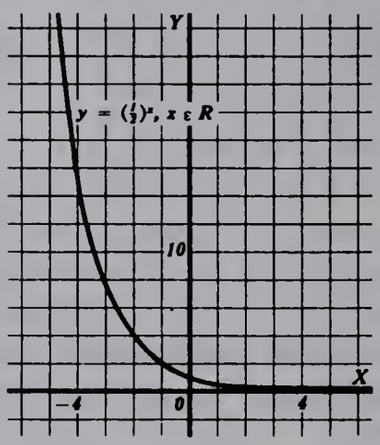
57. $3 \times 3 = 9, 9^{\frac{1}{2}} \times 9^{\frac{1}{2}} = 9^{\frac{1}{2}+\frac{1}{2}} = 9^1 = 9$ **58.** $1, 16^0 = 1$ **59.** $8, 8^1 = 8$ **60.** $6, (4 \times 9)^{\frac{1}{2}} = 6$ **61.** $\frac{2}{3}, \left(\frac{32}{243}\right)^{\frac{1}{5}} = \frac{2}{3}$ **62.** $1, 16^0 = 1$ **63.** $\frac{3}{4}, \left(\frac{27}{64}\right)^{\frac{1}{3}} = \frac{3}{4}$ **64.** $35, (25 \times 49)^{\frac{1}{2}} = 35$

Exercise 7-3 (page 165). **45.** $\frac{1}{8}$ **46.** 8 **47.** $\frac{1}{100}$ **48.** $\frac{27}{8}$ **49.** $\frac{9}{4}$ **50.** $\frac{1}{343}$ **51.** 2 **52.** $a^{\frac{9}{4}}$ **53.** $b^{\frac{1}{2}}$ **54.** $x^{\frac{23}{12}}$ **55.** y^{m+n+p} **56.** $a^{\frac{5}{14}}$ **57.** $b^{\frac{1}{6}}$ **58.** $12x^{\frac{19}{15}}$ **59.** $a^{\frac{2}{3}}$ **60.** 3 **61.** $a^{\frac{1}{3}}$ **62.** $x^{\frac{np+mp-mn}{mnp}}$ **63.** $\left(\frac{ab}{c}\right)^{\frac{2}{3}}$ **70.** 3 **71.** 16 **72.** 27 **73.** 64 **74.** 81 **75.** $\frac{27}{8}$

Exercise 7-4 (page 174). **1.** 52.5 **2.** 69 **3.** 15 **4.** 9.4 **5.** 8.5 **6.** 3.8 **7.** 28.4 **8.** 88 **9.** 4.2 **10.** 3 **11.** 8.8 **12.** 2.9

Exercise 7-5 (page 177). **1.** (i) (0.562, 3.648) (ii) (2 + 0.378, 238.8) (iii) (−1 + 0.732, 0.5395) (iv) (−4 + 0.816, 0.0006546) (v) (1 + 0.392, 24.66) (vi) (0.239, 1.734) (vii) (−2 + 0.260, 0.01820) (viii) (3 + 0.559, 3622) **2.** 663.7 **3.** 6.324 **4.** 4.457 **5.** 8730 **6.** 1.556 **7.** 2.518 **8.** 8.913 **9.** 10.54 **10.** 1.968 **11.** 0.2371 **12.** 29.51 **13.** 3.148

Exercise 7-6 (page 179). **1.**



Practice Exercise 7-7 (page 183). **1.** 5^{15} **2.** 3^{-1} **3.** 2^2 **4.** 5^8 **5.** $7^3 \times 5^3 \times 3^3$ **6.** $\frac{3^7}{5^7}$ **7.** $3^6 \times 5^{18}$ **8.** 1 or 7^0 **9.** −1 **10.** 3^{17} **11.** 3^4 **12.** $\frac{2^8 \times 3^{12}}{5^{12} \times 7^8}$ **13.** $\left(\frac{3}{7}\right)^3$ **14.** 1 or $(abc)^0$ **15.** $\left(\frac{38}{13}\right)^4$ **16.** $\frac{1}{a^3}$ **17.** 1 **18.** $\frac{1}{a^3}$ **19.** $\frac{a^2}{b^3}$ **20.** $\frac{an^3}{mb^2}$ **21.** $\frac{3y^3}{2x^5}$ **22.** $\frac{1}{-3}$ **23.** $\frac{(-5)^3}{(-5)^3}$ or 1 **24.** $\left(\frac{2b}{3a}\right)^4$ **25.** $\frac{2^{-1}b^6}{1}$ **26.** $\frac{3a^{-2}b^{-2}c^3}{1}$ **27.** $\frac{17x^2y^3m^4b^{-5}}{1}$ **28.** $\frac{3^{-1}b^5c^{10}}{1}$ **29.** $\frac{32a^{12}b^{28}}{1}$ **30.** $\frac{726^{-1}x^8}{1}$ **31.** a^8 **32.** $3x^2y^{-1}$ **33.** $\frac{a^6}{b^9}$ **34.** $a^{p-r}b^{q-s}$ **35.** $\frac{m^4p^{12}q^8}{r^8s^4}$ **36.** $\frac{1}{ca^4d^2}$ or $c^{-1}a^{-4}d^{-2}$ **37.** $\frac{3}{4}$ **38.** $\frac{1}{5}$ **39.** $3a^5b^{-2}$ **40.** a^3 **41.** $\frac{m^9}{n^9}$ **42.** m^{2b} **43.** $5^{12}, 7^{15}$ **44.** $5^{3n}, 7^{3n}$ **45.** 5^{-4n+9} **46.** 7^{-9n-2} **47.** 3.267×10 **48.** 5.4×10^{-3} **49.** 3.24675172×10^5 **50.** 7.26×10^{-5} **51.** 3.645×10^{11} **52.** 1.7×10^{-11}

Practice Exercise 7-8 (page 184). **1.** 3^{11} **2.** 9 **3.** 7^{-13} **4.** 2^{12} **5.** $3^5 \times 7^5$ or 21^5 **6.** $\frac{2^4}{3^4}$ **7.** $\frac{7^{13}}{9^{13}}$ **8.** 1 or 5^0 **9.** 1 **10.** 1 **11.** 2^9 **12.** 2^7 **13.** xyz **14.** $(7 \times 5)^4$ or 35^4 **15.** $\left(\frac{2 \cdot 3 \cdot 5}{7 \cdot 9 \cdot 11}\right)^5$ **16.** b^2 **17.** $\frac{q^2}{p^2}$ **18.** $\frac{1}{a^3}$ **19.** $\frac{2}{x^3}$ **20.** $\frac{4^3a^2b^2}{3^2}$ **21.** $(-2)^9$ **22.** 1 **23.** $\frac{3b^2}{5a^2}$ **24.** $\left(\frac{5y}{3x}\right)^7$ **25.** $\frac{2b^2c^3}{1}$ **26.** $\frac{2a^4b^{-7}}{1}$ **27.** $\frac{5x^{-2}y^2z^3}{1}$ **28.** $\frac{3x^8y^{-10}}{1}$ **29.** $\frac{3x^4(a-b)^{-2}}{1}$ **30.** $\frac{8x^8y^2}{1}$ **31.** $\frac{1}{ab}$ or $(ab)^{-1}$ **32.** $(-2)^{x+y-z}$ **33.** a^4b^8 **34.** 3^{x-2} **35.** $\frac{6}{35y^2}$ or $\frac{6y^{-2}}{35}$ **36.** $\frac{4a^9}{25b^{11}}$ **37.** $\left(\frac{2}{3}\right)^{x+y}$ **38.** $\frac{1}{3^{2x+2}}$ or 3^{-2x-2} **39.** $2m^4n^{-6}p^6$ **40.** $11m^{-5}q^2$ **41.** $\left(\frac{x}{y}\right)^{p+q}$ **42.** $a+b$ **43.** $5^{10}, 7^6$ **44.** $5^{2n}, 7^{2n}$ **45.** 5^{3n+9} **46.** 7^{7-4n} **47.** 3.42×10 **48.** 3.425×10^{-3} **49.** 3.46725×10^4 **50.** 7.4362×10^{-6} **51.** 7.456284134×10^7 **52.** 4.6278215×10^{-1}

Practice Exercise 7-9 (page 185). **1.** The principal square root of 3. **2.** The principal fifth root of -3. **3.** The principal nth root of -7 if $n \in \{1, 3, 5, \ldots\}$, not defined if $n \in \{2, 4, 6, \ldots\}$. **4.** The principal fifth root of 3 cubed. **5.** The principal seventh root of a raised to the exponent 4. **6.** The principal ninth root of x^3 raised to the exponent 5. **7.** $|x|$ **8.** $|x+4|$ **9.** $|2a-5b|$ **10.** x **11.** $x+y$ **12.** $|a-b|$ **13.** $\frac{1}{16}$ **14.** $\frac{1}{8}$ **15.** $\frac{x^6y^9}{a^{12}b^{15}}$ **16.** $\frac{x^6y^9}{m^{12}p^3}$ **17.** $4^{\frac{13}{12}}$ **18.** $a^{\frac{19}{30}}$ **19.** 16 **20.** 243 **21.** 125 **22.** 64 **23.** 16 **24.** 78125

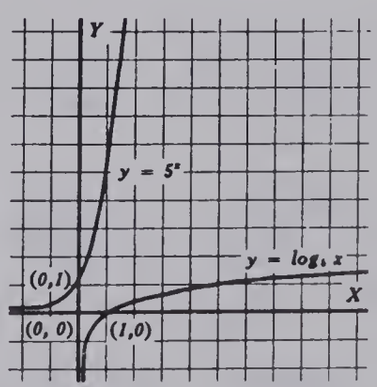
Exercise 8-1 (page 189). **4.** (i) 2 (ii) 3 (iii) 4 **5.** (i) 5 (ii) 3 (iii) 3 (iv) 4 (v) 3 (vi) 4 (vii) 3 (viii) 1 (ix) 1

6. $y = 5^x$

x	-1	0	1	2
y	.2	1	5	25

$y = \log_5 x$

x	.2	1	5	25
y	-1	0	1	2



7. (i) $\log_5 1 = 0$ (ii) $\log_{\frac{1}{8}} 16 = -\frac{4}{3}$ (iii) $\log_2 \frac{1}{8} = -3$ (iv) $\log_{10} 2 = .3010$ (v) $\log_{\frac{1}{81}} 27 = -\frac{3}{4}$ (vi) $\log_a c = b$ **8.** (i) $27 = 9^{\frac{3}{2}}$ (ii) $x = 4^{\frac{3}{4}}$ (iii) $M = b^a$ (iv) $\frac{1}{16} = x^{-\frac{4}{5}}$ (v) $16 = 8^x$ (vi) $\frac{1}{16} = 4^{-2}$ **9.** (i) 8 (ii) 1 (iii) $\frac{1}{3}$ (iv) 16 (v) $\frac{1}{49}$ (vi) 16 **10.** (i) 2^5 or 32 (ii) $\frac{8}{27}$ (iii) 7 (iv) $\frac{3}{5}$ (v) $\frac{1}{8}$ (vi) $\frac{1}{5}$ **11.** -3 **12.** $\frac{9}{2}$ **13.** -30

Exercise 8-2 (page 192). **8.** $\log_a 37 + \log_a 129$ **9.** $\log_b 537 - \log_b 29$ **10.** $5 \log_c 31$ **11.** $\frac{1}{k} \log_w 47$ **12.** $\log_x a + \log_x b$ **13.** $\log_y l - \log_y f$ **14.** $l \log_z k$ **15.** $\frac{r}{s} \log_p q$

Exercise 8-3 (page 195). **1.** (i) $\log_{10}48 - \log_{10}17.9$ (ii) $\log_5 53.4 + \log_5 27.8 + \log_5 36.9$ (iii) $\log_a x + \log_a y + \log_a z$ (iv) $\log_{10}53 + \log_{10}46.8 - \log_{10}91.6$ (v) $\log_a p + \log_a q + \log_a r - (\log_a s + \log_a t)$ (vi) $-(\log_a p + \log_a q)$ **2.** (i) $\frac{1}{3}(\log_{10}48 - \log_{10}37.6)$ (ii) $\frac{2}{3}(\log_{10}63.5)$ (iii) $\log_{10}2 + \log_{10}\pi + \frac{1}{2}(\log_{10}d - \log_{10}q)$ (iv) $\log_{10}7 + \frac{3}{2}(\log_{10}x) - \frac{1}{3}(\log_{10}17 + \log_{10}y)$ (v) $p\log_a x - \frac{2}{q}(\log_a y)$ (vi) $3\log_a x + \frac{1}{2}\log_a y - \frac{4}{3}\log_a z$ **3.** (i) -2 (ii) 9 (iii) 20 **4.** (i) .6990 (ii) 1.0791 (iii) .21582 (iv) $-.1761$ (v) .0512 (vi) 1.9542 **5.** (i) $\log_a \frac{2x^4}{y^3}$ (ii) $\log_2 \frac{10}{9}$ (iii) $\log_a \frac{\sqrt{x}}{\sqrt[3]{y^2}}$ (iv) $\log_4 600$ **6.** (i) 2^{10} or 1024 (ii) 2^4 or 16 **7.** (i) 5^9 or 1953125 (ii) 5^3 or 125 **8.** (i) $\log_3 14 \times \frac{1}{\log_3 2}$ (ii) $\log_5 156 \times \frac{1}{\log_5 7}$ (iii) $\log_{10}57.9 \times \frac{1}{\log_{10}6}$ (iv) $\log_{10}1587 \times \frac{1}{\log_{10}4}$ (v) $\log_q K \times \frac{1}{\log_q p}$ (vi) $\log_v A \times \frac{1}{\log_v u}$

Exercise 8-4 (page 198). **1.** 6.72×10^2, 2 **2.** 8.93×10^1, 1 **3.** 6.72×10^{-1}, -1 **4.** 4.63×10^{-3}, -3 **5.** 1.56×10^{-1}, -1 **6.** 2.376×10^1, 1 **7.** 5.84, 0 **8.** 7.6×10^{-6}, -6 **9.** 1.111×10^2, 2 **10.** 5.83×10^{-2}, -2 **11.** 4.963×10^3, 3 **12.** 9.006×10^{-1}, -1 **13.** 1.5, 0 **14.** 9.9, 0 **15.** 1.0073, 0 **16.** 4.518×10^3, 3 **17.** 5.16×10^{-3}, -3 **18.** 5.89×10^{-2}, -2

Exercise 8-5 (page 200). **1.** .8156 **2.** 1 + .8156 **3.** -3 + .8156 **4.** 1 + .9222 **5.** .8549 **6.** -1 + .3692 **7.** .7340 **8.** -1 + .0212 **9.** -4 + .6646 **10.** 2 + .6064 **11.** -2 + .0453 **12.** .4969

Exercise 8-6 (page 201). 5.51 **2.** 8.19×10^2 **3.** 5.44×10^{-1} **4.** 1.17×10^{-2} **5.** 3.07×10^1 **6.** 6.10 **7.** 9.93×10^{-1} **8.** 9.08×10^2 **9.** 2.12×10^{-2} **10.** 3.28×10^{-3} **11.** 2.45×10^3 **12.** 1.17×10^3 **13.** 9.13 **14.** 6.65×10^1 **15.** 5.68×10^{-1} **16.** 4.64×10^2 **17.** 3.65×10^{-2} **18.** 2.45×10^{-3}

Exercise 8-7 (page 202). **1.** 258 **2.** 32.0 **3.** 10.1 **4.** 0.618 **5.** 21.5 **6.** 2900 **7.** 0.0252 **8.** .0204 **9.** 1.35 **10.** .0176 **11.** 44.0 **12.** 0.476 **13.** 11.2 **14.** 623

Exercise 8-8 (page 204). **1.** 9.86 **2.** 2.04 **3.** .613 **4.** .692 **5.** 37.6 **6.** 1.85 **7.** 0.00290 **8.** .292 **9.** 1420 **10.** 76.9 **11.** 1.61 **12.** 5.04 **13.** 0.193 watts **14.** 1410 sq. units **15.** 1.26 **16.** 113 sq. units **17.** 8000 cu. in.

Exercise 8-9 (page 207). **1.** .4972 **2.** 2 + .7241 **3.** -1 + .2871 **4.** -3 + .3918 **5.** 1 + .6726 **6.** -4 + .5129 **7.** .0077 **8.** -2 + .9579 **9.** 3 + .1995

Exercise 8-10 (page 207). **1.** 6.634 **2.** 5.947×10^1 **3.** 2.676×10^{-2} **4.** 1.514×10^{-1} **5.** 1.225×10^2 **6.** 2.086×10^{-3} **7.** 3.623×10^3 **8.** 1.612 **9.** 1.064×10^1

Exercise 8-11 (page 209). **1.** 100 **2.** 1710 **3.** 1 **4.** 200 **5.** 0.1590 **6.** 3.29 **7.** 2.51 **8.** 4.10 **9.** 2.99 **10.** 13.9 **11.** 18 **12.** 0.277

Exercise 8-12 (page 212). **1.** 3.66% per year **2.** 5.5% per year **3.** 5.5% per year
4. \$1375 **5.** \$5120 **6.** \$2929.64 **7.** \$77.25 **8.** \$718.67 **9.** \$454.55
10. \$1179.94 **11.** \$961.54 **12.** \$70.75 **13.** \$336.34 **14.** \$917.43
15. $26\frac{2}{3}\%$ **16.** \$625, \$643.75 **17.** 1.5 years

Exercise 8-13 (page 216). **1.** \$425.43 **2.** \$2960.50 **3.** \$1792.64 **4.** \$955.24
5. $D(1.06)^y$ **6.** $K(1 + r)^x$ **7.** (i) \$1479.60 (ii) \$1480.20 **8.** 260,000

Exercise 8-14 (page 218). **1.** \$395.16 **2.** \$789.40 **3.** \$1024.20 **4.** \$374.76
5. \$1413.50 **6.** \$2810 **7.** \$18,420 **8.** \$1012

Exercise 8-15 (page 220). **1.** \$5061.30 **2.** \$3050.50 **3.** \$2202.80 **4.** \$2805.10
5. \$3804.70 **6.** \$2776.60 **7.** \$1076.70 **8.** \$1536.30 **9.** \$1148.10
10. \$530.60 **11.** \$2559.00 **12.** (i) 26.8% (ii) 12.6% (iii) 12.4%
(iv) 16% (v) 6.2%

Practice Exercise 8-16 (page 225). **1.** $\log_5 125 = 3$ **2.** $\log_3 1 = 0$ **3.** $\log_{27} 9 = \frac{2}{3}$
4. $\log_r q = p$ **5.** $\log_8 4 \doteq .6667$ **6.** $\log_{10} 6 \doteq .7782$ **7.** $8^{\frac{1}{3}} = 2$ **8.** $9^{.5} = 3$
9. $125^{.3333} \doteq 5$ **10.** $10^{.3010} \doteq 2$ **11.** $10^{.4771} \doteq 3$ **12.** $q^r = p$

Practice Exercise 8-17 (page 225). **1.** $\frac{1}{49}$ **2.** $\frac{1}{27}$ **3.** 5 **4.** $\frac{1}{36}$ **5.** 343 **6.** 81
7. q **8.** b **9.** $\log_p\left(\frac{q^2\sqrt[5]{r}}{s^3}\right)$ **10.** $\log_y\left(\frac{\sqrt[3]{x^2}}{\sqrt{z} \times w^2}\right)$ **11.** $\log_a\sqrt{\frac{b^2 \times \sqrt[3]{c}}{d^3}}$

Practice Exercise 8-18 (page 225). **1.** $1 + .5416$ **2.** $.5416$ **3.** $-4 + .5416$
4. $-2 + .4683$ **5.** .9943 **6.** $-1 + .8189$ **7.** 9.61×10^1 **8.** 1.66×10^{-2}
9. 1.92×10^2 **10.** 2.55 **11.** 3.06×10^{-3} **12.** 2.18×10^{-1}

Practice Exercise 8-19 (page 226). **1.** 3.41×10^{-1} **2.** 8.12×10^2 **3.** 2.25×10^{-1}
4. 5.16×10^{-2} **5.** 0.0160 **6.** 6.18 **7.** 2.92 **8.** 0.00108 **9.** 9.02
10. 0.00180

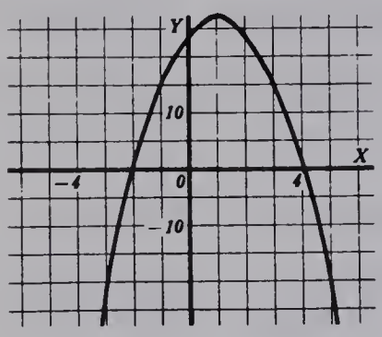
Practice Exercise 8-20 (page 226). **1.** 0.539 **2.** 57.5 **3.** 0.913 **4.** 0.781
5. 14.9 **6.** 9.08 **7.** 94.9 **8.** 40.8 **9.** 0.0329 **10.** 6.03

Review Exercise 8-21 (page 226). **5.** $\frac{1}{16}$ **6.** $\frac{5}{2}$ **7.** 125 **8.** 36 **9.** r **10.** $2\frac{6}{7}$
11. $\log_{10}\left(\frac{17 \times \sqrt[3]{23}}{6^2}\right)$ **12.** $\log_a\left(b^3 \times \sqrt{\frac{c}{d}}\right)$ **13.** 2190 **14.** 0.227 **15.** 0.158
16. 179 **17.** 14.3 **18.** 5.74 **19.** 0.227 **20.** 10.4 **21.** 3.17 **22.** 99.0
23. 8.62 **24.** 0.619 **25.** 0.759 **26.** 0.748 **27.** 0.228 **28.** 0.0773 **29.** 2.04
30. 1.07 **31.** 234 **32.** 0.267 **33.** 13.2 **34.** 40.4 **35.** 5.57 **36.** 1.5 in.
37. 7.8 in. **38.** \$6468 **39.** \$221 **40.** \$3649 **41.** \$813

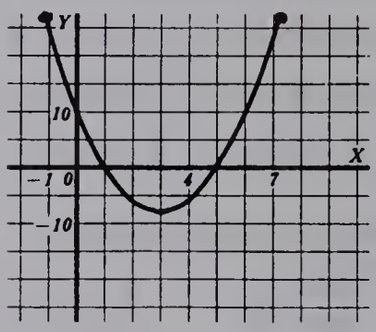
Exercise 9-1 (page 230). **9.** (i) $f(0) = 1$ (ii) $f(1) = 4$ (iii) $f(-1) = 2$
(iv) $f(10) = 211$ **10.** (i) $g(-2) = 13$ (ii) $g(0) = -3$ (iii) $g(2) = 13$
(iv) $g(4) = 61$ **11.** $f(3) = 0, f(-\frac{5}{2}) = 0$ **12.** $g(3) = -36, g(-\frac{5}{2}) = -25$
13. $a(3) = 0, a(2) = 1, a(4) = 1$ **14.** $b(-3) = 0, b(-2) = -1, b(-4) = -1$

Exercise 9-3 (page 233).

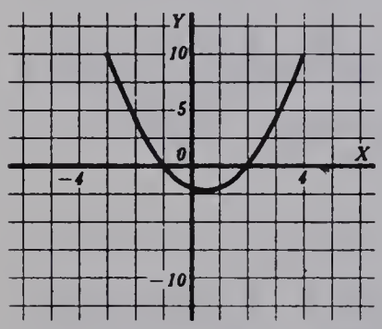
1. Vertex (1, 27)
Axis of symmetry $x = 1$
$a < 0$, opens downwards
Estimated roots -2, 4



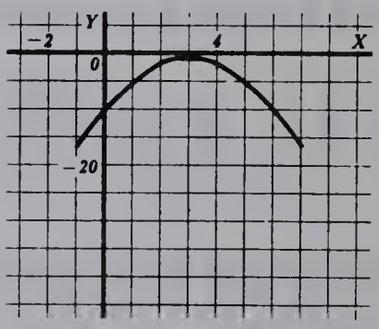
2. Vertex (3, -8)
Axis of symmetry $x = 3$
$a > 0$, opens upwards
Estimated roots 1, 5



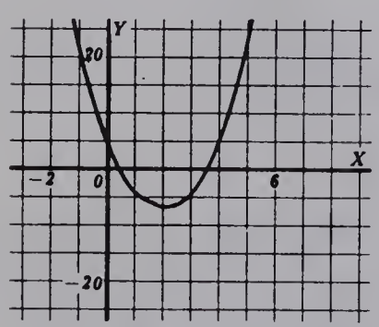
3. Vertex $(\frac{1}{2}, -\frac{9}{4})$
Axis of symmetry $x = \frac{1}{2}$
$a > 0$, opens upwards
Estimated roots -1, 2



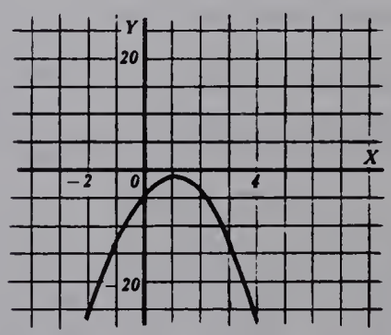
4. Vertex (3, -1)
Axis of symmetry $x = 3$
$a < 0$, opens downwards
No real roots



5. Vertex (2, -7)
Axis of symmetry $x = 2$
$a > 0$, opens upwards
Estimated roots 0.1, 3.7



6. Vertex (1, -1)
Axis of symmetry $x = 1$
$a < 0$, opens downwards
No real roots



Exercise 9-4 (page 234).

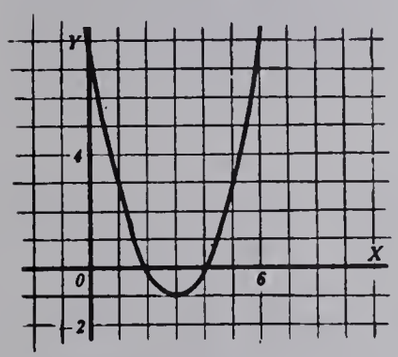
1. Roots 0, $\frac{5}{3}$ **2.** $-1, \frac{14}{3}$ **3.** -2, 5 **4.** 4, 7
5. $-\frac{3}{2}$, 5 **6.** -2, 10 **7.** $-1 \pm \frac{2}{3}\sqrt{3}$ **8.** $\frac{1}{2}(15 \pm \sqrt{241})$ **9.** No real roots
10. $\frac{1}{2} \pm \frac{\sqrt{2}}{2}$ **11.** $-\frac{3}{2}, \frac{1}{3}$ **12.** $1 \pm 2\sqrt{3}$ **13.** No real roots **14.** $\frac{3}{2} \pm \frac{1}{6}\sqrt{129}$
15. $1 \pm \frac{1}{2}\sqrt{14}$ **16.** $\frac{3}{2} \pm \frac{3}{10}\sqrt{5}$ **17.** $\frac{1}{2}(1 \pm \sqrt{5})$ **18.** No real roots **19.** $-\frac{5}{3}$, 3
20. $-\frac{4}{3}, \frac{8}{3}$ **21.** -1, 6 **22.** $\frac{9}{10}$, 1 **23.** $-3a$, $5a$ **24.** $-5b$, b **25.** -1, -2, 2, 3

Exercise 9-5 (page 236). **1.** $m, -n$ **2.** $0, c + d$ **3.** a, b **4.** $0, -\frac{b}{a}$ **5.** $2a, 3a$

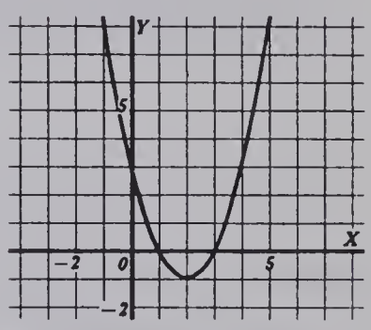
6. $a, \frac{1}{a}$ **7.** $\frac{1}{3}, m^2$ **8.** $\frac{b}{a}, -\frac{d}{c}$ **9.** $-\frac{c}{a}, -\frac{d}{b}$

Exercise 9-6 (page 239)

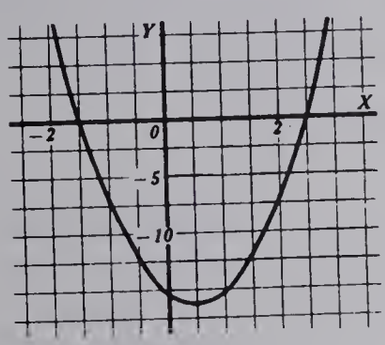
1. x-intercepts 2, 4
y-intercept 8
Range $\{y \mid y \geqq -1\}$
Axis of symmetry $x = 3$
Vertex $(3, -1)$



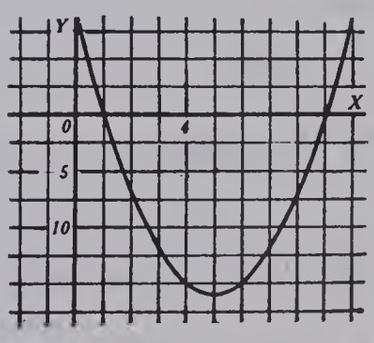
2. x-intercepts 1, 3
y-intercept 3
Range $\{y \mid y \geqq -1\}$
Axis of symmetry $x = 2$
Vertex $(2, -1)$



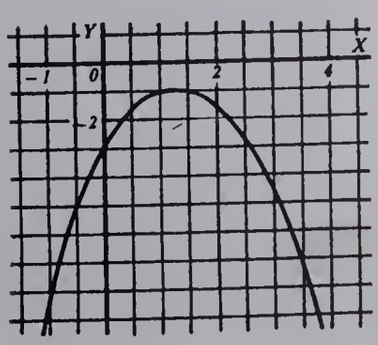
3. x-intercepts $-\frac{3}{2}, \frac{5}{2}$
y-intercept -15
Range $\{y \mid y \geqq -16\}$
Axis of symmetry $x = \frac{1}{2}$
Vertex $(\frac{1}{2}, -16)$



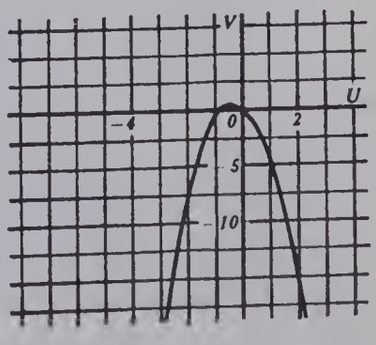
4. x-intercepts 1, 9
y-intercept 9
Range $\{y \mid y \geqq -16\}$
Axis of symmetry $x = 5$
Vertex $(5, -16)$



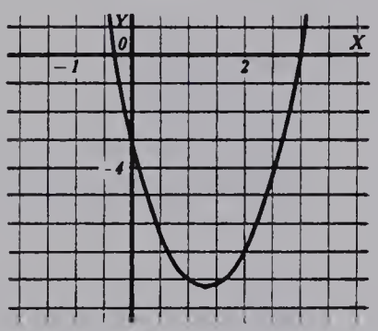
5. No x-intercepts
y-intercept -3
Range $\{y \mid y \leqq -1\}$
Axis of symmetry $x = \sqrt{2}$
Vertex $(\sqrt{2}, -1)$



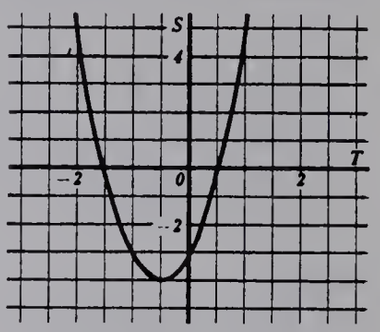
6. u-intercepts $-\frac{2}{3}, 0$
v-intercept 0
Range $\{v \mid v \leqq \frac{1}{3}\}$
Axis of symmetry $u = -\frac{1}{3}$
Vertex $(-\frac{1}{3}, \frac{1}{3})$



7. x-intercepts $-\frac{1}{3}$, 3
y-intercept -3
Range $\{y \mid y \geqq -\frac{25}{3}\}$
Axis of symmetry $x = \frac{4}{3}$
Vertex $(\frac{4}{3}, -\frac{25}{3})$



8. t-intercepts $-\frac{3}{2}$, $\frac{1}{2}$
s-intercept -3
Range $\{s \mid s \geqq -4\}$
Axis of symmetry $t = -\frac{1}{2}$
Vertex $(-\frac{1}{2}, -4)$



Exercise 9-7 (page 242).

12. Minimum value $-\frac{81}{8}$, for $x = \frac{1}{4}$

13. Maximum value $\frac{15}{2}$, for $x = -\frac{3}{2}$

14. Minimum value $-15\frac{1}{8}$, for $x = \frac{1}{4}$

15. Minimum value 2, for $x = \pm 1$

16. Minimum value $\frac{1}{2}$, for $x = \frac{1}{2}$

17. Minimum value 24, for $x = 3$

18. Minimum value -3, for $x = \frac{1}{3}$

19. 200′ and 300′; 60,000 sq. ft.

20. after 4 weeks

21. 10′ and 20′; 200 sq. ft.

22. $6\frac{1}{2}$ and $6\frac{1}{2}$

23. $6\frac{1}{2}$ and $6\frac{1}{2}$

25. 40 yds. (perpendicular to the bank) and 80 yds.

26. 32 cm. wide and 24 cm. high

27. 90 cents

28. 6 seconds; 582 ft.; 4 seconds and 8 seconds

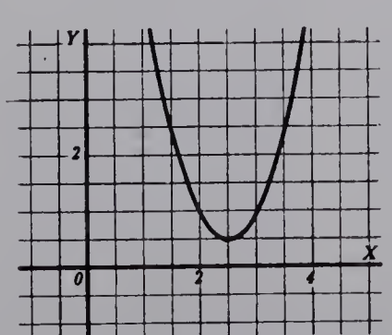
29. $\frac{600}{17}$ yds. and $\frac{600}{17}$ yds.; $\frac{450}{17}$ yds. and $\frac{900}{17}$ yds.

30. Radius of semicircle $\dfrac{30}{4+\pi}$ ft., rectangle $\dfrac{60}{4+\pi}$ ft. and $\dfrac{30}{4+\pi}$ ft.

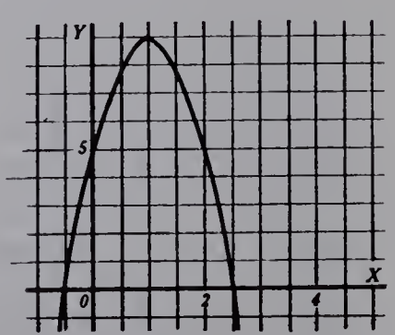
32. Yes. 79,500 sq. yds. (approx.)

Exercise 9-8 (page 247).

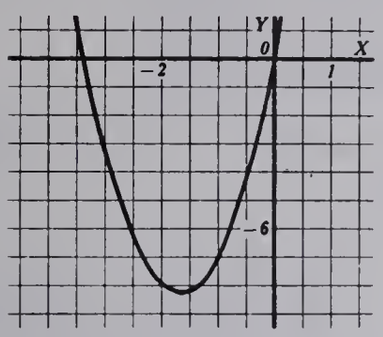
7. Range $\{y \mid y \geqq \frac{1}{2}\}$
Minimum value $\frac{1}{2}$
Axis of symmetry $x = \frac{5}{2}$
Vertex $(\frac{5}{2}, \frac{1}{2})$
No real roots



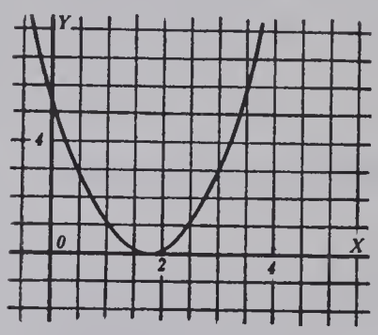
8. Range $\{y \mid y \leqq 9\}$
Maximum value 9
Axis of symmetry $x = 1$
Vertex (1, 9)
Roots $-\frac{1}{2}$, $\frac{5}{2}$



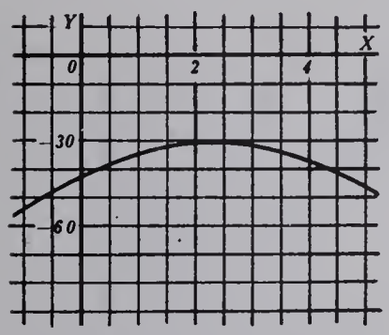
9. Range $\{y \mid y \geqq -\frac{25}{3}\}$
Minimum value $-\frac{25}{3}$
Axis of symmetry $x = -\frac{5}{3}$
Vertex $(-\frac{5}{3}, -\frac{25}{3})$
Roots $-\frac{10}{3}$, 0



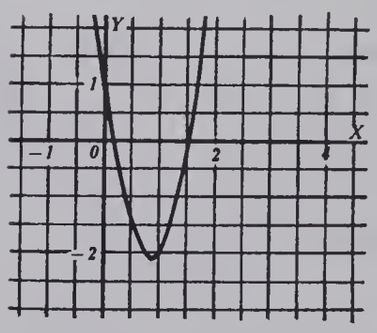
10. Range $\{y \mid y \geqq 0\}$
Minimum value 0
Axis of symmetry $x = \sqrt{3}$
Vertex $(\sqrt{3}, 0)$
Roots $\sqrt{3}$, $\sqrt{3}$



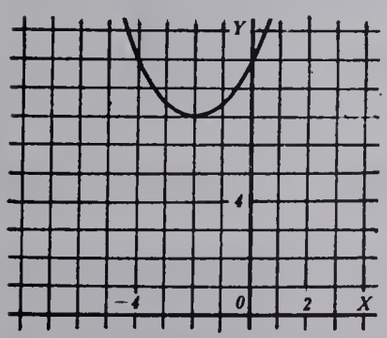
11. Range $\{y \mid y \leqq -\frac{247}{8}\}$
Maximum value $-\frac{247}{8}$
Axis of symmetry $x = \frac{9}{4}$
Vertex $(\frac{9}{4}, -\frac{247}{8})$
No real roots



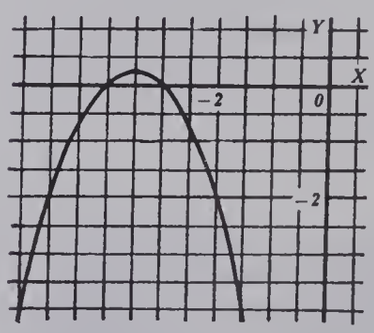
12. Range $\{y \mid y \geqq -\frac{11}{5}\}$
Minimum value $-\frac{11}{5}$
Axis of symmetry $x = \frac{4}{5}$
Vertex $(\frac{4}{5}, -\frac{11}{5})$
Roots $\frac{1}{5}(4 \pm \sqrt{11})$



13. Range $\{y \mid y \geqq 7\}$
Minimum value 7
Axis of symmetry $x = -1$
Vertex $(-1, 7)$
No real roots



14. Range $\{y \mid y \leqq \frac{1}{4}\}$
Maximum value $\frac{1}{4}$
Axis of symmetry $x = -\frac{7}{2}$
Vertex $(-\frac{7}{2}, \frac{1}{4})$
Roots -4, -3



Exercise 9-10 (page 251). **3.** 0; real and equal roots **4.** 81; real and unequal roots **5.** -3; no real roots **6.** 81; real and unequal roots **7.** 1; real and unequal roots **8.** 0; real and equal roots **9.** $a^2 + 4$; real and unequal roots **10.** 16; real and unequal roots **11.** $\pi^2 - 48$; no real roots **12.** 0; real and equal roots **13.** $\frac{1}{2}$, $\frac{2}{3}$ **14.** $\frac{3}{2} \pm \frac{1}{2}\sqrt{11}$ **15.** $\frac{1}{4}(13 \pm \sqrt{89})$ **16.** $-\frac{1}{40}$, $\frac{1}{30}$ **17.** $\frac{5}{2} \pm \frac{1}{2}\sqrt{15}$ **18.** 20, 21

19. $\frac{6}{5} \pm \frac{1}{5}\sqrt{26}$ **20.** $-b \pm \sqrt{b^2 + c^2}$ **21.** $(3 \pm \sqrt{6})a$ **22.** $\frac{1}{a}, -\frac{1}{2}$ **23.** $\frac{1}{\pi}, -\pi$
24. $2\sqrt{2}, -\sqrt{2}$

Exercise 9-11 (page 253). **1.** 20, 5 or $-20, -5$ **2.** 7, 13 **3.** $(25 - 5\sqrt{13})$ ft.
4. 66 **5.** 3, 4, 5 **6.** 18, 20, 22 **7.** 14, 15, 16

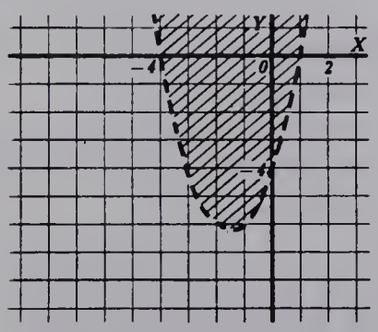
Exercise 9-12 (page 255). **1.** $\frac{3}{2} \pm \frac{\sqrt{11}}{2}i$ **2.** $-\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$ **3.** $1 \pm \frac{\sqrt{2}}{2}i$
4. $\frac{1}{2}, \frac{1}{2}$ **5.** $\frac{1}{3} \pm \frac{\sqrt{2}}{3}i$ **6.** $\frac{1 \pm \sqrt{13}}{2\sqrt{3}}$ **7.** $\frac{1}{4} \pm \frac{\sqrt{7}}{4}i$ **8.** $-\frac{1}{2}, 1$

Exercise 9-13 (page 258). **7.** $\frac{9}{4}, \frac{10}{3}$ **8.** ± 7 **9.** $-3, 5$ **10.** 1 **11.** $-\frac{2}{3} \pm \frac{2}{3}\sqrt{13}$
12. $\pm 2, \pm 3$ **13.** $5^{\frac{1}{3}}, 7^{\frac{1}{3}}$ **14.** $3, -4$ **15.** $\frac{1}{2}, 2, \frac{1}{3}, 3$ **16.** $-3, 1$ **17.** $\frac{1}{2}, 1$
18. $1, 1, -\frac{3}{2} \pm \frac{\sqrt{5}}{2}$ **19.** $-3, 6$ **20.** $1, 2$ **21.** $10^{10}, 10^{-\frac{5}{2}}$ **22.** $10^{\frac{1}{2}}, 10^{\frac{2}{3}}$
23. 1, 3 **24.** 52 m.p.h. and 44 m.p.h. **25.** 8 and 12 dollars **26.** $\frac{1}{2}$ **27.** 0, 1
28. 80 and 120 minutes

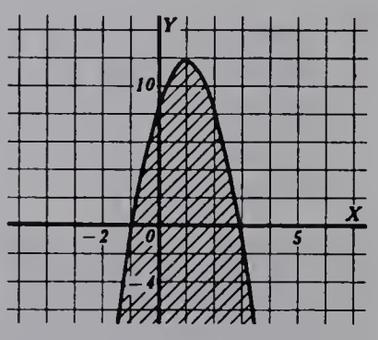
Exercise 9-14 (page 260). **9.** 7 **10.** no real root **11.** $-\frac{14}{3}$ **12.** 14 **13.** 3
14. no real roots **15.** $\frac{1}{81}$ **16.** 19 **17.** $\frac{1}{4}$ **18.** $-\frac{20}{3}, 5$ **19.** 9 **20.** $(10 + 10\sqrt{2})$ in.
21. $(-8 + 6\sqrt{3})$ in. **22.** 5 ft. and 12 ft. **23.** $-\frac{122}{9}$ or 22 **24.** $\frac{9}{4}$ **25.** $4, -\frac{1}{2}$

Exercise 9-15 (page 264).

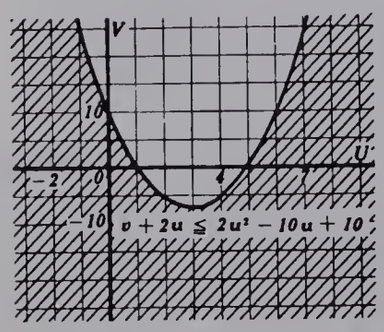
1.



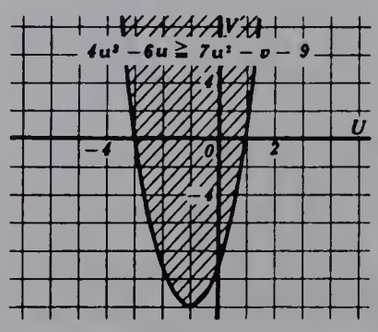
2.



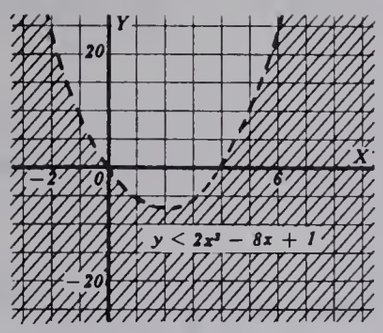
3.



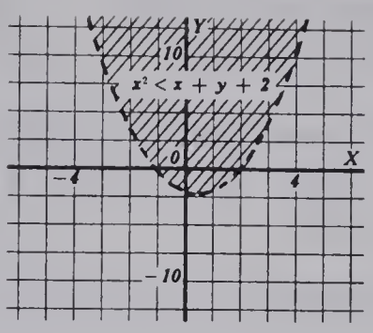
4.



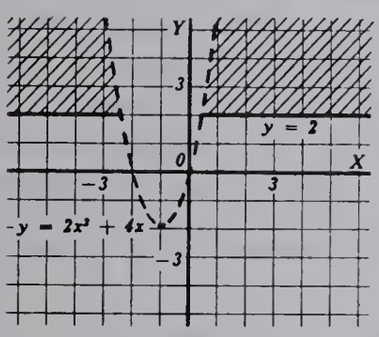
5.



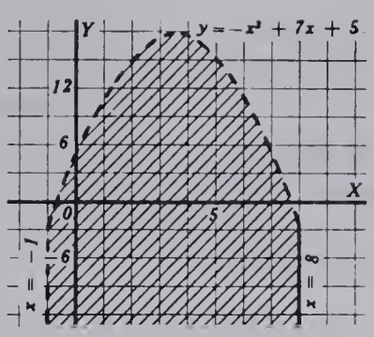
6.



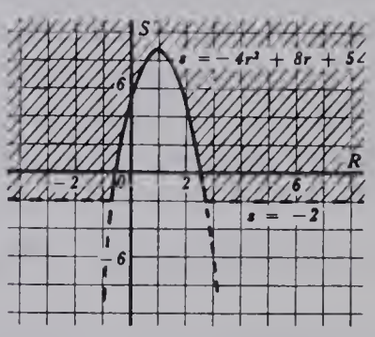
7.



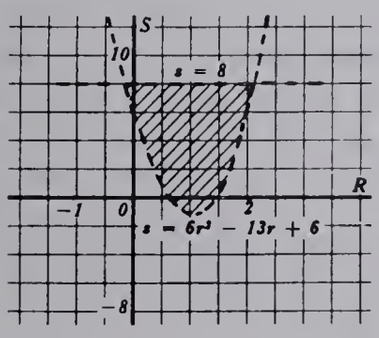
8.



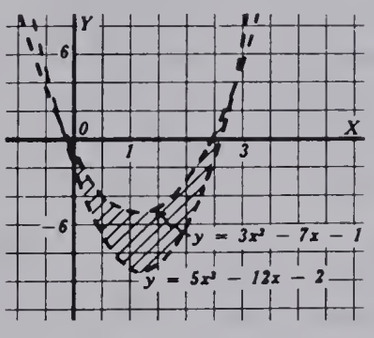
9.



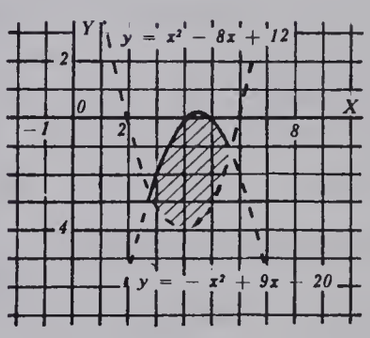
10.



11.



12.



13. $\left(\frac{5+\sqrt{33}}{4}, \frac{9+\sqrt{33}}{8}\right), \left(\frac{5-\sqrt{33}}{4}, \frac{9-\sqrt{33}}{8}\right)$

Practice Exercise 9-16 (page 264). **1.** 7, −6 **2.** 9, −8 **3.** $-\frac{1}{5}, -\frac{1}{5}$ **4.** $-\frac{1}{4}, -\frac{1}{4}$ **5.** $-\frac{1}{2}$, 1 **6.** $-\frac{1}{16}$, 1 **7.** $\frac{9}{2}$, −2 **8.** $\frac{4}{3}, -\frac{5}{4}$ **9.** −9, −9 **10.** $\frac{1}{16}$, 1

Practice Exercise 9-17 (page 265). **1.** $\frac{-1 \pm \sqrt{5}}{2}$ **2.** 2, −1 **3.** $\frac{-9 \pm \sqrt{30}}{3}$ **4.** $\frac{1 \pm \sqrt{57}}{4}$ **5.** $\frac{5}{2}$, −1 **6.** 2, $\frac{1}{3}$ **7.** 0, 16 **8.** $\frac{7 \pm \sqrt{55}}{3}$ **9.** $\frac{3 \pm \sqrt{21}}{6}$ **10.** 3, −1

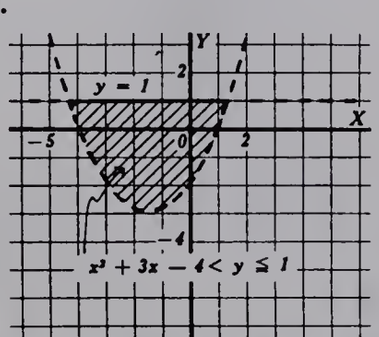
Practice Exercise 9-18 (page 265). **1.** $-\frac{1}{2}$, 1 **2.** $\frac{5 \pm \sqrt{21}}{2}$ **3.** $\frac{-5 \pm \sqrt{10}}{3}$ **4.** $\frac{-3 \pm \sqrt{17}}{4}$ **5.** $\frac{3 \pm \sqrt{69}}{5}$ **6.** $\frac{5 \pm \sqrt{17}}{4}$ **7.** $\frac{1 \pm \sqrt{7}i}{4}$ **8.** $1 \pm \frac{1}{2}\sqrt{2}i$ **9.** $-1 \pm \sqrt{5}$ **10.** $\frac{-1 \pm \sqrt{11}}{10}$

Practice Exercise 9-19 (page 265). **1.** minimum value $\frac{3}{2}$ **2.** minimum value $-\frac{15}{2}$ **3.** minimum value $-\frac{121}{8}$ **4.** maximum value 10 **5.** maximum value $\frac{57}{8}$ **6.** maximum value $\frac{100}{3}$

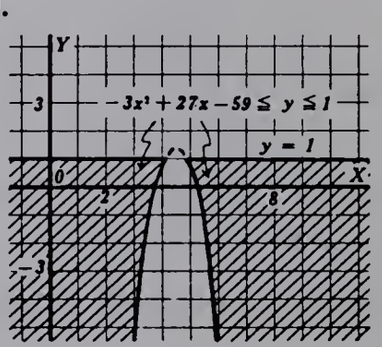
Practice Exercise 9-20 (page 265). **1.** $(0, -1)$ **2.** $(0, -1)$ **3.** $(\frac{1}{6}, -\frac{25}{12})$ **4.** $(-\frac{1}{6}, -\frac{23}{12})$ **5.** $(\frac{1}{2}, -\frac{1}{4})$ **6.** $(-\frac{1}{2}, \frac{1}{4})$ **7.** $(1, 1)$ **8.** $(-1, 3)$ **9.** $(-\frac{1}{4}, \frac{63}{16})$ **10.** $(-3, -8)$

Review Exercise 9-21 (page 266). **5.** x-intercepts $\frac{1}{2} \pm \frac{\sqrt{2}}{2}$, y-intercept -1, Range $\{y \mid y \geqq -2\}$, Axis of symmetry $x = \frac{1}{2}$, Vertex $(\frac{1}{2}, -2)$, Opens upward. **6.** x-intercepts 2, 8, y-intercept -16, Range $\{y \mid y \leqq 9\}$, Axis of symmetry $x = 5$, Vertex $(5, 9)$, Opens downward. **7.** u-intercepts -3, 1, v-intercept -9, Range $\{v \mid v \geqq -12\}$, Axis of symmetry $u = -1$, Vertex $(-1, -12)$, Opens upward. **8.** No r-intercepts, s-intercept 15, Range $\{s \mid s \geqq 12\}$, Axis of symmetry $r = 1$, Vertex $(1, 12)$, Opens upward. **9.** x-intercepts $-\frac{5}{2}, \frac{3}{2}$, y-intercept -15, Range $\{y \mid y \geqq -16\}$, Axis of symmetry $x = -\frac{1}{2}$, Vertex $(-\frac{1}{2}, -16)$, Opens upward. **10.** x-intercepts 83, 97, y-intercept 16, 102, Range $\{y \mid y \geqq -98\}$, Axis of symmetry $x = 90$, Vertex $(90, -98)$, Opens upward. **11.** x-intercepts $\frac{a-b}{2}, \frac{a+b}{2}$, y-intercept $a^2 - b^2$, Range $\{y \mid y \geqq -b^2\}$, Axis of symmetry $x = \frac{a}{2}$, Vertex $\left(\frac{a}{2}, -b^2\right)$ Opens upward. **12.** x-intercepts $-b \pm \sqrt{a^2 + b^2}$, y-intercept a^2, Range $\{y \mid y \leqq a^2 + b^2\}$, Axis of symmetry $x = -b$, Vertex $(-b, a^2 + b^2)$, Opens downward. **13.** No x-intercepts, y-intercept $2(a^2 + b^2)$, Range $\{y \mid y \geqq (a - b)^2\}$, Axis of symmetry $x = -(a + b)$, Vertex $(-(a + b), (a - b)^2)$, Opens upward. **14.** 2 **16.** 13,612.5 sq. yds. **17.** 25 sq. ins. **18.** 35 m.p.h. **19.** Width $\frac{1}{6}(33 + 11\sqrt{3})$ ft. **20.** 1100, \$12,100 **21.** 16, 17 or -17, -16 **22.** 4 in. and 6 in. **23.** 15, 20 and 25 units **24.** \$10 **25.** 15 in. **26.** $\sqrt{6} + 1$ ft. wide, $\sqrt{6} + 3$ ft. long **27.** 6 hrs. **28.** 12 **29.** 40 m.p.h., 56 m.p.h. **30.** 7 **31.** $-\frac{1}{2}, -2$ **32.** 7 **33.** $\frac{5 \pm \sqrt{29}}{2}, \frac{5 \pm \sqrt{41}}{2}$ **34.** 64 **35.** $(\frac{7}{2})^{\frac{4}{3}}$ **36.** No real roots **37.** $-3, 5$ **38.** $-1, 3$

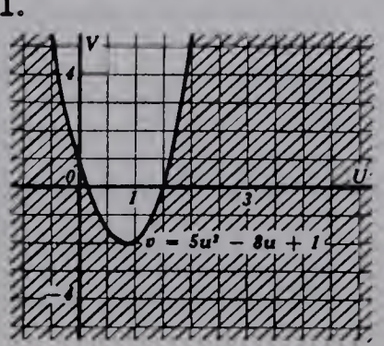
39.



40.



41.



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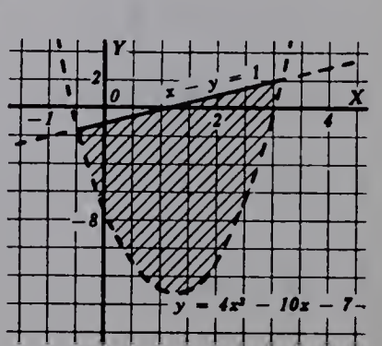
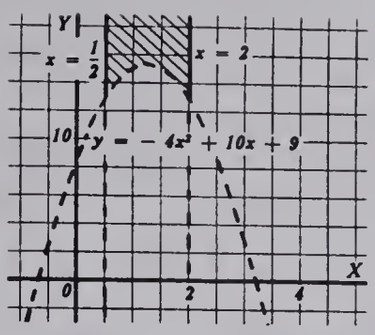
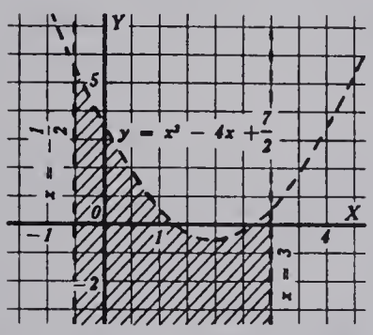


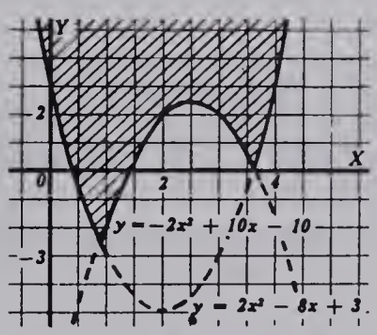
43.



44.



45.



46. $\left(\frac{11 + \sqrt{217}}{8}, \frac{3 + \sqrt{217}}{8}\right), \left(\frac{11 - \sqrt{217}}{8}, \frac{3 - \sqrt{217}}{8}\right)$

Exercise 10-1 (page 271). **1.** Real, unequal **2.** Real, unequal **3.** Non-real **4.** Real, unequal **5.** Real, equal **6.** Real, unequal **7.** Non-real **8.** Real, unequal **9.** Real, unequal **10.** Non-real **11.** Non-real **12.** Real, unequal **13.** Non-real **14.** Real, unequal **15.** Real, unequal **16.** $-\frac{13}{3}$ **17.** $\pm\frac{1}{2}\sqrt{29}$ **18.** $-\frac{10}{9}$, 2 **19.** 3, 5 **20.** 4 **21.** $\frac{1}{8}$ **22.** $-\frac{1}{4}$ **23.** $\frac{1}{3}$ (*Note:* if $k = 3$, then $x = -2$ which is inadmissible.) **24.** (i) $k > -\frac{13}{3}$ (ii) $k < -\frac{13}{3}$ **25.** (i) $|k| > \frac{1}{2}\sqrt{29}$; $k < 3$ or $k > 5$; $k \neq 4$; $k < -\frac{10}{9}$ or $k > 2$ (ii) $|k| < \frac{1}{2}\sqrt{29}$; $3 < k < 5$; no values; $-\frac{10}{9} < k < 2$ **26.** $-1 \pm i$ **27.** $\frac{3 \pm 4i}{2}$ **28.** $\frac{-3 \pm 5\sqrt{3}i}{6}$ **29.** $\frac{-1 \pm \sqrt{5\pi - 1}i}{\pi}$

Exercise 10-3 (page 273). **7.** (ii) -1 **8.** 2 **9.** 0 **10.** 8 **11.** $\frac{19}{2}$ **12.** $\frac{7}{2}$ and $\frac{7}{4}$ **13.** (i) and (iii) **14.** (i) $c = a$ (ii) $b = 0$ (iii) $c = 0$ **15.** $-2; \frac{1}{2}$ **16.** 1 **17.** $\frac{50}{9}$ **18.** $2(a + b - c)$, $a^2 + b^2 - c^2$ **19.** $b + c - a$ **20.** $\frac{a-b}{a+b}$ **21.** $2b^2 = 9ac$

Exercise 10-4 (page 277).

11. $6x^2 + 5x - 21 = 0$ **12.** $x^2 - 2x - 1 = 0$ **13.** $x^2 + (7 - \pi)x - 7\pi = 0$ **14.** $x^2 + 4x + 1 = 0$ **15.** $4x^2 - 8\sqrt{3}x + 11 = 0$ **16.** $8x^2 - 20x + 7 = 0$ **17.** $x^2 - 2ax + a^2 - b^2 = 0$ **18.** $x^2 - (m + n + mn)x + mn(m + n) = 0$ **19.** $x^3 - 6x^2 + 11x - 6 = 0$ **20.** $24x^3 - 26x^2 + 9x - 1 = 0$ **21.** $x^3 - 11x^2 + 37x - 35 = 0$ **22.** $x^3 - 2(a + b)x^2 + (a^2 + 3ab + b^2)x - ab(a + b) = 0$ **24.** (ii) and (iv) **25.** $(x - 3 - \sqrt{3})(x - 3 + \sqrt{3})$ **26.** $(y - 5)(y + 22)$ **27.** $\left(x - \frac{9}{2} - \frac{\sqrt{5}}{2}\right)\left(x - \frac{9}{2} + \frac{\sqrt{5}}{2}\right)$ **28.** $(2x - 5)(3x + 17)$ **29.** $3\left(x - 2 - \frac{\sqrt{33}}{3}\right)\left(x - 2 + \frac{\sqrt{33}}{3}\right)$ **30.** $(40y - 1)(45y + 1)$ **31.** $(a^2 - b^2)x^2 - 2(a^2 + b^2)x + a^2 - b^2 = 0$ **32.** $4x^2 - 28x + 45 = 0$

Exercise 10-5 (page 280).

1. $x^2 + 7x - 18 = 0$ **2.** $x^2 + 20x + 76 = 0$ **3.** $2x^2 - 5x + 3 = 0$ **4.** $9x^2 - 7x - 2 = 0$ **5.** $9x^2 - 13x + 4 = 0$ **6.** $4x^2 - 53x + 49 = 0$ **7.** $3x^2 - \sqrt{2}x - \pi = 0$ **8.** $x^2 - (4\pi^2 + 2\sqrt{2})x + 2 = 0$ **9.** $4x^2 + 2px + q = 0$ **10.** $qx^2 + px + 1 = 0$ **11.** $x^2 + (2q - p^2)x + q^2 = 0$ **12.** $x^2 + (4q - p^2)x + 4q^2 - 2p^2q = 0$ **13.** $x^2 + 3px + q + 2p^2 = 0$ **14.** $x^2 + (p - 2q)x + q(q - p + 1) = 0$

Exercise 10-7 (page 283).

1. Real, unequal **2.** Non-real **3.** Real, equal **4.** Real, unequal **5.** Real, equal **6.** Non-real **7.** $-\frac{29}{5}$ **8.** $\frac{-5 \pm \sqrt{10}}{2}$ **9.** $\frac{-2 \pm 2\sqrt{7}}{3}$ **10.** 3, −1 **11.** $\pm\sqrt{2}$ **12.** No real value

Exercise 10-8 (page 283).

1. $\frac{7}{2}, \frac{3}{2}$ **2.** $-\frac{3}{2}, -2$ **3.** $\frac{1}{2}, -\frac{3\pi}{2}$ **4.** $\frac{\pi}{6}, -\frac{\sqrt{2}}{6}$ **5.** $0, -\frac{3}{8}$ **6.** $1, \frac{3 - 7\pi}{3\pi}$ **7.** 5 **8.** $\frac{1}{36}$ **9.** $-\frac{20}{9}$ **10.** $\frac{3\pi}{7}$ **11.** $\frac{\pi}{\sqrt{2}}$ **12.** No value of k

Exercise 10-9 (page 284).

1. $x^2 - 4x + 1 = 0$ **2.** $x^2 - (2\pi - 3)x - 6\pi = 0$ **3.** $x^2 + 2x - 4 = 0$ **4.** $9x^2 - 18\sqrt{7}x + 62 = 0$ **5.** $9x^2 - 18x + 2 = 0$ **6.** $x^3 - 9x^2 + 23x - 15 = 0$ **7.** $12x^3 - 19x^2 + 8x - 1 = 0$ **8.** $x^3 - 13x^2 + 46x - 28 = 0$ **9.** $x^3 - 3x^2 - 16x + 30 = 0$

Review Exercise 10-10 (page 284).

1. Real and equal **2.** Real and unequal **3.** Non-real **4.** Real and unequal **5.** $\frac{1}{5}$ **6.** $\frac{11}{4}$ **7.** $\frac{13}{12}$ **8.** $3, \frac{7}{3}$ **9.** $0, -\frac{7}{4}$ **10.** $-\frac{7}{3}, 1$ **11.** $-2, -\frac{9}{2}$ **12.** 2 **13.** 0 **14.** 36 **15.** $9x^2 - 18\sqrt{5}x + 44 = 0$ **16.** $x^2(-\sqrt{5} + \sqrt{3} - 1)x + \sqrt{15} - 3\sqrt{3} + 2\sqrt{5} - 6 = 0$ **17.** $(x - 4 - \sqrt{3})(x - 4 + \sqrt{3})$ **18.** $5\left(x - \frac{6 + \sqrt{26}}{5}\right)\left(x - \frac{6 - \sqrt{26}}{5}\right)$ **19.** (i) 1 (ii) $k < 1$ or $k > 1$ **20.** $2x^2 - 25x + 77 = 0$ **21.** $2, \frac{-5 + \sqrt{5}}{2}, \frac{-5 - \sqrt{5}}{2}$

Exercise 11-2 (page 293). **3.** (i) 120° (ii) 40° (iii) 90° (iv) 70° (v) 70° (vi) 90° **4.** (i) 30° (ii) 80° (iii) 70° (iv) 70° **5.** 135° **6.** 110° **7.** 50° **8.** 180° **9.** 70° **10.** 106°, 66°, 74°, 114°

Exercise 11-5 (page 303). **17.** 20°, 60°, 100° **18.** $52\frac{1}{2}°$, 60°, $67\frac{1}{2}°$ **19.** 70°, 70°, 40° **23.** 2 cm., 6 cm.

Exercise 11-7 (page 314).

1.

6 *sides*	(x) 12 *sides*
(i) 60°, 30°	(i) 30°, 15°
(ii) 5.2 in.	(ii) 5.8 in.
(iii) 6 in.	(iii) 3.1 in.
(iv) 36 in.	(iv) 37.2 in.
(v) 37.7 in.	(v) 37.7 in.
(vi) 1.7 in.	(vi) 0.5 in.
(vii) 93.6 sq. in.	(vii) 107.9 in.
(viii) 113.0 sq. in.	(viii) 113.0 sq. in.
(ix) 19.4 sq. in.	(ix) 5.1 sq. in.
(xi) very much less in each case.	
(xii) 120°, 150°	

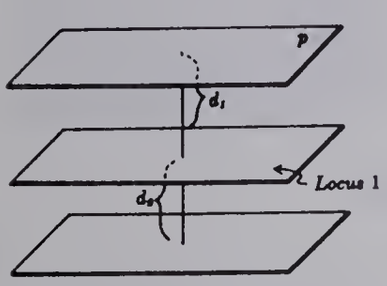
2.

6 *sides*	(vi) 12 *sides*
(i) 6.9 in.	(i) 3.2 in.
(ii) 41.5 in.	(ii) 38.5 in.
(iii) 3.8 in.	(iii) 0.8 in.
(iv) 124.7 sq. in.	(iv) 115.6 sq. in.
(v) 11.7 sq. in.	(v) 2.6 sq. in.
(vii) very much less in each case.	
(viii) The circumference and area of a circle is the limit of the perimeter and area of the circumscribed polygon respectively, as the number of sides is increased without bound.	

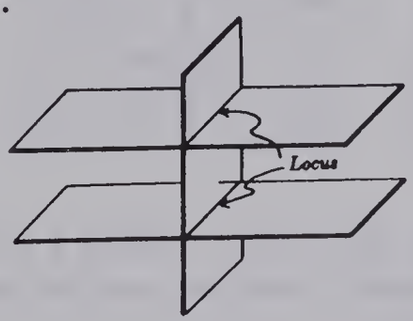
Exercise 11-8 (page 317). **1.** (i) 20.9 in. (ii) 31.4 in. (iii) 12.6 in. (iv) 47.1 in.
2. (i) 2.4 sq. in. (ii) 5.7 sq. in. (iii) 9.4 sq. in. (iv) 11.8 sq. in.
3. (i) 60 sq. cm. (ii) 72 sq. cm. (iii) 150 sq. cm. (iv) 360 sq. cm.
4. 1.9 cm., 1.9 cm. **5.** (i) 0.5 sq. in. (ii) 23.9 sq. in. (iii) 35.4 sq. in.
6. (i) 9.4 sq. in. (ii) no **7.** $2a^2$ sq. units

Exercise 11-10 (page 326).

1.



2.



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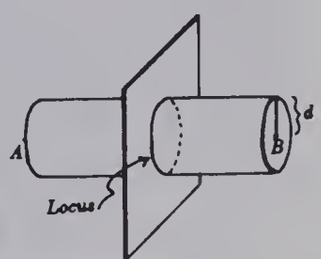
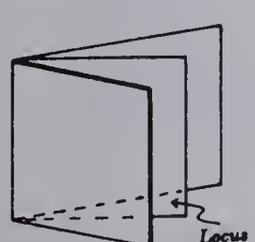
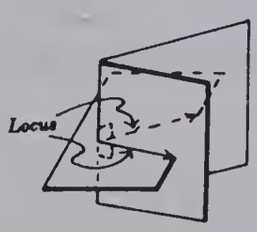


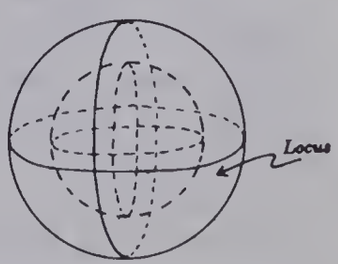
4.



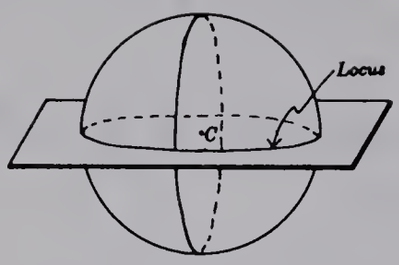
5.



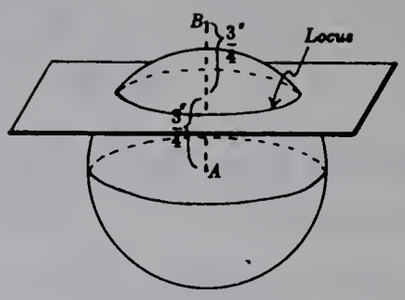
6.



7.

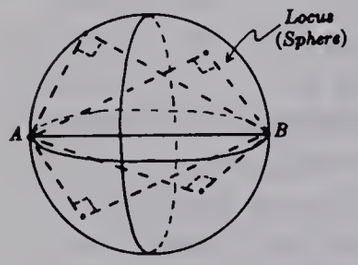


8

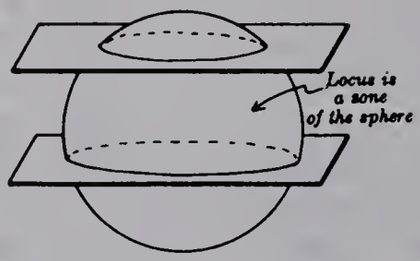


The locus is called a great circle of the sphere.

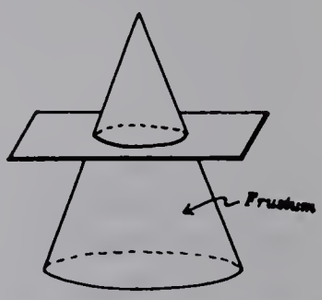
9.



10.



11.



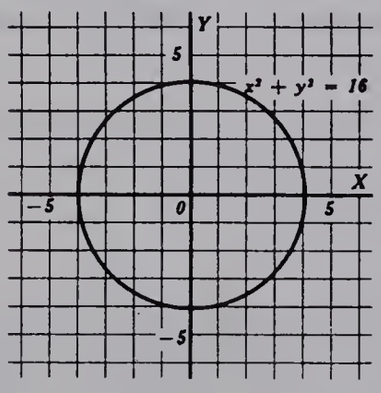
Exercise 11-11 (page 330).

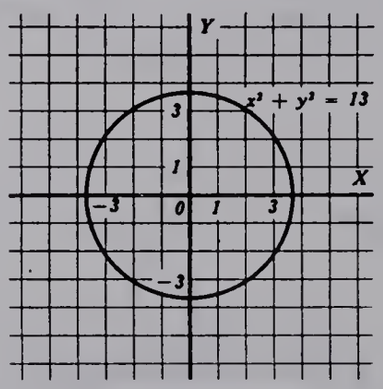
1. (i) $\pm 4, \pm 4$;

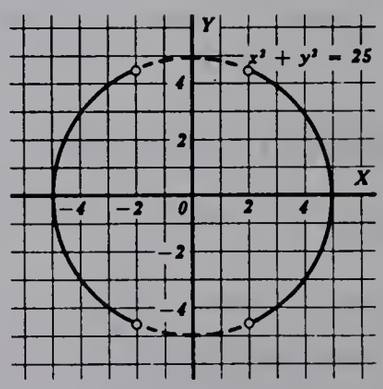
(ii) $\{x \mid -4 \leqq x \leqq 4\}, \{y \mid -4 \leqq y \leqq 4\}$;

(iii) symmetric with respect to both axes and the origin.

(iv) See graph at left below.







2. (i) $\pm\sqrt{13}, \pm\sqrt{13}$;

(ii) $\{x \mid -\sqrt{13} \leqq x \leqq \sqrt{13}\}, \{y \mid -\sqrt{13} \leqq y \leqq \sqrt{13}\}$;

(iii) symmetric with respect to both axes and the origin.

(iv) See graph centre above.

3. (i) x-intercepts ± 5, no y-intercepts.

(ii) $\{x \mid 2 < |x| \leqq 5\}, \{y \mid -\sqrt{21} < y < \sqrt{21}\}$;

(iii) symmetric with respect to both axes and the origin.

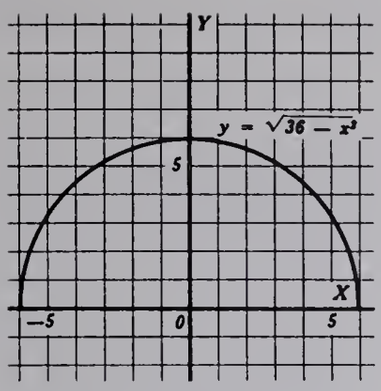
(iv) See graph at the right above.

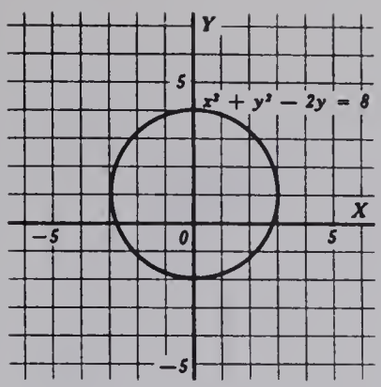
4. (i) $\pm 6, 6$;

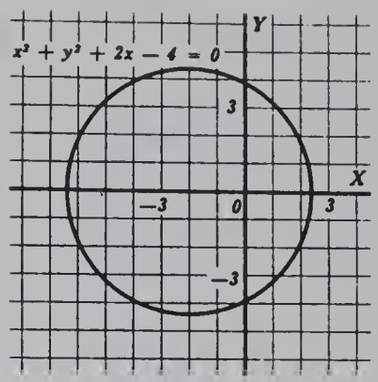
(ii) $\{x \mid -6 \leqq x \leqq 6\}, \{y \mid 0 \leqq y \leqq 6\}$;

(iii) symmetric with respect to the y-axis.

(iv) See graph on the left at the top of page 503







5. (i) $\pm 2\sqrt{2}$, 4 or -2;
(ii) $\{x \mid -3 \leqq x \leqq 3\}$, $\{y \mid -2 \leqq y \leqq 4\}$;
(iii) symmetric with respect to the y-axis.
(iv) See graph centre above.

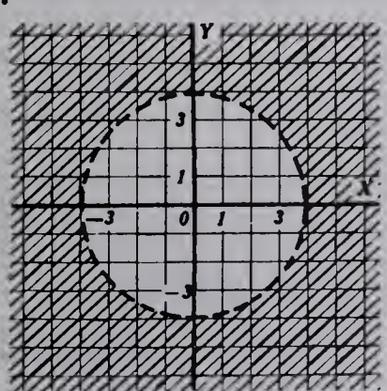
6. (i) $-1 \pm \sqrt{5}$, ± 2;
(ii) $\{x \mid -1 - \sqrt{5} \leqq x \leqq -1 + \sqrt{5}\}$, $\{y \mid -\sqrt{5} \leqq y \leqq \sqrt{5}\}$;
(iii) symmetric with respect to the x-axis.
(iv) See graph at the right above.

Exercise 11-12 (page 332).

2. (i) $x^2 + y^2 = 81$; (ii) $x^2 + y^2 = 75$; (iii) $x^2 + y^2 = 25$; (iv) $x^2 + y^2 = 7$; (v) $x^2 + y^2 = 100$; (vi) $x^2 + y^2 = 4$.
3. (i) $x^2 + y^2 = 9$; (ii) $x^2 + y^2 = 36$; (iii) $x^2 + y^2 = \frac{49}{2}$; (iv) $x^2 + y^2 = 2$.
4. (i) on (ii) interior (iii) on (iv) on (v) on (vi) on (vii) exterior

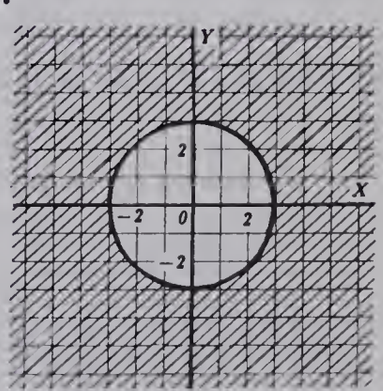
Exercise 11-13 (page 334).

1.



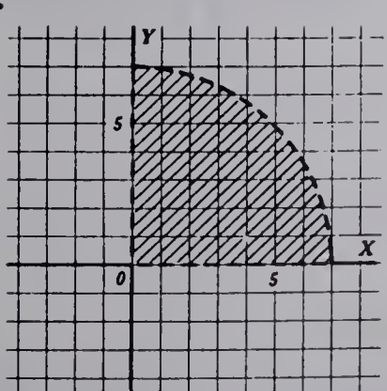
$x^2 + y^2 > 16$

2.



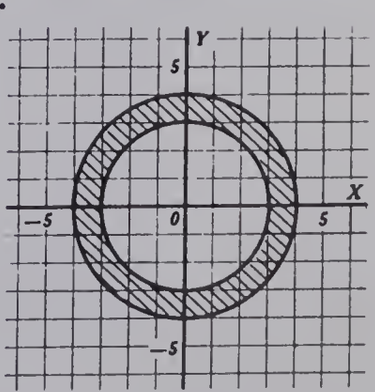
$x^2 + y^2 \geqq 9$

3.



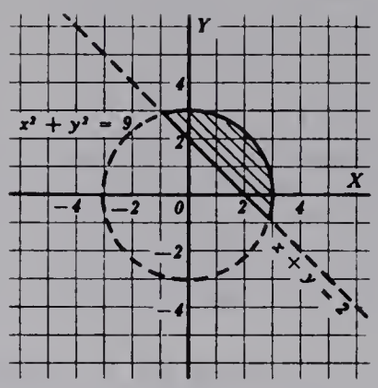
$x^2 + y^2 < 49$, $x > 0$, $y > 0$

4.



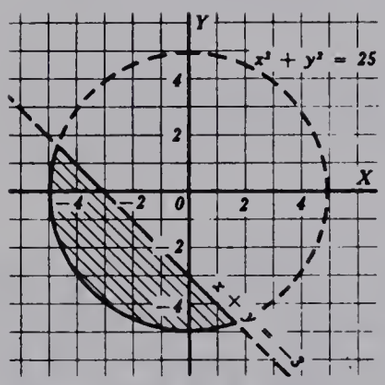
$x^2 + y^2 \leqq 16$, $x^2 + y^2 \geqq 9$

5.



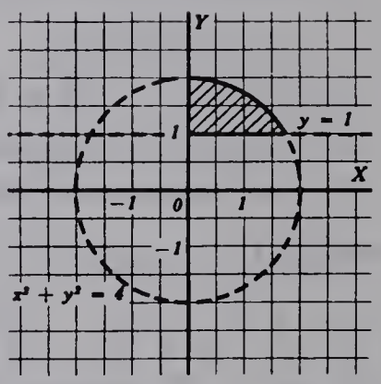
$x^2 + y^2 \leqq 9,\ x + y \geqq 2$

6.



$x^2 + y^2 \leqq 25,\ x + y \leqq -3$

7.



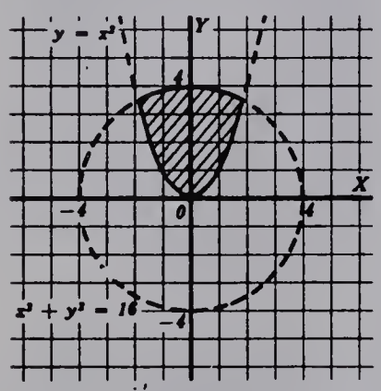
$x^2 + y^2 \leqq 4,\ x \geqq 0,\ y \geqq 1$

8. $\{(x, y) \mid x^2 + y^2 \leqq 25,\ y \geqq -2,\ x \geqq -3\}$ **9.** $\{(x, y) \mid x^2 + y^2 \leqq 25,\ x^2 + y^2 \geqq 4\}$

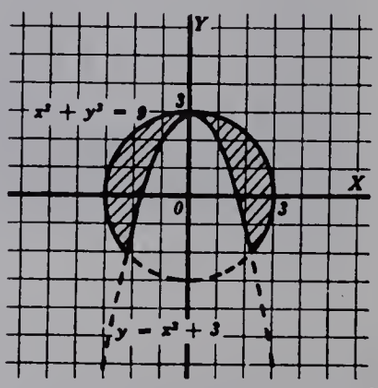
10. $\{(x, y) \mid x^2 + y^2 \leqq 25,\ x + 2y \leqq -2\}$

11. $\{(x, y) \mid x^2 + y^2 \leqq 9,\ x^2 + y^2 \geqq 4,\ x + y \geqq 1\}$

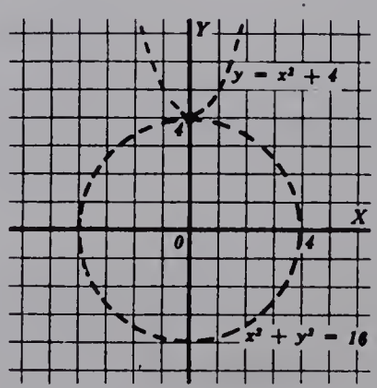
12.



13.

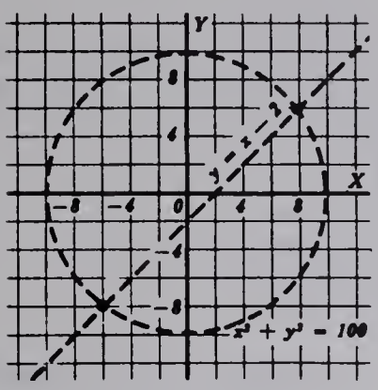


14.

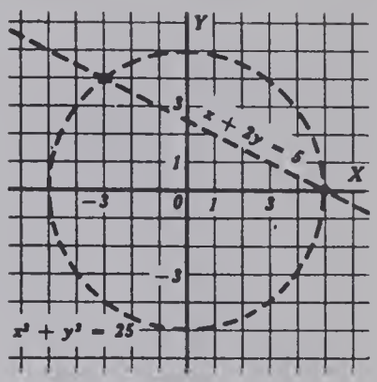


Exercise 11-14 (page 338).

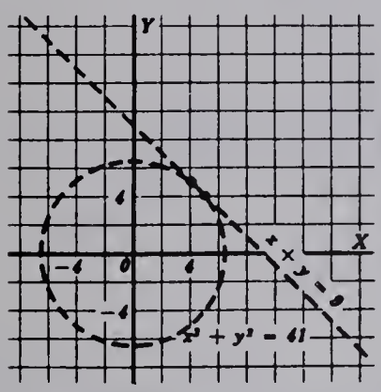
1. $\{(8, 6), (-6, -8)\}$



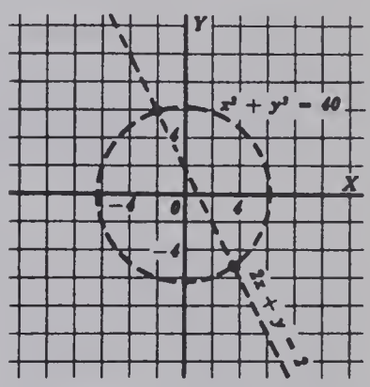
2. $\{(-3, 4), (5, 0)\}$



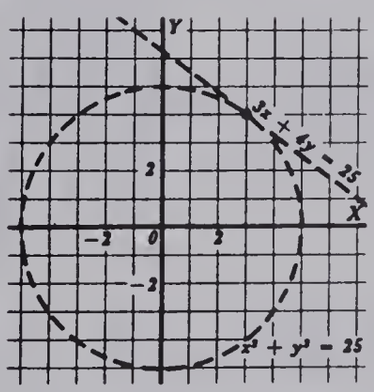
3. $\{(4, 5), (5, 4)\}$



4. $\{(-2, 6), (\frac{18}{5}, -\frac{26}{5})\}$

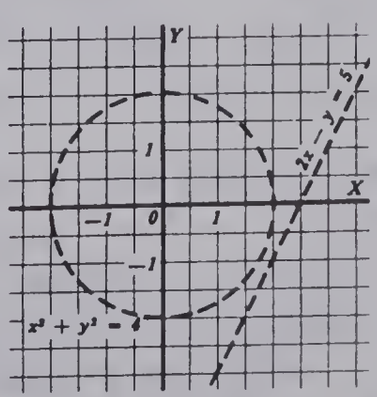


5. $\{(3, 4)\}$



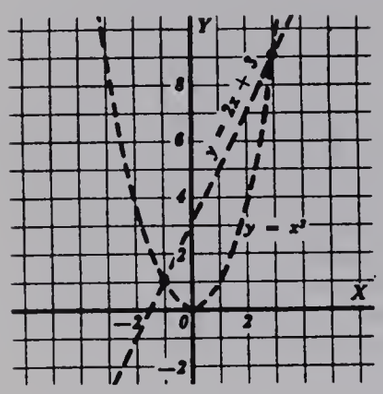
6. For $x, y \in R$, $F = \phi$.
For $x, y \in C$,

$$F = \left\{\left(\frac{10 + \sqrt{5}i}{5}, \frac{-5 + 2\sqrt{5}i}{5}\right), \left(\frac{10 - \sqrt{5}i}{5}, \frac{-5 - 2\sqrt{5}i}{5}\right)\right\}$$

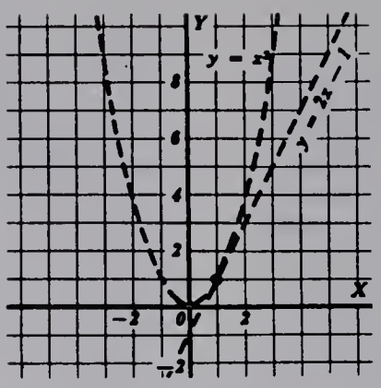


7. secant in 1, 2, 3, 4; tangent in 5; does not intersect in 6.

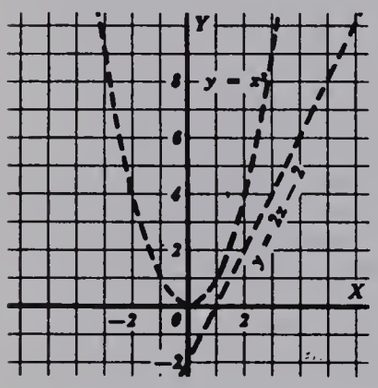
8. $\{(3, 9), (-1, 1)\}$



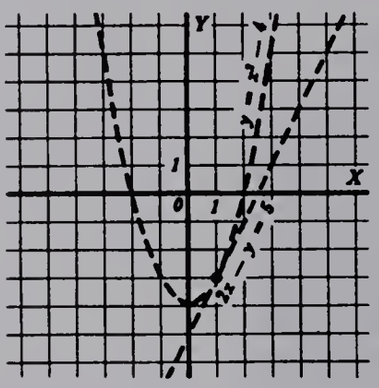
9. $\{(1, 1)\}$



10. For $x, y \in R$, $L = \phi$.
For $x, y \in C$,
$L = \{(1 + i, 2i), (1 - i, -2i)\}$.

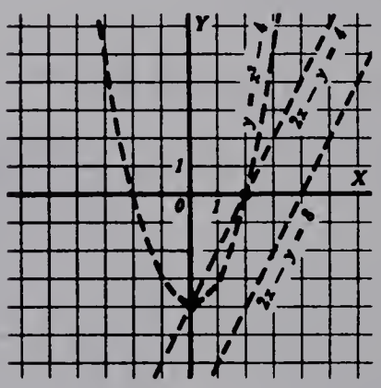


11. $\{(1, -3)\}$



12. $\{(2, 0), (0, -4)\}$

13. For $x, y \in R$, $N = \phi$.
For $x, y \in C$,
$N = \{(1 + \sqrt{3}i, -6 + 2\sqrt{3}i),$
$(1 - \sqrt{3}\,i, -6 - 2\sqrt{3}\,i\}$.

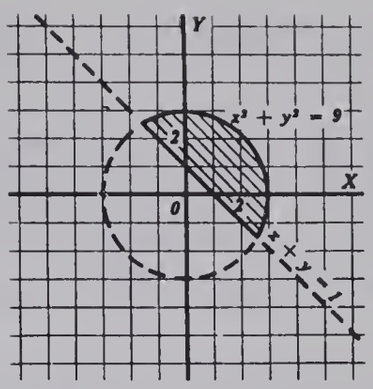


Exercise 11-15 (page 341). **5.** $\sqrt{10}$ **6.** $2\sqrt{6}$ **7.** $\frac{1}{2}\sqrt{43}$ **9.** $\sqrt{x_1^2 + y_1^2 - r^2}$

Exercise 11-17 (page 343). **1.** (i) $2x - 7y + 53 = 0$ (ii) $20x - 2y = 101$
(iii) $18x + 3y + 37 = 0$ (iv) $2x - 3\sqrt{5}y = 49$ (v) $16x - 4\sqrt{3}y = 39$

Review Exercise 11-18 (page 343). **1.** 3 cm. **2.** 8.5 in., 3.5 in. **3.** (i) 50° (ii) 40°
4. 120°, 135°, 60°, 45° **5.** 13° **6.** 110°, 40° **7.** 12 cm. **8.** 40° **9.** 24 cm.
10. (i) 45° (ii) 1.4 cm. (iii) 2 cm. **11.** (i) 3.1 in. (ii) 6.3 in. (iii) 7.1 in.
(iv) 7.8 in. **12.** (i) 1.0 sq. in. (ii) 3.1 sq. in. (iii) 4.7 sq. in. (iv) 5.2 sq. in.
13. 3.8 cm. **23.** (i) $x^2 + y^2 = 49$
(ii) $x^2 + y^2 = 3$ (iii) $x^2 + y^2 = 41$
(iv) $x^2 + y^2 = 36$ (v) $x^2 + y^2 = 64$

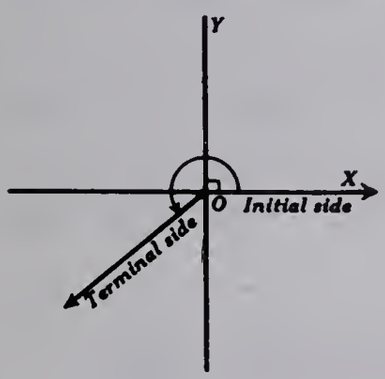
24. See graph at the right.



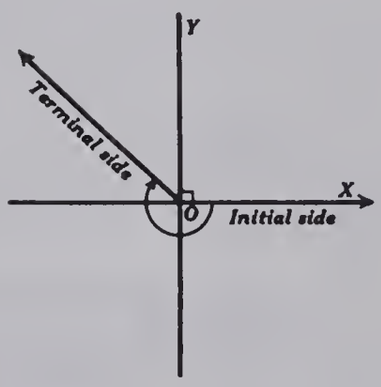
27. $2\sqrt{5}$ **28.** $2x - y = 5$

Exercise 12-1 (page 347).

1.



2.



3. (i) 90° (ii) 180° (iii) 270° (iv) 360° (v) 540° (vi) 990° (vii) −90° (viii) −180° (ix) −270° (x) −360° (xi) −450° (xii) −720°
4. (i) $\angle XOP$, −225° (ii) $\angle XOQ$, 210° (iii) $\angle XOR$, −315° (iv) $\angle XOM$, 300°
5. (i) −240° (ii) −120° (iii) −290° (iv) −70° (v) −90° **6.** (i) 340° (ii) 180° (iii) 160° (iv) 225° (v) 60° **7.** (i) 420°, 780°, 1140° (ii) −300°, −660°, −1020° (iii) $[60 + n(360)]°$, $n \in N_0$ (iv) $-[300 + n(360)]°$, $n \in N_0$ or $[60 - n(360)]°$, $n \in N_0$ (v) $[60 \pm n(360)]°$, $n \in N_0$

Exercise 12-2 (page 353). **31.** $\frac{\pi}{2}$ **32.** $\frac{\pi}{4}$ **33.** $\frac{\pi}{3}$ **34.** 2π **35.** $\frac{3\pi}{4}$ **36.** $\frac{5\pi}{6}$
37. 4π **38.** $\frac{\pi}{18}$ **39.** $-\frac{\pi}{36}$ **40.** $-\frac{\pi}{10}$ **41.** $-\pi$ **42.** $-\frac{\pi}{5}$ **43.** 180°
44. 30° **45.** 45° **46.** 120° **47.** 240° **48.** −540° **49.** 72° **50.** −1800°
51. 572.958° (approx.) **52.** −28.6479° (approx.) **53.** 57.2958° (approx.) **54.** −2°

Exercise 12-3 (page 359). **1.** 1 **2.** 1 **3.** $\frac{\sqrt{3}-1}{2\sqrt{2}}$ **4.** $\frac{\sqrt{3}+1}{2\sqrt{2}}$ **5.** $\frac{1}{4}$ **6.** $\frac{1}{4}$
7. $1\frac{1}{2}$ **8.** 5 **9.** $4\frac{1}{3}$ **10.** 0 **11.** $\frac{\sqrt{3}+3}{2}$ **12.** $\frac{3\sqrt{2}}{2}$ **13.** $\frac{1}{\sqrt{3}}$ **14.** 1
15. 1 **16.** 1 **17.** 0 **18.** 1 **19.** $-\frac{1}{4}$ **20.** $\sqrt{3}$

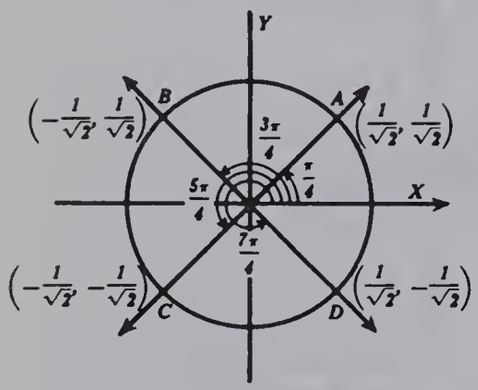
Exercise 12-5 (page 364). **1.** $\frac{1}{\sqrt{2}}$ **2.** $\sqrt{3}$ **3.** $\sqrt{3}$ **4.** $\frac{1}{2}$ **5.** $\frac{\sqrt{3}}{2}$ **6.** $-\sqrt{3}$

7. $-\frac{\sqrt{3}}{2}$ **8.** $-\frac{1}{\sqrt{3}}$ **9.** $-\frac{1}{2}$ **10.** $\frac{\sqrt{3}}{2}$ **11.** $\frac{1}{2}$ **12.** $\frac{\sqrt{3}}{2}$ **13.** $\frac{1}{\sqrt{3}}$ **14.** $-\frac{\sqrt{3}}{2}$ **15.** $-\sqrt{3}$

Exercise 12-6 (page 365). **1.** (i) $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ (ii) $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ (iii) $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$

2. (i) $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ (ii) $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ (iii) $\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$

3. (i)



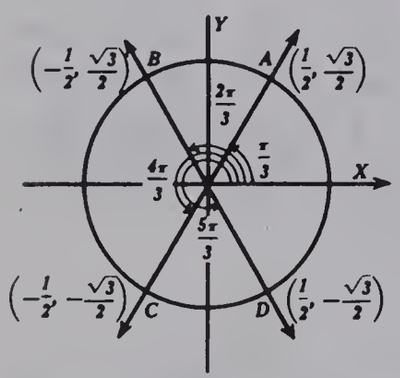
(ii) $\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}$ (iii) $\left(\frac{\pi}{4}, \frac{1}{\sqrt{2}}\right), \left(\frac{3\pi}{4}, \frac{1}{\sqrt{2}}\right)$; $\left(\frac{5\pi}{4}, -\frac{1}{\sqrt{2}}\right), \left(\frac{7\pi}{4}, -\frac{1}{\sqrt{2}}\right)$ (iv) $\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}$

(v) $\left(\frac{\pi}{4}, \frac{1}{\sqrt{2}}\right), \left(\frac{3\pi}{4}, -\frac{1}{\sqrt{2}}\right)$; $\left(\frac{5\pi}{4}, -\frac{1}{\sqrt{2}}\right), \left(\frac{7\pi}{4}, \frac{1}{\sqrt{2}}\right)$

(vi) 1, −1, 1, −1

(vii) $\left(\frac{\pi}{4}, 1\right), \left(\frac{3\pi}{4}, -1\right)$; $\left(\frac{5\pi}{4}, 1\right), \left(\frac{7\pi}{4}, -1\right)$

4. (i)

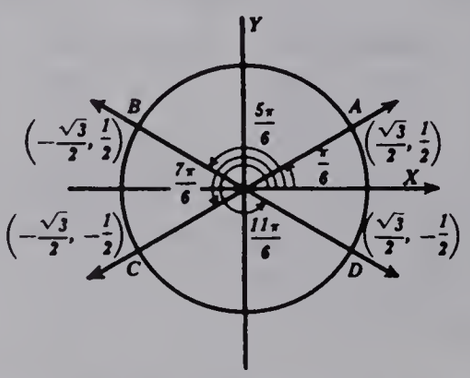


(ii) $\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2}$

(iii) $\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}$

(iv) $\sqrt{3}, -\sqrt{3}, \sqrt{3}, -\sqrt{3}$

5. (i)

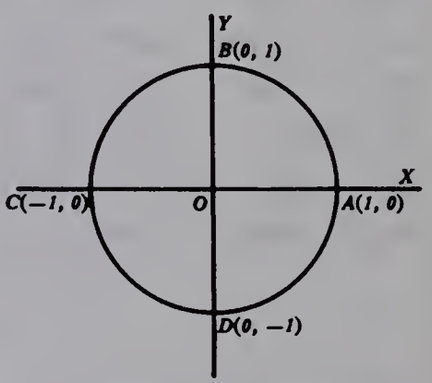


(ii) $\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}$

(iii) $\frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}$

(iv) $\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}$

6. (i)



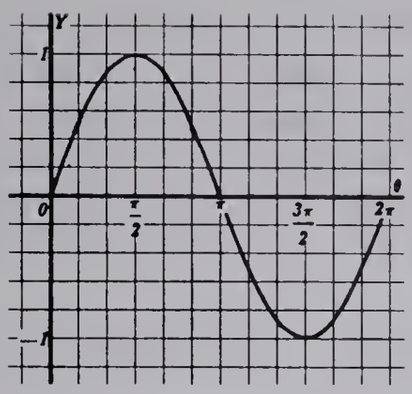
(ii) 0, 1, 0, −1, 0

(iii) 1, 0, −1, 0, 1

7. (i)

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{\sqrt{2}}$	$\frac{-\sqrt{3}}{2}$	−1	$\frac{-\sqrt{3}}{2}$	$\frac{-1}{\sqrt{2}}$	$-\frac{1}{2}$	0
			0.7	0.9		0.9	0.7				−0.7	−0.9		−0.9	−0.7		

(ii)

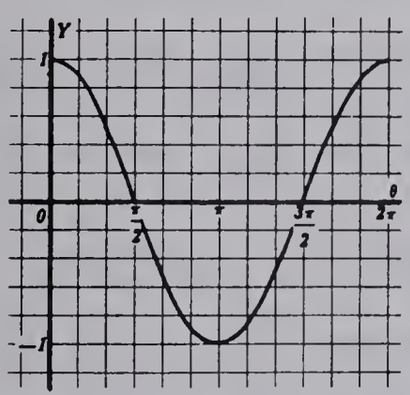


(iii) The extension of the graph is a repetition of the part already drawn.

8. (i)

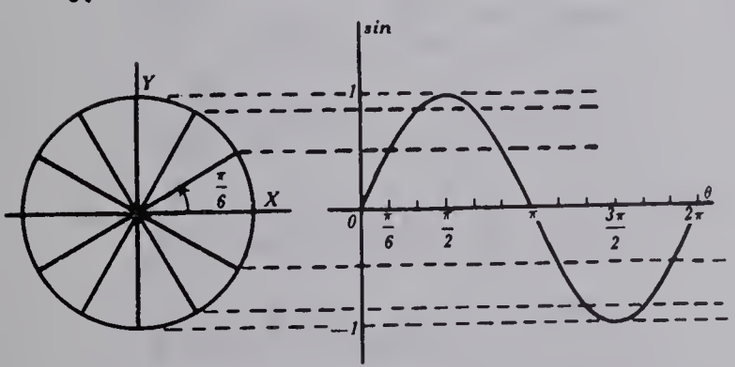
θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{\sqrt{3}}{2}$	−1	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
		0.9	0.7				−0.7	−0.9		−0.9	−0.7				0.7	0.9	

(ii)

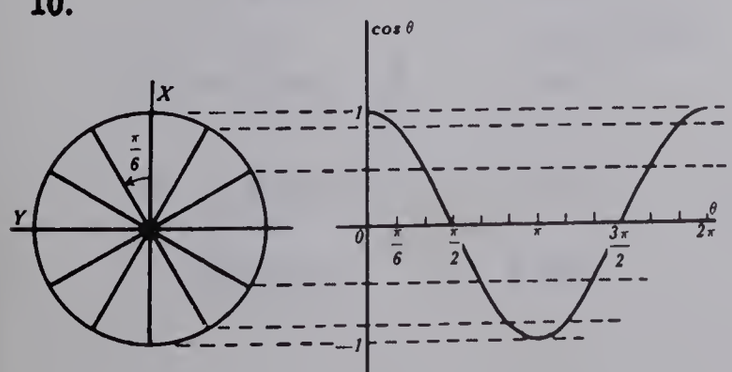


(iii) The graphs of the sine and cosine functions appear to have the same shape. The graph of the cosine function is the graph of the sine function displaced a distance $\frac{\pi}{2}$ to the left.

9.



10.



Exercise 12-7 (page 371).

16. .8746 **17.** 3.7321 **18.** .8480 **19.** 2.790
20. .5095 **21.** .0872 **22.** 1.2868 **23.** .9004
24. .1736 **25.** 1.2690 **26.** 28.654 **27.** 1.0000
28. 49° **29.** 65° **30.** 27° **31.** 68°
32. 35° **33.** 84° **34.** 53° **35.** 21°
36. 5° **37.** 78°

38. .7071 **39.** −.5000 **40.** 1.0000 **41.** −.7071
42. +1.1918 **43.** 1.7321 **44.** −.8660 **45.** −.5774 **46.** −.5000 **47.** .8660.
48. .5000 **49.** −1.7321 **50.** .8660 **51.** −.8660 **52.** .9012 **53.** .3827.
54. −.9848 **55.** −.3420

Exercise 12-8 (page 374).

2. (i) $\sin A = \frac{2}{\sqrt{13}}$, $\sin C = \frac{3}{\sqrt{13}}$; $\cos A = \frac{3}{\sqrt{13}}$, $\cos C = \frac{2}{\sqrt{13}}$; $\tan A = \frac{2}{3}$, $\tan C = \frac{3}{2}$; $\csc A = \frac{\sqrt{13}}{2}$, $\csc C = \frac{\sqrt{13}}{3}$; $\sec A = \frac{\sqrt{13}}{3}$, $\sec C = \frac{\sqrt{13}}{2}$; $\cot A = \frac{3}{2}$, $\cot C = \frac{2}{3}$

(ii) $\sin B = \frac{2}{\sqrt{29}}$, $\sin C = \frac{5}{\sqrt{29}}$; $\cos B = \frac{5}{\sqrt{29}}$, $\cos C = \frac{2}{\sqrt{29}}$; $\tan B = \frac{2}{5}$, $\tan C = \frac{5}{2}$; $\csc B = \frac{\sqrt{29}}{2}$, $\csc C = \frac{\sqrt{29}}{5}$; $\sec B = \frac{\sqrt{29}}{5}$, $\sec C = \frac{\sqrt{29}}{2}$; $\cot B = \frac{5}{2}$, $\cot C = \frac{2}{5}$

(iii) $\sin A = \frac{4}{5}$, $\sin B = \frac{3}{5}$; $\cos A = \frac{3}{5}$, $\cos B = \frac{4}{5}$; $\tan A = \frac{4}{3}$, $\tan B = \frac{3}{4}$; $\csc A = \frac{5}{4}$, $\csc B = \frac{5}{3}$; $\sec A = \frac{5}{3}$, $\sec B = \frac{5}{4}$; $\cot A = \frac{3}{4}$, $\cot B = \frac{4}{3}$

(iv) $\sin A = \frac{1}{\sqrt{2}}$, $\sin B = \frac{1}{\sqrt{2}}$; $\cos A = \frac{1}{\sqrt{2}}$, $\cos B = \frac{1}{\sqrt{2}}$; $\tan A = 1$, $\tan B = 1$; $\csc A = \sqrt{2}$, $\csc B = \sqrt{2}$; $\sec A = \sqrt{2}$, $\sec B = \sqrt{2}$; $\cot A = 1$, $\cot B = 1$

3. (i) $\sin \angle PQT = \frac{\sqrt{3}}{2}$, $\csc \angle PQT = \frac{2}{\sqrt{3}}$

$\cos \angle PQT = \frac{1}{2}$, $\sec \angle PQT = 2$

$\tan \angle PQT = \sqrt{3}$, $\cot \angle PQT = \frac{1}{\sqrt{3}}$

(ii) $\sin \angle QPT = \frac{1}{2}$, $\cos \angle QPT = \frac{\sqrt{3}}{2}$, $\tan \angle QPT = \frac{1}{\sqrt{3}}$,

$\csc \angle QPT = 2$, $\sec \angle QPT = \frac{2}{\sqrt{3}}$, $\cot \angle QPT = \sqrt{3}$

4. (i) $\sin A = \frac{BC}{AB}$, $\csc A = \frac{AB}{BC}$ (ii) $\sin A = \frac{DC}{AC}$, $\csc A = \frac{AC}{DC}$

$\cos A = \frac{AC}{AB}$, $\sec A = \frac{AB}{AC}$ $\cos A = \frac{AD}{AC}$, $\sec A = \frac{AC}{AD}$

$\tan A = \frac{BC}{AC}$, $\cot A = \frac{AC}{BC}$ $\tan A = \frac{DC}{AD}$, $\cot A = \frac{AD}{DC}$

5. (i) $\sin \angle ACB = \frac{AB}{BC} = \frac{AD}{AC}$ (ii) $\sin \angle AEB = \frac{AB}{AE} = \frac{BD}{BE}$

$\cos \angle ACB = \frac{AC}{BC} = \frac{DC}{AC}$ $\cos \angle AEB = \frac{BE}{AE} = \frac{DE}{BE}$

$\tan \angle ACB = \frac{AB}{AC} = \frac{AD}{DC}$ $\tan \angle AEB = \frac{AB}{BE} = \frac{BD}{DE}$

$\csc \angle ACB = \frac{BC}{AB} = \frac{AC}{AD}$ $\csc \angle AEB = \frac{AE}{AB} = \frac{BE}{BD}$

$\sec \angle ACB = \frac{BC}{AC} = \frac{AC}{DC}$ $\sec \angle AEB = \frac{AE}{BE} = \frac{BE}{DE}$

$\cot \angle ACB = \frac{AC}{AB} = \frac{DC}{AD}$ $\cot \angle AEB = \frac{BE}{AB} = \frac{DE}{BD}$

(iii) $\sin \angle BAE = \frac{BE}{AE} = \frac{BD}{AB}$ $\csc \angle BAE = \frac{AE}{BE} = \frac{AB}{BD}$

$\cos \angle BAE = \frac{AB}{AE} = \frac{AD}{AB}$ $\sec \angle BAE = \frac{AE}{AB} = \frac{AB}{AD}$

$\tan \angle BAE = \frac{BE}{AB} = \frac{BD}{AD}$ $\cot \angle BAE = \frac{AB}{BE} = \frac{AD}{BD}$

6. (i) csc 40° (ii) sin 15° (iii) cot 42°

(iv) cos 33° (v) sec 30° (vi) tan 24°

7. $\sin \angle ABD = \frac{1}{\sqrt{2}}$ $\csc \angle ABD = \sqrt{2}$

$\cos \angle ABD = \frac{1}{\sqrt{2}}$ $\sec \angle ABD = \sqrt{2}$

$\tan \angle ABD = 1$ $\cot \angle ABD = 1$

8. (i) $\sin A = \frac{3}{5}$, $\sin C = \frac{4}{5}$ (ii) $\sin A = \frac{2}{\sqrt{5}}$, $\sin C = \frac{1}{\sqrt{5}}$

$\cos A = \frac{4}{5}$, $\cos C = \frac{3}{5}$ $\cos A = \frac{1}{\sqrt{5}}$, $\cos C = \frac{2}{\sqrt{5}}$

$\tan A = \frac{3}{4}$, $\tan C = \frac{4}{3}$ $\tan A = 2$, $\tan C = \frac{1}{2}$

$\csc A = \frac{5}{3}$, $\csc C = \frac{5}{4}$ $\csc A = \frac{\sqrt{5}}{2}$, $\csc C = \sqrt{5}$

$\sec A = \frac{5}{4}$, $\sec C = \frac{5}{3}$ $\sec A = \sqrt{5}$, $\sec C = \frac{\sqrt{5}}{2}$

$\cot A = \frac{4}{3}$, $\cot C = \frac{3}{4}$ $\cot A = \frac{1}{2}$, $\cot C = 2$

(iii) $\sin A = \frac{8}{17}$, $\sin C = \frac{15}{17}$ (iv) $\sin A = \frac{21}{29}$, $\sin C = \frac{20}{29}$

$\cos A = \frac{15}{17}$, $\cos C = \frac{8}{17}$ $\cos A = \frac{20}{29}$, $\cos C = \frac{21}{29}$

$\tan A = \frac{8}{15}$, $\tan C = \frac{15}{8}$ $\tan A = \frac{21}{20}$, $\tan C = \frac{20}{21}$

$\csc A = \frac{17}{8}$, $\csc C = \frac{17}{15}$ $\csc A = \frac{29}{21}$, $\csc C = \frac{29}{20}$

$\sec A = \frac{17}{15}$, $\sec C = \frac{17}{8}$ $\sec A = \frac{29}{20}$, $\sec C = \frac{29}{21}$

$\cot A = \frac{15}{8}$, $\cot C = \frac{8}{15}$ $\cot A = \frac{20}{21}$, $\cot C = \frac{21}{20}$

Exercise 12-9 (page 377).

3. (i) $c = 5$, $\angle B = 67°$, $\angle C = 23°$ (ii) $b = 2.9$, $\angle A = 59°$, $\angle C = 31°$
(iii) $a = 16.0$, $b = 18.9$, $\angle C = 32°$ (iv) $a = 10.1$, $c = 11.6$, $\angle A = 61°$
(v) $a = 69.4$, $\angle A = 66°$, $\angle C = 24°$ (vi) $b = 11.8$, $c = 12.8$, $\angle B = 67°$
(vii) $b = 6.1$, $c = 5.1$, $\angle B = 50°$ (viii) $b = 33.2$, $\angle A = 44°$, $\angle C = 46°$
(ix) $a = 2.6$, $b = 8.0$, $\angle A = 18°$ (x) $a = 20.7$, $c = 5.4$, $\angle B = 75°$

4. 8.6 in.

Exercise 12-10 (page 380).

1. 700 ft. **2.** 321 ft. **3.** 56 ft. **4.** 480 ft. **5.** 56 ft.
6. 37 mi. **7.** 163 mi. **8.** 70 ft. **9.** 1,225 ft. **10.** 520 ft.
11. 190 ft. **12.** 432 yd. **13.** 1,963 yd. **14.** 3,565 ft.

Exercise 12-11 (page 384).

1. $\angle C = 78°$, $AC = 5.0$ in., $BC = 7.8$ in.
2. $\angle C = 10°$, $AC = 56$ ft., $AB = 14$ ft. **3.** $\angle B = 74°$, $AB = 44$ ft., $BC = 24$ ft.
4. 22.6 ft., 10.2 ft. **5.** 914 yd. from A; 1231 yd. from B **6.** 133 yd., 76 yd.

Exercise 12-12 (page 387). **1.** 1.2 in. **2.** 9.1 ft. **3.** 43° **4.** 30°, 45°
6. $\angle A = 29°$, $\angle C = 80°$, $AC = 4.8$ yd. **7.** 4.5 mi. **8.** 35°

Practice Exercise 12-13 (page 387). **1.** 30° **2.** 30° **3.** $\frac{\pi}{6}$ **4.** 30° **5.** $\frac{\pi}{5}$ **6.** $\frac{\pi}{3}$
7. 60° **8.** 30° **9.** $\frac{-\sqrt{3}}{2}$ **10.** $-\frac{1}{\sqrt{2}}$ **11.** $-\sqrt{3}$ **12.** -1 **13.** $\sqrt{3}$ **14.** $\frac{1}{2}$
15. $-\frac{1}{2}$ **16.** 0 **17.** $\frac{\sqrt{3}}{2}$

Practice Exercise 12-14 (page 388). **1.** 2 **2.** -1 **3.** 0 **4.** 1 **5.** $2\frac{1}{2}$ **6.** $4\frac{1}{3}$

Review Exercise 12-15 (page 388). **1.** $\frac{\pi}{6}$ **2.** $\frac{3\pi}{4}$ **3.** $\frac{3\pi}{2}$ **4.** $\frac{2\pi}{9}$ **5.** $-\frac{\pi}{18}$
6. $\frac{7\pi}{9}$ **7.** $\frac{\pi}{180}$ **8.** $-\frac{17\pi}{18}$ **9.** $\frac{5\pi}{2}$ **10.** 4π **11.** 30° **12.** 225° **13.** $\frac{180°}{\pi}$
14. $\frac{1800°}{\pi}$ **15.** 108° **16.** $-210°$ **17.** $-135°$ **18.** 420° **19.** $-\frac{360°}{\pi}$ **20.** 1440°

21. $5 + \sqrt{3}$ **22.** 6 **23.** $\frac{1}{2}$ **24.** 1

25. $\frac{1}{3}$ **26.** (i) $\pm\frac{\sqrt{5}}{2}$ (ii) $\pm\frac{3}{\sqrt{5}}$

29. 703 ft. **30.** 167 ft. **31.** 155 ft. **32.** 0.2 ft.

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